

FOURIER SERIES

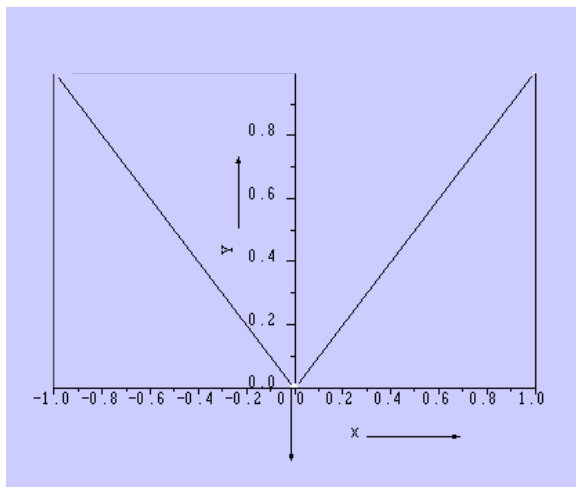
Introduction: In many engineering problems, especially in the study of periodic phenomena in conduction of heat, electro-dynamics and acoustics, it is necessary to express a function in a series of sines and cosines. Such a series is known as the **Fourier series**.

DEFINITIONS:

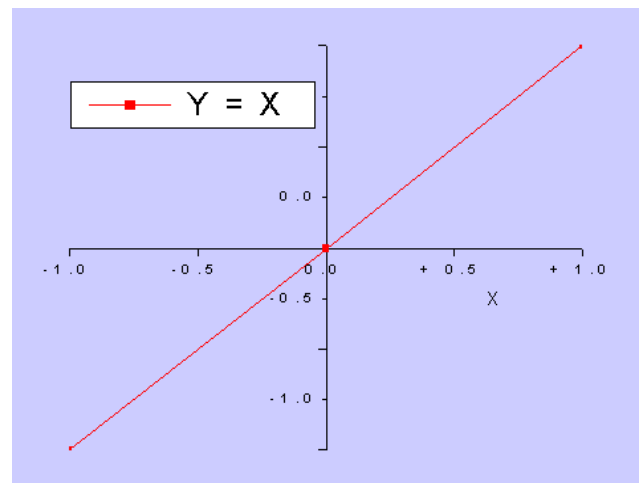
A function $y = f(x)$ is said to be even, if $f(-x) = f(x)$. The graph of the even function is always symmetrical about the y-axis.

A function $y = f(x)$ is said to be odd, if $f(-x) = -f(x)$. The graph of the odd function is always symmetrical about the origin.

For example, the function $f(x) = |x|$ in $[-1, 1]$ is even as $f(-x) = |-x| = |x| = f(x)$ and the function $f(x) = x$ in $[-1, 1]$ is odd as $f(-x) = -x = -f(x)$. The graphs of these functions are shown below:



Graph of $f(x) = |x|$



Graph of $f(x) = x$

Note that the graph of $f(x) = |x|$ is symmetrical about the y-axis and the graph of $f(x) = x$ is symmetrical about the origin.

1. If $f(x)$ is even and $g(x)$ is odd, then

- $h(x) = f(x) \times g(x)$ is odd
- $h(x) = f(x) \times f(x)$ is even
- $h(x) = g(x) \times g(x)$ is even

For example,

1. $h(x) = x^2 \cos x$ is even, since both x^2 and $\cos x$ are even functions
2. $h(x) = x \sin x$ is even, since x and $\sin x$ are odd functions
3. $h(x) = x^2 \sin x$ is odd, since x^2 is even and $\sin x$ is odd.

2. If $f(x)$ is even, then
$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

3. If $f(x)$ is odd, then
$$\int_{-a}^a f(x) dx = 0$$

For example,

$$\int_{-a}^a \cos x dx = 2 \int_0^a \cos x dx, \text{ as } \cos x \text{ is even}$$

$$\text{and } \int_{-a}^a \sin x dx = 0, \text{ as } \sin x \text{ is odd}$$

PERIODIC FUNCTIONS:-

A function $f(x)$ is said to be periodic function with period T if $f(x+T) = f(x)$

Here $f(x)$ is a real-valued function and T is a positive real number.

As a consequence, it follows that

$$f(x) = f(x+T) = f(x+2T) = f(x+3T) = \dots = f(x+nT)$$

Thus, $f(x) = f(x+nT)$, $n = 1, 2, 3, 4, \dots$

The function $f(x) = \sin x$ is periodic of period 2π since

$$\sin(x+2n\pi) = \sin x, \quad n=1,2,3,\dots$$

Note that the graph of the function between 0 and 2π is the same as that between 2π and 4π and so on. It may be verified that a linear combination of periodic functions having period T is also periodic of period T .

EULER'S FORMULAE:

The Fourier series for the function $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\text{Where } a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx, \quad b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx dx,$$

These values of a_0, a_n, b_n are known as Euler's formulae.

Proof: Let $f(x)$ be represented in the interval $(\alpha, \alpha + 2\pi)$ by the Fourier series-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{-----} (1)$$

To find a_0, a_n, b_n , assume that the series (1) can be integrated term by term from $x = \alpha$ to $x = \alpha + 2\pi$.

$$\begin{aligned} \text{To find } a_0: \int_{\alpha}^{\alpha+2\pi} f(x) dx &= \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} a_n \cos nx \right) dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} b_n \sin nx \right) dx \\ &= \frac{1}{2} a_0 (\alpha + 2\pi - \alpha) + 0 + 0 = a_0 \pi \end{aligned}$$

$$\text{Since } \int_{\alpha}^{\alpha+2\pi} \cos nx dx = \frac{\sin nx}{n} \Big|_{\alpha}^{\alpha+2\pi} = 0 \text{ \& } \int_{\alpha}^{\alpha+2\pi} \sin nx dx = -\frac{\cos nx}{n} \Big|_{\alpha}^{\alpha+2\pi} = 0$$

$$\therefore a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) dx$$

To find a_n , multiply each side of (1) by $\cos nx$ and integrate from $x = \alpha$ to $x = \alpha + 2\pi$, we get

$$\begin{aligned} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx &= \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} \cos nx dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} a_n \cos nx \right) \cos nx dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} b_n \sin nx \right) \cos nx dx \\ &= 0 + \pi a_n + 0 = \pi a_n \end{aligned}$$

Since

$$\begin{aligned} \int_{\alpha}^{\alpha+2\pi} \cos^2 nx dx &= \int_{\alpha}^{\alpha+2\pi} \frac{1 + \cos 2nx}{2} dx = \frac{1}{2} \left(x + \frac{\sin 2nx}{2n} \right) \Big|_{\alpha}^{\alpha+2\pi} = \frac{1}{2} (\alpha + 2\pi - \alpha + 0 - 0) = \pi \\ \& \int_{\alpha}^{\alpha+2\pi} \sin nx \cos nx dx &= \frac{\sin^2 nx}{2n} \Big|_{\alpha}^{\alpha+2\pi} = 0 \end{aligned}$$

$$\text{Hence } a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx$$

Similarly To find b_n , multiply each side of (1) by $\sin nx$ and integrate from $x = \alpha$ to $x = \alpha + 2\pi$, we get

$$\begin{aligned} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx dx &= \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} \sin nx dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} a_n \cos nx \right) \sin nx dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} b_n \sin nx \right) \sin nx dx \\ &= 0 + 0 + \pi b_n = \pi b_n \end{aligned}$$

Since

$$\begin{aligned} \int_{\alpha}^{\alpha+2\pi} \sin^2 nx dx &= \int_{\alpha}^{\alpha+2\pi} \frac{1 - \cos 2nx}{2} dx = \frac{1}{2} \left(x - \frac{\sin 2nx}{2n} \right) \Big|_{\alpha}^{\alpha+2\pi} = \frac{1}{2} (\alpha + 2\pi - \alpha + 0 - 0) = \pi \\ \& \int_{\alpha}^{\alpha+2\pi} \sin nx \cos nx dx &= \frac{\sin^2 nx}{2n} \Big|_{\alpha}^{\alpha+2\pi} = 0 \end{aligned}$$

$$\text{Hence } b_n = \frac{1}{\pi} \int_a^{a+2\pi} f(x) \sin nx dx$$

FOURIER SERIES:

A Fourier series of a periodic function consists of a sum of sine and cosine terms. Sines and cosines are the most fundamental periodic functions.

The Fourier series is named after the French Mathematician and Physicist Jacques Fourier (1768 – 1830).

FORMULA FOR FOURIER SERIES

Consider a real-valued function $f(x)$ which obeys the following conditions called Dirichlet's conditions:

1. $f(x)$ is defined in an interval $(a, a+2l)$, and $f(x+2l) = f(x)$ so that $f(x)$ is a periodic function of period $2l$.
2. $f(x)$ is continuous or has only a finite number of discontinuities in the interval $(a, a+2l)$.
3. $f(x)$ has no or only a finite number of maxima & minima in the interval $(a, a+2l)$.

$$\text{Also, let } a_0 = \frac{1}{l} \int_a^{a+2l} f(x) dx \text{ ----- (1)}$$

$$a_n = \frac{1}{l} \int_a^{a+2l} f(x) \cos\left(\frac{n\pi}{l}\right) x dx, n = 1, 2, 3, \dots \text{ ----- (2)}$$

$$b_n = \frac{1}{l} \int_a^{a+2l} f(x) \sin\left(\frac{n\pi}{l}\right) x dx, n = 1, 2, 3, \dots \text{ ----- (3)}$$

Then, the infinite series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right) x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right) x \text{ ----- (4)}$$

is called the Fourier series of $f(x)$ in the interval $(a, a+2l)$. Also, the real numbers $a_0, a_1, a_2, \dots, a_n$, and $b_1, b_2, \dots, b_n$ are called the Fourier coefficients of $f(x)$. The formulae (1), (2) and (3) are called Euler's formulae.

It can be proved that the sum of the series (4) is $f(x)$ if $f(x)$ is continuous at x . Thus we have $f(x)$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right) x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right) x \dots \dots \dots (5)$$

Suppose $f(x)$ is discontinuous at x , then the sum of the series (4) would be

$$\frac{1}{2} [f(x^+) + f(x^-)]$$

where $f(x^+)$ & $f(x^-)$ are respectively right hand and left hand limits of $f(x)$ given by.

$$f(x^+) = \lim_{h \rightarrow 0} f(x+h), \quad f(x^-) = \lim_{h \rightarrow 0} f(x-h), \quad h > 0$$

Particular Cases:

Case (i)

Suppose $a = 0$. Then $f(x)$ is defined over the interval $(0, 2l)$. Formulae (1), (2), (3) reduce to

$$\begin{aligned} a_0 &= \frac{1}{l} \int_0^{2l} f(x) dx \\ a_n &= \frac{1}{l} \int_0^{2l} f(x) \cos\left(\frac{n\pi}{l}x\right) dx, \quad n = 1, 2, \dots, \infty \text{ ----- (6)} \\ b_n &= \frac{1}{l} \int_0^{2l} f(x) \sin\left(\frac{n\pi}{l}x\right) dx, \end{aligned}$$

Then the right-hand side of (5) is the Fourier expansion of $f(x)$ over the interval $(0, 2l)$.

If we set $l = \pi$, then $f(x)$ is defined over the interval $(0, 2\pi)$. Formulae (6) reduce to

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x) dx \\ a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx, \quad n = 1, 2, \dots, \infty \text{ ----- (7)} \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx \quad n = 1, 2, \dots, \infty \end{aligned}$$

$$\text{Also, in this case, (5) becomes: } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{ ----- (8)}$$

Case (ii)

Suppose $a = -l$. Then $f(x)$ is defined over the interval $(-l, l)$. Formulae (1), (2) (3) reduce to

$$\begin{aligned} a_0 &= \frac{1}{l} \int_{-l}^l f(x) dx \\ a_n &= \frac{1}{l} \int_{-l}^l f(x) \cos\left(\frac{n\pi}{l}x\right) dx, \quad n = 1, 2, \dots, \infty \text{ ----- (9)} \\ b_n &= \frac{1}{l} \int_{-l}^l f(x) \sin\left(\frac{n\pi}{l}x\right) dx \quad n = 1, 2, \dots, \infty \end{aligned}$$

Then the right-hand side of (5) is the Fourier expansion of $f(x)$ over the interval $(-l, l)$.

If we set $l = \pi$, then $f(x)$ is defined over the interval $(-\pi, \pi)$. Formulae (9) reduce to

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad n=1,2,\dots,\infty \quad \text{-----} \quad (10) \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \quad n=1,2,\dots,\infty \end{aligned}$$

$$\text{Putting } l = \pi \text{ in (5), we get } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Some useful results:

1. The following rule called Bernoulli's generalized rule of integration by parts is useful in evaluating the Fourier coefficients.

$$\int u v dx = u v_1 - u' v_2 + u'' v_3 + \dots$$

Here $u', u'', \dots$ are the successive derivatives of u and $v_1 = \int v dx, v_2 = \int v_1 dx, \dots$

We illustrate the rule, through the following examples:

$$\begin{aligned} \int x^2 \sin nx dx &= x^2 \left(\frac{-\cos nx}{n} \right) - 2x \left(\frac{-\sin nx}{n^2} \right) + 2 \left(\frac{\cos nx}{n^3} \right) \\ \int x^3 e^{2x} dx &= x^3 \left(\frac{e^{2x}}{2} \right) - 3x^2 \left(\frac{e^{2x}}{4} \right) + 6x \left(\frac{e^{2x}}{8} \right) - 6 \left(\frac{e^{2x}}{16} \right) \end{aligned}$$

2. The following integrals are also useful:

$$\begin{aligned} \int e^{ax} \cos bxdx &= \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx] \\ \int e^{ax} \sin bxdx &= \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx] \end{aligned}$$

3. If 'n' is integer, then

$$\begin{aligned} \sin n\pi &= 0, \quad \cos n\pi = (-1)^n, \quad \sin 2n\pi = 0, \quad \cos 2n\pi = 1 \\ \sin(n+1/2)\pi &= (-1)^n, \quad \cos(n+1/2)\pi = 0 \end{aligned}$$

Examples

1. Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$

$$\text{We have } a_0 = \frac{1}{\pi} \int_0^{2\pi} e^{-x} dx = \frac{e^{-x}}{-1} \Big|_0^{2\pi} = -\frac{1}{\pi} (e^{-2\pi} - 1) = \frac{1 - e^{-2\pi}}{\pi}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_0^{2\pi} e^{-x} \cos nx dx = \frac{1}{\pi} \left[\frac{e^{-x}}{1+n^2} (-\cos nx + n \sin nx) \right]_0^{2\pi} \\
&= \frac{1}{\pi(1+n^2)} \left[e^{-2\pi} (-\cos 2n\pi + n \sin 2n\pi) - e^0 (-\cos 0 + n \sin 0) \right] \\
&= \frac{1}{\pi(1+n^2)} \left[-e^{-2\pi} + 1 \right] = \left(\frac{1-e^{-2\pi}}{\pi} \right) \cdot \frac{1}{1+n^2} \\
b_n &= \frac{1}{\pi} \int_0^{2\pi} e^{-x} \sin nx dx = \frac{1}{\pi} \left[\frac{e^{-x}}{1+n^2} (-\sin nx - n \cos nx) \right]_0^{2\pi} \\
&= \frac{1}{\pi(1+n^2)} \left[e^{-2\pi} (-\sin 2n\pi - n \cos 2n\pi) - e^0 (-\sin 0 - n \cos 0) \right] \\
&= \frac{1}{\pi(1+n^2)} \left[-ne^{-2\pi} + n \right] = \left(\frac{1-e^{-2\pi}}{\pi} \right) \cdot \frac{n}{1+n^2} \\
\therefore e^{-x} &= \frac{1-e^{-2\pi}}{2\pi} + \sum_{n=1}^{\infty} \left(\frac{1-e^{-2\pi}}{\pi} \cdot \frac{1}{1+n^2} \right) \cos nx + \sum_{n=1}^{\infty} \left(\frac{1-e^{-2\pi}}{\pi} \cdot \frac{n}{1+n^2} \right) \sin nx \\
&= \frac{1-e^{-2\pi}}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{1}{1+n^2} \right) \cos nx + \sum_{n=1}^{\infty} \left(\frac{n}{1+n^2} \right) \sin nx \right]
\end{aligned}$$

2. Obtain the Fourier expansion of $f(x) = \frac{1}{2}(\pi - x)$ in $-\pi < x < \pi$

$$\text{We have, } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2}(\pi - x) dx = \frac{1}{2\pi} \left[\pi x - \frac{x^2}{2} \right]_{-\pi}^{\pi} = \pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2}(\pi - x) \cos nx dx$$

Here we use integration by parts, so that

$$a_n = \frac{1}{2\pi} \left[(\pi - x) \frac{\sin nx}{n} - (-1) \left(\frac{-\cos nx}{n^2} \right) \right]_{-\pi}^{\pi} = \frac{1}{2\pi} [0] = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2}(\pi - x) \sin nx dx = \frac{1}{2\pi} \left[(\pi - x) \frac{-\cos nx}{n} - (-1) \left(\frac{-\sin nx}{n^2} \right) \right]_{-\pi}^{\pi} = \frac{(-1)^n}{n}$$

Using the values of a_0 , a_n and b_n in the Fourier expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\text{we get, } f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$$

This is the required Fourier expansion of the given function.

3. Obtain the Fourier expansion of $f(x) = e^{-ax}$ in the interval $(-\pi, \pi)$. Deduce that

$$\frac{\pi}{\sinh \pi} = 2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 + 1}$$

$$\text{Here, } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-ax} dx = \frac{1}{\pi} \left[\frac{e^{-ax}}{-a} \right]_{-\pi}^{\pi} = \frac{e^{a\pi} - e^{-a\pi}}{a\pi} = \frac{2 \sinh a\pi}{a\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-ax} \cos nx dx = \frac{1}{\pi} \left[\frac{e^{-ax}}{a^2 + n^2} \{-a \cos nx + n \sin nx\} \right]_{-\pi}^{\pi} = \frac{2a}{\pi} \left[\frac{(-1)^n \sinh a\pi}{a^2 + n^2} \right]$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-ax} \sin nx dx = \frac{1}{\pi} \left[\frac{e^{-ax}}{a^2 + n^2} \{-a \sin nx - n \cos nx\} \right]_{-\pi}^{\pi} = \frac{2n}{\pi} \left[\frac{(-1)^n \sinh a\pi}{a^2 + n^2} \right]$$

$$\text{Thus, } f(x) = \frac{\sinh a\pi}{a\pi} + \frac{2a \sinh a\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2 + n^2} \cos nx + \frac{2}{\pi} \sinh a\pi \sum_{n=1}^{\infty} \frac{n(-1)^n}{a^2 + n^2} \sin nx$$

For $x = 0$, $a = 1$, the series reduces to

$$f(0) = 1 = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 1}$$

$$\text{or } 1 = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \left[-\frac{1}{2} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 + 1} \right]$$

$$\text{or } 1 = \frac{2 \sinh \pi}{\pi} \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 + 1}$$

$$\text{Thus, } \frac{\pi}{\sinh \pi} = 2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 + 1}. \text{ This is the desired deduction.}$$

4. Obtain the Fourier expansion of $f(x) = x^2$ over the interval $(-\pi, \pi)$. Deduce that

$$\text{a) } \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots + \infty \quad \text{b) } \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \infty$$

$$\text{c) } \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots + \infty$$

The function $f(x)$ is even. Hence

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx, \text{ since } f(x) \cos nx \text{ is even} \\ &= \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx, \text{ Integrating by parts, we get} \end{aligned}$$

$$= \frac{2}{\pi} \left[x^2 \left(\frac{\sin nx}{n} \right) - 2x \left(\frac{-\cos nx}{n^2} \right) + 2 \left(\frac{-\sin nx}{n^3} \right) \right]_0^\pi = \frac{4(-1)^n}{n^2}$$

Also, $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = 0$, since $f(x) \sin nx$ is odd.

Thus

$$f(x) = x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2} \text{----- (i)}$$

a) By putting $x = 0$, in (i) we get,

$$0 = f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos 0}{n^2} = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\text{i.e. } -\frac{\pi^2}{3} = 4 \left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots \right)$$

$$\text{i.e. } \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

b) by putting $x = \pi$, we get,

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\text{Hence, } \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

c) by adding (a) and (b) we get,

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots + 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

$$\text{i.e. } \frac{3\pi^2}{12} = 2 + \frac{2}{3^2} + \frac{2}{5^2} + \dots \Rightarrow \frac{\pi^2}{4} = 2 \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right)$$

$$\text{Or } \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

Functions having points of discontinuity:

5. Obtain the Fourier expansion of

$$f(x) = \begin{cases} x, 0 \leq x \leq \pi \\ 2\pi - x, \pi \leq x \leq 2\pi \end{cases}$$

Deduce that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

$$\text{Here, } a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \left\{ \int_0^{\pi} f(x) dx + \int_{\pi}^{2\pi} f(x) dx \right\} = \frac{1}{\pi} \left\{ \int_0^{\pi} x dx + \int_{\pi}^{2\pi} (2\pi - x) dx \right\}$$

$$\begin{aligned}
&= \frac{1}{\pi} \left\{ \frac{x^2}{2} \Big|_0^\pi + \left(2\pi x - \frac{x^2}{2} \right) \frac{2\pi}{\pi} \right\} = \frac{1}{\pi} \left\{ \frac{\pi^2}{2} + 4\pi^2 - \frac{4\pi^2}{2} - (2\pi^2 - \frac{\pi^2}{2}) \right\} \\
&= \frac{1}{2\pi} \left\{ \pi^2 + 8\pi^2 - 4\pi^2 - 4\pi^2 + \pi^2 \right\} = \pi \Rightarrow \frac{a_0}{2} = \frac{\pi}{2}.
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx = \frac{1}{\pi} \left\{ \int_0^\pi f(x) \cos nx dx + \int_\pi^{2\pi} f(x) \cos nx dx \right\} \\
&= \frac{1}{\pi} \left\{ \int_0^\pi x \cos nx dx + \int_\pi^{2\pi} (2\pi - x) \cos nx dx \right\}, \text{ integrating using Bernoulli's generalized rule} \\
&= \frac{1}{\pi} \left\{ \left[x \left(\frac{\sin nx}{n} \right) - (1) \left(\frac{-\cos nx}{n^2} \right) \right]_0^\pi + \left[(2\pi - x) \left(\frac{\sin nx}{n} \right) - (-1) \left(\frac{-\cos nx}{n^2} \right) \right]_\pi^{2\pi} \right\} \\
&= \frac{1}{\pi} \left\{ \left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^\pi + \left[(2\pi - x) \left(\frac{\sin nx}{n} \right) - \frac{\cos nx}{n^2} \right]_\pi^{2\pi} \right\} \\
&= \frac{1}{\pi} \left\{ \left(\frac{(-1)^n}{n^2} - \frac{1}{n^2} \right) + \left(0 - \frac{1}{n^2} - \left(0 - \frac{(-1)^n}{n^2} \right) \right) \right\} \\
&= \frac{1}{\pi} \left\{ \frac{2(-1)^n - 2}{n^2} \right\} = \frac{2}{\pi n^2} \{ (-1)^n - 1 \}
\end{aligned}$$

$$\begin{aligned}
\text{Also, } b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx = \frac{1}{\pi} \left\{ \int_0^\pi f(x) \sin nx dx + \int_\pi^{2\pi} f(x) \sin nx dx \right\}, \\
&= \frac{1}{\pi} \left\{ \int_0^\pi x \sin nx dx + \int_\pi^{2\pi} (2\pi - x) \sin nx dx \right\}, \text{ integrating using Bernoulli's generalized rule} \\
&= \frac{1}{\pi} \left\{ \left[x \left(\frac{-\cos nx}{n} \right) - (1) \left(\frac{-\sin nx}{n^2} \right) \right]_0^\pi + \left[(2\pi - x) \left(\frac{-\cos nx}{n} \right) - (-1) \left(\frac{-\sin nx}{n^2} \right) \right]_\pi^{2\pi} \right\} \\
&= \frac{1}{\pi} \left\{ \left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^\pi + \left[-(2\pi - x) \frac{\cos nx}{n} - \frac{\sin nx}{n^2} \right]_\pi^{2\pi} \right\} \\
&= \frac{1}{\pi} \left\{ \frac{-\pi(-1)^n}{n} + 0 - \left(\frac{-\pi(-1)^n}{n} - 0 \right) \right\} = \frac{1}{\pi} \left\{ \frac{-\pi(-1)^n}{n} + \frac{\pi(-1)^n}{n} \right\} = 0
\end{aligned}$$

Thus the Fourier series of $f(x)$ is $f(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} [(-1)^n - 1] \cos nx$

For $x = \pi$, we get

$$f(\pi) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} [(-1)^n - 1] \cos n\pi$$

But $(-1)^n - 1 = 0$, if n is even & $(-1)^n - 1 = -2$, if n is odd.

(since $f(x)$ is discontinuous at $x = \pi$, $f(\pi) = \frac{1}{2}[f(\pi-0) + f(\pi+0)] = \frac{1}{2}(x + 2\pi - x) = \pi$)

$$\therefore \pi = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{-2 \cos(2n-1)\pi}{(2n-1)^2}$$

$$\text{Thus, } \frac{\pi}{2} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{-2(-1)}{(2n-1)^2} \Rightarrow \frac{\pi^2}{4} = \sum_{n=1}^{\infty} \frac{2}{(2n-1)^2} \Rightarrow \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

This is the series as required.

6. Obtain the Fourier expansion of

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases} \quad \text{hence deduce that } \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

$$\begin{aligned} \text{Here, } a_0 &= \frac{1}{\pi} \left[\int_{-\pi}^0 -\pi dx + \int_0^{\pi} x dx \right] = \frac{1}{\pi} \left\{ \left[-\pi x \right]_{-\pi}^0 + \left[\frac{x^2}{2} \right]_0^{\pi} \right\} = \frac{1}{\pi} \left[-\pi(0 - (-\pi)) + \frac{1}{2}(\pi^2 - 0) \right] \\ &= \frac{1}{\pi} \left(-\pi^2 + \frac{\pi^2}{2} \right) = -\frac{1}{\pi} \times \frac{\pi^2}{2} = -\frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \left[\int_{-\pi}^0 -\pi \cos nx dx + \int_0^{\pi} x \cos nx dx \right] \\ &= \frac{1}{\pi} \left[\left(-\pi \frac{\sin nx}{n} \right)_{-\pi}^0 + \left(x \frac{\sin nx}{n} - (1) \frac{-\cos nx}{n^2} \right)_{\pi}^0 \right] \\ &= \frac{1}{\pi} \left[0 + 0 + \frac{(-1)^n}{n^2} - \frac{1}{n^2} \right] = \frac{1}{\pi n^2} [(-1)^n - 1] \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \left[\int_{-\pi}^0 -\pi \sin nx dx + \int_0^{\pi} x \sin nx dx \right] \\ &= \frac{1}{\pi} \left[\left(-\pi \frac{-\cos nx}{n} \right)_{-\pi}^0 + \left\{ x \left(\frac{-\cos nx}{n} \right) - (1) \left(\frac{-\sin nx}{n^2} \right) \right\}_{\pi}^0 \right] \\ &= \frac{1}{\pi} \left[\frac{\pi}{n} (\cos 0 - (-1)^n) - \frac{\pi}{n} (-1)^n + 0 \right] = \frac{1}{\pi} \left[\frac{\pi}{n} (1 - 2(-1)^n) \right] = \frac{1}{n} [1 - 2(-1)^n] \end{aligned}$$

$\therefore$ Fourier series is

$$f(x) = \frac{-\pi}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} [(-1)^n - 1] \cos nx + \sum_{n=1}^{\infty} \left[\frac{1 - 2(-1)^n}{n} \right] \sin nx$$

Note that the point $x = 0$ is a point of discontinuity of $f(x)$. Here $f(x^+) = 0$, $f(x^-) = -\pi$ at $x = 0$. Hence $\frac{1}{2}[f(x^+) + f(x^-)] = \frac{1}{2}(0 - \pi) = \frac{-\pi}{2}$

The Fourier expansion of $f(x)$ at $x = 0$ becomes

$$\frac{-\pi}{2} = \frac{-\pi}{4} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} [(-1)^n - 1]$$

$$\text{or } \frac{\pi^2}{4} = \sum_{n=1}^{\infty} \frac{1}{n^2} [(-1)^n - 1]$$

But $(-1)^n - 1 = 0$, if n is even & $(-1)^n - 1 = -2$, if n is odd.

$$\therefore \frac{\pi^2}{4} = \sum_{n=1}^{\infty} \frac{2}{(2n-1)^2} \Rightarrow \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

$$\text{Hence } \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

7. Obtain the Fourier series of $f(x) = 1 - x^2$ over the interval $(-1, 1)$.

The given function is even as $f(-x) = f(x)$. Also period of $f(x)$ is $1 - (-1) = 2 \Rightarrow 2l = 2 \Rightarrow l = 1$

$$\text{Here } a_0 = \frac{1}{1} \int_{-1}^1 f(x) dx = 2 \int_0^1 f(x) dx = 2 \int_0^1 (1 - x^2) dx = 2 \left[x - \frac{x^3}{3} \right]_0^1 = \frac{4}{3} \text{ as } f(x) \text{ is even.}$$

$$a_n = \frac{1}{1} \int_{-1}^1 f(x) \cos(n\pi x) dx = 2 \int_0^1 f(x) \cos(n\pi x) dx$$

$$= 2 \int_0^1 (1 - x^2) \cos(n\pi x) dx \text{ [as } f(x) \cos n\pi x \text{ is even. Integrating by parts, we get}$$

$$= 2 \left[\left(1 - x^2 \right) \left(\frac{\sin n\pi x}{n\pi} \right) - (-2x) \left(\frac{-\cos n\pi x}{(n\pi)^2} \right) + (-2) \left(\frac{-\sin n\pi x}{(n\pi)^3} \right) \right]_0^1$$

$$= 2 \left[[0 - 0 - \frac{2}{(n\pi)^2} [(-1)^n - 0] + \frac{2}{(n\pi)^3} (0)] \right] = \frac{4(-1)^{n+1}}{n^2 \pi^2}$$

$$b_n = \frac{1}{1} \int_{-1}^1 f(x) \sin(n\pi x) dx = 0, \text{ since } f(x) \sin(n\pi x) \text{ is odd.}$$

$$\text{The Fourier series of } f(x) \text{ is } f(x) = \frac{2}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos(n\pi x)$$

8. Obtain the Fourier expansion of

$$f(x) = \begin{cases} 1 + \frac{4x}{3} & -\frac{3}{2} < x \leq 0 \\ 1 - \frac{4x}{3} & 0 \leq x < \frac{3}{2} \end{cases} \quad \text{Hence deduce that } \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

$$\text{The period of } f(x) \text{ is } \frac{3}{2} - \left(-\frac{3}{2} \right) = 3 \Rightarrow 2l = 3 \Rightarrow l = 3/2$$

Also $f(-x) = f(x)$. Hence $f(x)$ is even

$$a_0 = \frac{1}{3/2} \int_{-3/2}^{3/2} f(x) dx = \frac{2}{3/2} \int_0^{3/2} f(x) dx = \frac{4}{3} \int_0^{3/2} \left(1 - \frac{4x}{3}\right) dx = 0$$

$$= \frac{4}{9} \int_0^{3/2} (3 - 4x) dx = \frac{4}{9} \left[3x - \frac{4x^2}{2} \right]_0^{3/2} = \frac{4}{9} \left[3x - 2x^2 \right]_0^{3/2} = \frac{4}{9} \left[\frac{9}{2} - 2 \times \frac{9}{4} \right] = 0$$

$$a_n = \frac{1}{3/2} \int_{-3/2}^{3/2} f(x) \cos\left(\frac{n\pi x}{3/2}\right) dx = \frac{2}{3/2} \int_0^{3/2} f(x) \cos\left(\frac{2n\pi x}{3}\right) dx$$

$$= \frac{4}{3} \int_0^{3/2} \left(1 - \frac{4x}{3}\right) \cos\left(\frac{2n\pi x}{3}\right) dx$$

$$= \frac{4}{3} \left[\left(1 - \frac{4x}{3}\right) \left(\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)} \right) - \left(-\frac{4}{3}\right) \left(\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2} \right) \right]_0^{3/2}$$

$$= \frac{4}{3} \left[0 - \frac{4}{3} \times \frac{9}{4n^2\pi^2} ((-1)^n - 1) \right] = \frac{4}{n^2\pi^2} [1 - (-1)^n] = \frac{8}{n^2\pi^2}, n = 1, 3, 5, \dots$$

Also, $b_n = \frac{1}{3/2} \int_{-3/2}^{3/2} f(x) \sin\left(\frac{n\pi x}{3/2}\right) dx = 0$, as $f(x) \sin\left(\frac{n\pi x}{3/2}\right)$ is an odd function.

Thus $f(x) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{2n\pi x}{3}\right), n = 1, 3, 5, \dots$

putting $x=0$, we get $f(0) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$

or $1 = \frac{8}{\pi^2} \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$

Thus, $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

Exercises: 1) Obtain the Fourier series of $f(x) = \frac{\pi-x}{2}$ in $0 < x < 2\pi$. Hence deduce

that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$

2) If $f(x) = x(2\pi - x)$ in $0 \leq x \leq 2\pi$ show that

$$f(x) = \frac{2\pi^2}{3} - 4 \left(\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \dots \right) \text{ hence deduce that}$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

- 3) Find a Fourier series in $(-\pi, \pi)$ to represent a) $f(x) = x - x^2$ b) $f(x) = x + x^2$

Hence deduce that

$$a) \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots + \infty$$

$$b) \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \infty$$

$$c) \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots + \infty$$

- 4) Obtain the Fourier series for the function $f(x) = \begin{cases} 0 & \text{in } -\pi \leq x \leq 0 \\ \sin x & \text{in } 0 \leq x \leq \pi \end{cases}$

Hence deduce that

$$1. \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots = \frac{1}{2} \quad 2. \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} + \dots = \frac{\pi - 2}{4}$$

- 5) If $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the range 0 to 2π , show that $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$

- 6) An alternating current after passing through a rectifier has the form

$$i = \begin{cases} I_0 \sin x & \text{for } 0 \leq x \leq \pi \\ 0 & \text{for } \pi \leq x \leq 2\pi \end{cases}$$

Where I_0 is the maximum current and the period is 2π . Express i as a Fourier series.

- 7) Obtain the Fourier series for the function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 < x < \pi \end{cases} \quad \text{Where } f(x + 2\pi) = f(x)$$

- 8) Obtain the Fourier series for the function

$$f(x) = \begin{cases} x^2, & 0 \leq x \leq \pi \\ -x^2, & -\pi \leq x \leq 0 \end{cases}$$

- 9) Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-l, l)$

- 10) If $f(x) = \begin{cases} \pi x, & 0 \leq x \leq 1 \\ \pi(2 - x), & 1 \leq x \leq 2 \end{cases}$; show that in the interval $(0, 2)$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

Answers: 1) $f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$

$$3) a) x - x^2 = \frac{-\pi^2}{3} + 4 \sum_1^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx + 2 \sum_1^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

$$b) x - x^2 = \frac{\pi^2}{3} + 4 \sum_1^{\infty} \frac{(-1)^n}{n^2} \cos nx + 2 \sum_1^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

$$4) f(x) = \frac{1}{\pi} + \sum_2^{\infty} \frac{-1}{\pi(n^2 - 1)} \{1 + (-1)^n\} \cos nx + \frac{1}{2} \sin x.$$

$$6) i = \frac{I_0}{\pi} + \frac{1}{2} I_0 \sin x - \frac{2I_0}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}$$

$$7) f(x) = \frac{\pi^2}{6} - 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos nx}{n^2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{2}{n^3} - \frac{\pi^2}{n} \right) \sin nx$$

$$8) f(x) = 2 \left(\pi - \frac{4}{\pi} \right) \sin x - \pi \sin 2x + \frac{2}{3} \left(\pi - \frac{4}{9\pi} \right) \sin 3x - \frac{\pi}{2} \sin 4x + \dots$$

$$9) e^{-x} = \sinh l \left\{ \frac{1}{l} - 2l \sum_{n=1}^{\infty} \frac{1}{l^2 + n^2 \pi^2} \cos \frac{n\pi x}{l} - 2\pi \sum_{n=1}^{\infty} \frac{n}{l^2 + n^2 \pi^2} \sin \frac{n\pi x}{l} \right\}$$

HALF-RANGE FOURIER SERIES

The Fourier expansion of the periodic function $f(x)$ of period $2l$ may contain both sine and cosine terms. Many a time it is required to obtain the Fourier expansion of $f(x)$ in the interval $(0, l)$ which is regarded as half interval. The definition can be extended to the other half in such a manner that the function becomes even or odd. This will result in cosine series or sine series only.

Sine series :

Suppose $f(x) = \phi(x)$ is given in the interval $(0, l)$. Then we define $f(x) = -\phi(-x)$ in $(-l, 0)$. Hence $f(x)$ becomes an odd function in $(-l, l)$. The Fourier series then is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \left(\frac{n\pi x}{l} \right) \quad (11)$$

where
$$b_n = \frac{2}{l} \int_0^l f(x) \sin \left(\frac{n\pi x}{l} \right) dx$$

The series (11) is called half-range sine series over $(0, l)$.

Putting $l=\pi$ in (11), we obtain the half-range sine series of $f(x)$ over $(0, \pi)$ given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$$

Cosine series :

Let us define

$$f(x) = \begin{cases} \phi(x) & \text{in } (0, l) \dots \dots \text{given} \\ \phi(-x) & \text{in } (-l, 0) \dots \dots \text{in order to make the function even} \end{cases}$$

Then the Fourier series of f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad (12)$$

where,

$$a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

The series (12) is called half-range cosine series over (0,l)

Putting $l = \pi$ in (12), we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx \quad n = 1, 2, 3, \dots$$

Examples :

1. Expand $f(x) = x(\pi - x)$ as half-range sine series over the interval $(0, \pi)$.

We have,

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \sin nx dx$$

Integrating by parts, we get

$$b_n = \frac{2}{\pi} \left[\left(\pi x - x^2 \right) \left(\frac{-\cos nx}{n} \right) - (\pi - 2x) \left(\frac{-\sin nx}{n^2} \right) + (-2) \left(\frac{\cos nx}{n^3} \right) \right]_0^\pi$$

$$= \frac{4}{n^3 \pi} [1 - (-1)^n]$$

The sine series of f(x) is

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^3} [1 - (-1)^n] \sin nx$$

2. Obtain the cosine series of

$$f(x) = \begin{cases} x, 0 < x < \frac{\pi}{2} \\ \pi - x, \frac{\pi}{2} < x < \pi \end{cases} \quad \text{over}(0, \pi)$$

Here

$$a_0 = \frac{2}{\pi} \left[\int_0^{\pi/2} x dx + \int_{\pi/2}^{\pi} (\pi - x) dx \right] = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi} \left[\int_0^{\pi/2} x \cos nx dx + \int_{\pi/2}^{\pi} (\pi - x) \cos nx dx \right]$$

Performing integration by parts and simplifying, we get

$$a_n = -\frac{2}{n^2 \pi} \left[1 + (-1)^n - 2 \cos \left(\frac{n\pi}{2} \right) \right]$$

$$= -\frac{8}{n^2 \pi}, n = 2, 6, 10, \dots$$

Thus, the Fourier cosine series is

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi} \left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \dots \infty \right]$$

3. Obtain the half-range cosine series of f(x) = c-x in 0 < x < c

$$\text{Here } a_0 = \frac{2}{c} \int_0^c (c-x) dx = c, \quad a_n = \frac{2}{c} \int_0^c (c-x) \cos \left(\frac{n\pi x}{c} \right) dx$$

$$\text{Integrating by parts and simplifying we get, } a_n = \frac{2c}{n^2 \pi^2} [1 - (-1)^n]$$

The cosine series is given by

$$f(x) = \frac{c}{2} + \frac{2c}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} [1 - (-1)^n] \cos \left(\frac{n\pi x}{c} \right)$$

Exercises:

I. Obtain the half-range sine series of the following functions over the specified intervals:

1. $f(x) = \cos x$ over $(0, \pi)$ 2. $f(x) = \sin^3 x$ over $(0, \pi)$ 3. $f(x) = lx - x^2$ over $(0, l)$

II. Obtain the half-range cosine series of the following functions over the specified intervals:

1. $f(x) = x^2$ over $(0, \pi)$ 2. $f(x) = x \sin x$ over $(0, \pi)$ 3. $f(x) = (x-1)^2$ over $(0, 1)$

$$4. f(x) = \begin{cases} kx, 0 \leq x \leq \frac{l}{2} \\ k(l-x), \frac{l}{2} \leq x \leq l \end{cases}$$

HARMONIC ANALYSIS

The Fourier series of a **known** function $f(x)$ in a given interval may be found by finding the Fourier coefficients. The method described cannot be employed when $f(x)$ is not known explicitly, but defined through the values of the function at some equidistant points. In such a case, the integrals in Euler's formulae cannot be evaluated. Harmonic analysis is the process of finding the Fourier coefficients numerically.

To derive the relevant formulae for Fourier coefficients in Harmonic analysis, we employ the following result:

The mean value of a continuous function $f(x)$ over the interval (a, b) denoted by $[f(x)]$ is defined

as
$$[f(x)] = \frac{1}{b-a} \int_a^b f(x) dx.$$

The Fourier coefficients defined through Euler's formulae, (1), (2), (3) may be redefined as

$$a_0 = 2 \left[\frac{1}{2l} \int_a^{a+2l} f(x) dx \right] = 2[f(x)]$$

$$a_n = 2 \left[\frac{1}{2l} \int_a^{a+2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx \right] = 2 \left[f(x) \cos\left(\frac{n\pi x}{l}\right) \right]$$

$$b_n = 2 \left[\frac{1}{2l} \int_a^{a+2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx \right] = 2 \left[f(x) \sin\left(\frac{n\pi x}{l}\right) \right]$$

Using these in (5), we obtain the Fourier series of $f(x)$. The term $a_1 \cos x + b_1 \sin x$ is called the first harmonic or fundamental harmonic, the term $a_2 \cos 2x + b_2 \sin 2x$ is called the second harmonic and so on. The amplitude of the first harmonic is $\sqrt{a_1^2 + b_1^2}$ and that of second harmonic is $\sqrt{a_2^2 + b_2^2}$ and so on.

Examples

1. Find the first two harmonics of the Fourier series of $f(x)$ given the following table:

X	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
f(x)	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Note that the values of $y = f(x)$ are spread over the interval $0 \leq x \leq 2\pi$ and $f(0) = f(2\pi) = 1.0$.

Hence the function is periodic and so we omit the last value $f(2\pi) = 0$. We prepare the following table to compute the first two harmonics.

x°	$y = f(x)$	$\cos x$	$\cos 2x$	$\sin x$	$\sin 2x$	$y \cos x$	$y \cos 2x$	$y \sin x$	$y \sin 2x$
0°	1.0	1	1	0	0	1	1	0	0
60°	1.4	0.5	-0.5	0.866	0.866	0.7	-0.7	1.2124	1.2124
120°	1.9	-0.5	-0.5	0.866	-0.866	-0.95	-0.95	1.6454	-1.6454
180°	1.7	-1	1	0	0	-1.7	1.7	0	0
240°	1.5	-0.5	-0.5	-0.866	0.866	-0.75	-0.75	-1.299	1.299
300°	1.2	0.5	-0.5	-0.866	-0.866	0.6	-0.6	-1.0392	-1.0392
Total						-1.1	-0.3	0.519	-0.1732

$$\text{We have } a_n = 2 \left[f(x) \cos \left(\frac{n\pi x}{l} \right) \right] = 2[y \cos nx] \quad b_n = 2 \left[f(x) \sin \left(\frac{n\pi x}{l} \right) \right] = 2[y \sin nx]$$

as the length of interval $= 2l = 2\pi$ or $l = \pi$

Putting, $n = 1, 2$, we get

$$a_1 = 2[y \cos x] = \frac{2 \sum y \cos x}{6} = \frac{2(-1.1)}{6} = -0.367$$

$$a_2 = 2[y \cos 2x] = \frac{2 \sum y \cos 2x}{6} = \frac{2(-0.3)}{6} = -0.1$$

$$b_1 = 2[y \sin x] = \frac{2 \sum y \sin x}{6} = 0.173$$

$$b_2 = 2[y \sin 2x] = \frac{2 \sum y \sin 2x}{6} = -0.0577$$

The first two harmonics are $a_1 \cos x + b_1 \sin x$ and $a_2 \cos 2x + b_2 \sin 2x$. That is

$(-0.367 \cos x + 1.0392 \sin x)$ and $(-0.1 \cos 2x - 0.0577 \sin 2x)$

2. Express y as a Fourier series up to the third harmonic given the following values:

X	0	1	2	3	4	5
Y	4	8	15	7	6	2

The values of y at $x = 0, 1, 2, 3, 4, 5$ are given and hence the interval of x should be

$0 \leq x < 6$. The length of the interval $= 6 - 0 = 6$, so that $2l = 6$ or $l = 3$.

The Fourier series up to the third harmonic is

$$y = \frac{a_0}{2} + \left(a_1 \cos \frac{\pi x}{l} + b_1 \sin \frac{\pi x}{l} \right) + \left(a_2 \cos \frac{2\pi x}{l} + b_2 \sin \frac{2\pi x}{l} \right) + \left(a_3 \cos \frac{3\pi x}{l} + b_3 \sin \frac{3\pi x}{l} \right) \text{ or}$$

$$y = \frac{a_0}{2} + \left(a_1 \cos \frac{\pi x}{3} + b_1 \sin \frac{\pi x}{3} \right) + \left(a_2 \cos \frac{2\pi x}{3} + b_2 \sin \frac{2\pi x}{3} \right) + \left(a_3 \cos \frac{3\pi x}{3} + b_3 \sin \frac{3\pi x}{3} \right)$$

Put $\theta = \frac{\pi x}{3}$, then

$$y = \frac{a_0}{2} + (a_1 \cos \theta + b_1 \sin \theta) + (a_2 \cos 2\theta + b_2 \sin 2\theta) + (a_3 \cos 3\theta + b_3 \sin 3\theta) \quad (1)$$

We prepare the following table using the given values:

x	$\theta = \frac{\pi x}{3}$	y	y cos θ	y cos 2 θ	y cos 3 θ	y sin θ	y sin 2 θ	y sin 3 θ
0	0	04	4	4	4	0	0	0
1	60°	08	4	-4	-8	6.928	6.928	0
2	120°	15	-7.5	-7.5	15	12.99	-12.99	0
3	180°	07	-7	7	-7	0	0	0
4	240°	06	-3	-3	6	-5.196	5.196	0
5	300°	02	1	-1	-2	-1.732	-1.732	0
Total		42	-8.5	-4.5	8	12.99	-2.598	0

$$a_0 = 2[f(x)] = 2[y] = \frac{2\sum y}{6} = \frac{1}{3}(42) = 14, \quad a_1 = 2[y \cos \theta] = \frac{2}{6}(-8.5) = -2.833,$$

$$b_1 = 2[y \sin \theta] = \frac{2}{6}(12.99) = 4.33, \quad a_2 = 2[y \cos 2\theta] = \frac{2}{6}(-4.5) = -1.5,$$

$$b_2 = 2[y \sin 2\theta] = \frac{2}{6}(-2.598) = -0.866, \quad a_3 = 2[y \cos 3\theta] = \frac{2}{6}(8) = 2.667, \quad b_3 = 2[y \sin 3\theta] = 0$$

Using these in (1), we get

$$y = 7 - 2.833 \cos \left(\frac{\pi x}{3} \right) + (4.33) \sin \left(\frac{\pi x}{3} \right) - 1.5 \cos \left(\frac{2\pi x}{3} \right) - 0.866 \sin \left(\frac{2\pi x}{3} \right) + 2.667 \cos \pi x$$

This is the required Fourier series up to the third harmonic.

3. The following table gives the variations of a periodic current A over a period T:

T (secs)	0	T/6	T/3	T/2	2T/3	5T/6	T
A (amp)	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Show that there is a constant part of 0.75amp. in the current A and obtain the amplitude of the first harmonic.

Note that the values of A at $t=0$ and $t=T$ are the same. Hence A (t) is a periodic function of period T. Let us denote $\theta = \left(\frac{2\pi}{T}\right)t$. We have

$$a_0 = 2[A], \quad a_1 = 2\left[A \cos\left(\frac{2\pi}{T}\right)t\right] = 2[A \cos \theta], \quad b_1 = 2\left[A \sin\left(\frac{2\pi}{T}\right)t\right] = 2[A \sin \theta] \text{----- (1)}$$

We prepare the following table:

t	$\theta = \frac{2\pi t}{T}$	A	cos θ	sin θ	A cos θ	A sin θ
0	0	1.98	1	0	1.98	0
T/6	60°	1.30	0.5	0.866	0.65	1.1258
T/3	120°	1.05	-0.5	0.866	-0.525	0.9093
T/2	180°	1.30	-1	0	-1.30	0
2T/3	240°	-0.88	-0.5	-0.866	0.44	0.7621
5T/6	300°	-0.25	0.5	-0.866	-0.125	0.2165
Total		4.5			1.12	3.0137

Using the values of the table in (1), we get

$$a_0 = \frac{2\sum A}{6} = \frac{4.5}{3} = 1.5, \quad a_1 = \frac{2\sum A \cos \theta}{6} = \frac{1.12}{3} = 0.3733, \quad b_1 = \frac{2\sum A \sin \theta}{6} = \frac{3.0137}{3} = 1.0046$$

The Fourier expansion up to the first harmonic is

$$A = \frac{a_0}{2} + a_1 \cos\left(\frac{2\pi t}{T}\right) + b_1 \sin\left(\frac{2\pi t}{T}\right) = 0.75 + 0.3733 \cos\left(\frac{2\pi t}{T}\right) + 1.0046 \sin\left(\frac{2\pi t}{T}\right)$$

The expression shows that A has a constant part 0.75 in it. Also the amplitude of the first harmonic is $\sqrt{a_1^2 + b_1^2} = 1.0717$.

ASSIGNMENT:

1. The displacement y of a part of a mechanism is tabulated with corresponding angular movement x^0 of the crank. Express y as a Fourier series up to the third harmonic.

x^0	0	30	60	90	120	150	180	210	240	270	300	330
y	1.80	1.10	0.30	0.16	1.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00

2. Obtain the Fourier series of y up to the second harmonic using the following table:

x°	45	90	135	180	225	270	315	360
y	4.0	3.8	2.4	2.0	-1.5	0	2.8	3.4

3. Obtain the constant term and the coefficients of the first sine and cosine terms in the Fourier expansion of y as given in the following table:

x	0	1	2	3	4	5
y	9	18	24	28	26	20

4. Find the Fourier series of y up to the second harmonic from the following table:

x	0	2	4	6	8	10	12
Y	9.0	18.2	24.4	27.8	27.5	22.0	9.0

5. Obtain the first three coefficients in the Fourier cosine series for y, where y is given in the following table:

x	0	1	2	3	4	5
y	4	8	15	7	6	2

6. The turning moment T is given for a series of values of the crank angle $\theta^\circ = 75^\circ$.

θ°	0	30	60	90	120	150	180
T	0	5224	8097	7850	5499	2626	0

Obtain the first four terms in a series of sines to represent T and calculate T at $\theta = 75^\circ$.

FOURIER TRANSFORMS

Introduction

Fourier Transform is a technique employed to solve ODE's, PDE's, IVP's, BVP's and Integral equations.

Infinite Fourier Transform

Let $f(x)$ be a real valued, differentiable function that satisfies the following conditions:

- 1) $f(x)$ and its derivative $f'(x)$ are continuous, or have only a finite number of simple discontinuities in every finite interval, and
- 2) The integral $\int_{-\infty}^{\infty} |f(x)| dx$ exists.

Also, let s be non – zero real parameter. Then infinite Fourier Transform of $f(x)$ denoted by $\hat{f}(s)$ or $F[f(x)]$ or $F(s)$ is defined by

$$\hat{f}(s) = F(s) = F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{isx} dx, \text{ provided the integral exists.}$$

The infinite Fourier Transform is also called complex Fourier Transform or just the Fourier Transform. The Inverse Fourier Transform of $\hat{f}(s)$ denoted by $F^{-1}[\hat{f}(s)]$ is defined by

$$F^{-1}[\hat{f}(s)] = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(s)e^{-isx} ds$$

Note: The function $f(x)$ is said to be self reciprocal with respect to Fourier transform

$$\text{if } \hat{f}(s) = f(s).$$

Basic Properties:

Below we prove some basic properties of Fourier Transforms:

1. Linearity Property

For any two functions $f(x)$ and $\phi(x)$ (whose Fourier Transforms exist) and any two constants a and b , $F[af(x) + b\phi(x)] = aF[f(x)] + bF[\phi(x)]$

Proof: By definition, we have

$$\begin{aligned} F[af(x) + b\phi(x)] &= \int_{-\infty}^{\infty} [af(x) + b\phi(x)]e^{isx} dx = a \int_{-\infty}^{\infty} f(x)e^{isx} dx + b \int_{-\infty}^{\infty} \phi(x)e^{isx} dx \\ &= aF[f(x)] + bF[\phi(x)]. \text{ This is the desired property.} \end{aligned}$$

In particular, if $a = b = 1$, we get $F[f(x) + \phi(x)] = F[f(x)] + F[\phi(x)]$

Again if $a = -b = 1$, we get $F[f(x) - \phi(x)] = F[f(x)] - F[\phi(x)]$

2. Change of Scale Property

If $\hat{f}(s) = F[f(x)]$, then for any non-zero constant a , we have $F[f(ax)] = \frac{1}{a} \hat{f}\left(\frac{s}{a}\right)$, $a \neq 0$

Proof: By definition, we have $F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$\begin{aligned} \therefore F[f(ax)] &= \int_{-\infty}^{\infty} f(ax) e^{isx} dx \quad \text{put } ax = t \Rightarrow adx = dt \Rightarrow dx = \frac{dt}{a} \\ &= \int_{-\infty}^{\infty} f(t) e^{is\left(\frac{t}{a}\right)} \frac{dt}{a} = \frac{1}{a} \int_{-\infty}^{\infty} f(t) e^{i(s/a)t} dt = \frac{1}{a} \hat{f}\left(\frac{s}{a}\right) \end{aligned}$$

3. Shifting Properties:

For any real constant 'a', (i) $F[f(x-a)] = e^{isa} \hat{f}(s)$ (ii) $F[e^{iax} f(x)] = \hat{f}(s+a)$, Where $\hat{f}(s) = F[f(x)]$

Proof: (i) We have $F[f(x)] = \hat{f}(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx$

Hence, $F[f(x-a)] = \int_{-\infty}^{\infty} f(x-a) e^{isx} dx$, set $x-a = t \Rightarrow dx = dt$, then

$$F[f(x-a)] = \int_{-\infty}^{\infty} f(t) e^{is(t+a)} dt = e^{ias} \int_{-\infty}^{\infty} f(t) e^{ist} dt = e^{ias} \hat{f}(s)$$

$$\text{ii) We have } \hat{f}(s+a) = \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} dx = \int_{-\infty}^{\infty} [f(x) e^{iax}] e^{isx} dx$$

$$= \int_{-\infty}^{\infty} g(x) e^{isx} dx, \text{ where } g(x) = f(x) e^{iax}$$

$$= F[g(x)] = F[e^{iax} f(x)] \text{ This is the desired result.}$$

4. **Modulation Property:** If $F[f(x)] = \hat{f}(s)$, then $F[f(x) \cos ax] = \frac{1}{2} [\hat{f}(s+a) + \hat{f}(s-a)]$

where 'a' is a real constant.

Proof: We have $\cos ax = \frac{e^{iax} + e^{-iax}}{2}$

$$\begin{aligned} \text{Hence } F[f(x) \cos ax] &= F\left[f(x) \left(\frac{e^{iax} + e^{-iax}}{2}\right)\right] \\ &= \frac{1}{2} F[f(x) e^{iax} + f(x) e^{-iax}] = \frac{1}{2} [F[f(x) e^{iax}] + F[f(x) e^{-iax}]] \end{aligned}$$

$$= \frac{1}{2} [\hat{f}(s+a) + \hat{f}(s-a)] \text{ by using linearity and shift properties.}$$

This is desired property.

Note: Similarly $F[f(x)\sin ax] = \frac{1}{2} [\hat{f}(s+a) - \hat{f}(s-a)]$

Examples

1. Find the Fourier transform of $f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$ hence evaluate $\int_0^{\infty} \frac{\sin x}{x} dx$

<< The Fourier transform of $f(x)$ is given by

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(x)e^{isx} dx = \int_{-\infty}^{-1} 0 \cdot e^{isx} dx + \int_{-1}^1 1 \cdot e^{isx} dx + \int_1^{\infty} 0 \cdot e^{isx} dx = \frac{e^{isx}}{is} \Big|_{-1}^1 = \frac{e^{is} - e^{-is}}{is} = \frac{2 \sin s}{s}, s \neq 0$$

& for $s = 0$, $\hat{f}(s) = 2$.

Now by the inverse Fourier transform we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(s)e^{-isx} ds \text{ Or } \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin s}{s} e^{-isx} ds = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

By putting $x = 0$ we get,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin s}{s} ds = 1 \Rightarrow \int_{-\infty}^{\infty} \frac{\sin s}{s} ds = \pi \Rightarrow 2 \int_0^{\infty} \frac{\sin s}{s} ds = \pi, \text{ as the integrand is even}$$

$$\Rightarrow \int_0^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2} \text{ Or } \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

2. Find the Fourier transform of $f(x) = \begin{cases} 1-x^2 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases}$ Hence evaluate

$$\int_0^{\infty} \frac{x \cos x - \sin x}{x^3} \cos \frac{x}{2} dx$$

<< The Fourier transform of $f(x)$ is given by

$$\begin{aligned} \hat{f}(s) &= \int_{-\infty}^{\infty} f(x)e^{isx} dx = \int_{-\infty}^{-1} 0 \cdot e^{isx} dx + \int_{-1}^1 (1-x^2) \cdot e^{isx} dx + \int_1^{\infty} 0 \cdot e^{isx} dx \\ &= (1-x^2) \frac{e^{isx}}{is} \Big|_{-1}^1 - (-2x) \frac{e^{isx}}{(is)^2} \Big|_{-1}^1 + (-2) \frac{e^{isx}}{(is)^3} \Big|_{-1}^1 \\ &= 0 + \frac{2}{-s^2} (e^{is} + e^{-is}) - \frac{2}{-is^3} (e^{is} - e^{-is}) = \frac{-4 \cos s}{s^2} + \frac{4 \sin s}{s^3} = -\frac{4}{s^3} (s \cos s - \sin s) \end{aligned}$$

Now by inversion formula, we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(s)e^{-isx} ds$$

$$\text{Or } \frac{1}{2\pi} \int_{-\infty}^{\infty} -\frac{4}{s^3} (s \cos s - \sin s) e^{-isx} ds = \begin{cases} 1-x^2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

Putting $x = 1/2$, we get

$$-\frac{4}{2\pi} \int_{-\infty}^{\infty} \frac{1}{s^3} (s \cos s - \sin s) e^{-is/2} ds = \frac{3}{4}$$

$$\text{Or } \int_{-\infty}^{\infty} \frac{s \cos s - \sin s}{s^3} \left(\cos \frac{s}{2} - i \sin \frac{s}{2} \right) ds = -\frac{3\pi}{8}$$

$$\text{Or } \int_{-\infty}^{\infty} \frac{s \cos s - \sin s}{s^3} \cos \frac{s}{2} ds = -\frac{3\pi}{8} \Rightarrow \int_0^{\infty} \frac{s \cos s - \sin s}{s^3} \cos \frac{s}{2} ds = -\frac{3\pi}{16}, \text{ since the integrand is even}$$

$$\int_0^{\infty} \frac{x \cos x - \sin x}{x^3} \cdot \cos \frac{x}{2} dx = -\frac{3\pi}{16}$$

3. Find the Fourier Transform of the function $f(x) = e^{-a|x|}$ where $a > 0$

The Fourier transform of $f(x)$ is given by

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx = \int_{-\infty}^{\infty} e^{-a|x|} e^{isx} dx = \left[\int_{-\infty}^0 e^{-a|x|} e^{isx} dx + \int_0^{\infty} e^{-a|x|} e^{isx} dx \right]$$

Using the fact that $|x| = x, 0 \leq x < \infty$ & $|x| = -x, -\infty < x \leq 0$, we get

$$\begin{aligned} \hat{f}(s) &= \left[\int_{-\infty}^0 e^{ax} e^{isx} dx + \int_0^{\infty} e^{-ax} e^{isx} dx \right] = \left[\int_{-\infty}^0 e^{(a+is)x} dx + \int_0^{\infty} e^{-(a-is)x} dx \right] \\ &= \left[\left\{ \frac{e^{(a+is)x}}{(a+is)} \right\}_{-\infty}^0 + \left\{ \frac{e^{-(a-is)x}}{-(a-is)} \right\}_0^{\infty} \right] = \left[\frac{1}{(a+is)} + \frac{1}{(a-is)} \right] = \left[\frac{2a}{a^2 + s^2} \right] \end{aligned}$$

Exercises:

4. Find the Fourier Transform of the function $f(x) = \begin{cases} 1, & |x| \leq a \\ 0, & |x| > a \end{cases}$

where 'a' is a positive constant. Hence evaluate

$$(i) \int_{-\infty}^{\infty} \frac{\sin s a \cos s x}{s} ds \quad (ii) \int_0^{\infty} \frac{\sin s}{s} ds$$

<< For the given function, we have

$$\begin{aligned} F[f(x)] &= \left[\int_{-\infty}^{\infty} f(x) e^{isx} dx \right] \\ &= \left[\int_{-a}^a f(x) e^{isx} dx + \int_{-\infty}^{-a} f(x) e^{isx} dx + \int_a^{\infty} f(x) e^{isx} dx \right] \\ &= \left[\int_{-a}^a e^{isx} dx \right] = 2 \left[\frac{\sin sa}{s} \right] \end{aligned}$$

$$\text{Thus } F[f(x)] = \hat{f}(s) = 2 \left(\frac{\sin s a}{s} \right) \dots\dots\dots(1)$$

Inverting $\hat{f}(s)$ by employing inversion formula, we get

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2 \left[\frac{\sin sa}{s} \right] e^{-isx} ds \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin sa (\cos sx - i \sin sx)}{s} ds \\ &= \frac{1}{\pi} \left[\int_{-\infty}^{\infty} \frac{\sin sa (\cos s x)}{s} ds - i \int_{-\infty}^{\infty} \frac{\sin s a \sin sx}{s} ds \right] \end{aligned}$$

Here, the integrand in the first integral is even and the integrand in the second integral is odd. Hence using the relevant properties of integral here, we get

$$\begin{aligned} f(x) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin sa \cos sx}{s} ds \quad \text{or} \\ \int_{-\infty}^{\infty} \frac{\sin sa \cos sx}{s} ds &= \pi f(x) = \begin{cases} \pi, & |x| \leq a \\ 0, & |x| > a \end{cases} \end{aligned}$$

$$\text{For } x = 0, a = 1, \text{ this yields } \int_{-\infty}^{\infty} \frac{\sin s}{s} ds = \pi$$

$$\text{Since the integrand is even, we have } 2 \int_0^{\infty} \frac{\sin s}{s} ds = \pi$$

$$\text{Or } \int_0^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2}$$

5. Find the Fourier Transform of $f(x) = e^{-a^2 x^2}$ where 'a' is a positive constant.

Deduce that $f(x) = e^{-x^2/2}$ is self reciprocal with respect to Fourier transform.
Here

$$\begin{aligned} F[f(x)] &= \int_{-\infty}^{\infty} e^{-a^2 x^2} e^{isx} dx = \int_{-\infty}^{\infty} e^{-\left(a^2 x^2 - isx\right)} dx \\ &= \int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}\right]} dx \\ &= e^{-\left(\frac{s^2}{4a^2}\right)} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx \end{aligned}$$

$$\text{Setting } t = ax - \frac{is}{2a}, \text{ we get } F[f(x)] = e^{-\left(\frac{s^2}{4a^2}\right)} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a}$$

$$\begin{aligned}
&= \frac{1}{a} e^{-\left(\frac{s^2}{4a^2}\right)} 2 \int_0^\infty e^{-t^2} dt \quad \text{but } \int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2} \\
&= \frac{1}{a} e^{-\left(\frac{s^2}{4a^2}\right)} \sqrt{\pi}, \text{ using gamma function.} \\
\hat{f}(s) &= \frac{\sqrt{\pi}}{a} e^{-\left(\frac{s^2}{4a^2}\right)}
\end{aligned}$$

This is the desired Fourier Transform of $f(x)$.

$$\text{For } a^2 = 1/2 \text{ in } f(x) = e^{-a^2 x^2}$$

$$\text{we get } f(x) = e^{-x^2/2} \text{ and hence,}$$

$$\hat{f}(s) = \sqrt{2\pi} e^{-s^2/2}$$

$$\text{Also putting } x = s \text{ in } f(x) = e^{-x^2/2}, \text{ we get } f(s) = e^{-s^2/2}.$$

Hence, $f(s)$ and $\hat{f}(s)$ are same but for constant multiplication by $\sqrt{2\pi}$.

Thus $f(s) = \hat{f}(s)$

It follows that $f(x) = e^{-x^2/2}$ is self reciprocal

6) Find the inverse Fourier Transform of $\hat{f}(s) = e^{-s^2}$

Exercises:

Find the Complex Fourier Transforms of the following functions:

$$(1) f(x) = \begin{cases} x, & |x| \leq a \\ 0, & |x| > a \end{cases} \text{ where 'a' is a positive constant}$$

$$(2) f(x) = \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

$$(3) f(x) = \begin{cases} 0, & x < a \\ 1, & a \leq x \leq b \\ 0, & x > b \end{cases} \text{ where 'a' and 'b' are positive constants}$$

$$(4) f(x) = \begin{cases} a^2 - x^2, & |x| \leq a \\ 0, & |x| > a \end{cases}$$

$$(5) f(x) = x e^{-a|x|} \text{ where 'a' is a positive constant} \quad (6) f(x) = e^{-|x|} \quad (7) f(x) = \cos 2x^2$$

$$(8) f(x) = \sin 3x^2$$

FOURIER SINE TRANSFORMS:

Let $f(x)$ be defined for all positive values of x .

The integral $\int_0^{\infty} f(x) \sin sx dx$ is called the Fourier sine Transform of $f(x)$. This is denoted by

$$\hat{f}_s(s) \text{ or } F_s[f(x)], \text{ thus } \hat{f}_s(s) = F_s[f(x)] = \int_0^{\infty} f(x) \sin sx dx$$

The inverse Fourier sine Transform of $\hat{f}_s(s)$ is defined through the integral $\frac{2}{\pi} \int_0^{\infty} \hat{f}_s(s) \sin sx ds$.

This is denoted by $f(x)$ or $F_s^{-1}[\hat{f}_s(s)]$. Thus

$$f(x) = F_s^{-1}[\hat{f}_s(s)] = \frac{2}{\pi} \int_0^{\infty} \hat{f}_s(s) \sin sx ds$$

Properties

The following are the basic properties of Sine Transforms.

(1) LINEARITY PROPERTY

If 'a' and 'b' are two constants, then for two functions $f(x)$ and $\phi(x)$, we have

$$F_s[af(x) + b\phi(x)] = aF_s[f(x)] + bF_s[\phi(x)]$$

Proof : By definition, we have

$$\begin{aligned} F_s[af(x) + b\phi(x)] &= \int_0^{\infty} [af(x) + b\phi(x)] \sin sx dx = \int_0^{\infty} af(x) \sin sx dx + \int_0^{\infty} b\phi(x) \sin sx dx \\ &= a \int_0^{\infty} f(x) \sin sx dx + b \int_0^{\infty} \phi(x) \sin sx dx = aF_s[f(x)] + bF_s[\phi(x)] \end{aligned}$$

This is the desired result. In particular, we have

$$F_s[f(x) + \phi(x)] = F_s[f(x)] + F_s[\phi(x)] \text{ and } F_s[f(x) - \phi(x)] = F_s[f(x)] - F_s[\phi(x)]$$

(2) CHANGE OF SCALE PROPERTY

If $F_s[f(x)] = \hat{f}_s(s)$, then for $a \neq 0$, we have $F_s[f(ax)] = \frac{1}{a} \hat{f}_s\left(\frac{s}{a}\right)$

Proof : We have $F_s[f(ax)] = \int_0^{\infty} f(ax) \sin sx dx$

Setting $ax = t$, we get $F_s[f(ax)] = \int_0^{\infty} f(t) \sin\left(\frac{s}{a}t\right) t \left(\frac{dt}{a}\right)$

$$\frac{1}{a} \int_0^{\infty} f(t) \sin\left(\frac{s}{a}t\right) t dt = \frac{1}{a} \hat{f}_s\left(\frac{s}{a}\right)$$

(3) **MODULATION PROPERTY**

If $F_s[f(x)] = \hat{f}_s(s)$, then for $a \neq 0$, we have $F_s[f(x)\cos ax] = \frac{1}{2}[\hat{f}_s(s+a) + \hat{f}_s(s-a)]$

Proof: We have $F_s[f(x)\cos ax] = \int_0^\infty f(x)\cos ax \sin sx \, dx$

$$\begin{aligned} &= \frac{1}{2} \left[\int_0^\infty f(x) \{ \sin(s+a)x + \sin(s-a)x \} dx \right] \\ &= \frac{1}{2} \left[\int_0^\infty f(x) \sin(s+a)x \, dx + \int_0^\infty f(x) \sin(s-a)x \, dx \right] \\ &= \frac{1}{2} [\hat{f}_s(s+a) + \hat{f}_s(s-a)] \text{ by using Linearity property.} \end{aligned}$$

FOURIER COSINE TRANSFORMS:

Let $f(x)$ be defined for positive values of x . Then the integral $\int_0^\infty f(x) \cos sx \, dx$ is called the

Fourier Cosine Transform of $f(x)$ and is denoted by $\hat{f}_c(s)$ or $F_c[f(x)]$. Thus

$$\hat{f}_c(s) = F_c[f(x)] = \frac{2}{\pi} \int_0^\infty f(x) \cos sx \, dx$$

The inverse Fourier Cosine Transform of $\hat{f}_c(s)$ is defined through the integral

$$\frac{2}{\pi} \int_0^\infty \hat{f}_c(s) \cos sx \, ds. \text{ This is denoted by } f(x) \text{ or } F_c^{-1}[\hat{f}_c(s)]. \text{ Thus}$$

$$f(x) = F_c^{-1}[\hat{f}_c(s)] = \frac{2}{\pi} \int_0^\infty \hat{f}_c(s) \cos sx \, ds$$

Basic Properties:

The following are the basic properties of cosine transforms:

(1) **Linearity property:** If 'a' and 'b' are two constants, then for two functions $f(x)$ and

$$\phi(x), \text{ we have } F_c[af(x) + b\phi(x)] = aF_c(f(x)) + bF_c(\phi(x))$$

(2) **Change of scale property:** If $F_c\{f(x)\} = \hat{f}_c(s)$, then for $a \neq 0$, we have $F_c[f(ax)] = \frac{1}{a} \hat{f}_c\left(\frac{s}{a}\right)$

(3) **Modulation property:** If $F_c\{f(x)\} = \hat{f}_c(s)$, then for $a \neq 0$, we have

$$F_c[f(x)\cos ax] = \frac{1}{2} [\hat{f}_c(s+a) + \hat{f}_c(s-a)]$$

The proofs of these properties are similar to the proofs of the corresponding properties of Fourier Sine Transforms.

EXAMPLES

1. Find the Fourier Sine transform of $e^{-|x|}$. Hence show that $\int_0^{\infty} \frac{x \sin mx}{1+x^2} dx = \frac{\pi e^{-m}}{2}, m > 0$

$$\begin{aligned} \ll F_s[f(x)] &= \int_0^{\infty} f(x) \sin sxdx = \int_0^{\infty} e^{-|x|} \sin sxdx = \int_0^{\infty} e^{-x} \sin sxdx \quad [\text{since } |x| = -x \text{ in } (0, \infty)] \\ &= \frac{e^{-x}}{1+s^2} (-\sin sx - s \cos sx) \Big|_0^{\infty} = 0 - \frac{1}{1+s^2} (0 - s) = \frac{s}{1+s^2} \end{aligned}$$

By using inverse formula for Fourier sine transform, we get

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s[f(x)] \sin sxdx \Rightarrow e^{-x} = \frac{2}{\pi} \int_0^{\infty} \frac{s}{1+s^2} \sin sxdx, \text{ By changing } x \text{ to } m, \text{ we get}$$

$$e^{-m} = \frac{2}{\pi} \int_0^{\infty} \frac{s \sin ms}{1+s^2} ds = \frac{2}{\pi} \int_0^{\infty} \frac{x \sin mx}{1+x^2} dx \Rightarrow \int_0^{\infty} \frac{x \sin mx}{1+x^2} dx = \frac{\pi e^{-m}}{2}$$

2. Find the Fourier Cosine transform of $f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2-x & \text{for } 1 < x < 2 \\ 0 & \text{for } x > 2 \end{cases}$

$$\begin{aligned} \ll F_c[f(x)] &= \int_0^{\infty} f(x) \cos sxdx = \int_0^1 f(x) \cos sxdx + \int_1^2 f(x) \cos sxdx + \int_2^{\infty} f(x) \cos sxdx \\ &= \int_0^1 x \cos sxdx + \int_1^2 (2-x) \cos sxdx + \int_2^{\infty} 0 \cdot \cos sxdx \\ &= \left[x \cdot \frac{\sin sx}{s} - 1 \cdot \left(\frac{-\cos sx}{s^2} \right) \right]_0^1 + \left[(2-x) \frac{\sin sx}{s} - (-1) \left(\frac{-\cos sx}{s^2} \right) \right]_1^2 \\ &= \frac{\sin s}{s} + \frac{\cos s}{s^2} - \left(0 - \frac{1}{s^2} \right) - \frac{\cos 2s}{s^2} - \left(\frac{\sin s}{s} - \frac{\cos s}{s^2} \right) \\ &= \frac{2 \cos s}{s^2} - \frac{1}{s^2} - \frac{\cos 2s}{s^2} = \frac{1}{s^2} (2 \cos s - \cos 2s - 1) \end{aligned}$$

3. Find the Fourier Sine transform of e^{-ax}/x

$\ll$ Let $f(x) = e^{-ax}/x$, then its Fourier sine transform is

$$F_s[f(x)] = \int_0^{\infty} f(x) \sin sxdx = \int_0^{\infty} \frac{e^{-ax}}{x} \sin sxdx = F(s), \text{ say}$$

Differentiate both sides w.r.t. s we get,

$$\frac{d}{ds} \{F(s)\} = \int_0^{\infty} \frac{\partial}{\partial s} \left(\frac{e^{-ax}}{x} \sin sx \right) dx = \int_0^{\infty} \frac{x e^{-ax}}{x} \cos sxdx = \int_0^{\infty} e^{-ax} \cos sxdx$$

$$= \frac{e^{-ax}}{a^2 + s^2} (-a \cos sx + s \sin sx) \Big|_0^\infty$$

$$= \frac{1}{s^2 + a^2} [0 - (-a)] = \frac{a}{s^2 + a^2}$$

Integrating w.r.t. s, we get,

$$F(s) = \int \frac{a}{s^2 + a^2} ds = \tan^{-1} \frac{s}{a} + c$$

But $F(0) = 0$ when $s = 0 \Rightarrow c = 0$. Hence $F(s) = \tan^{-1} \frac{s}{a}$

4. Find the Fourier Cosine Transform of $f(x) = \frac{1}{1+x^2}$. Hence derive Fourier sine transform of

$$\phi(x) = \frac{x}{1+x^2}$$

$$<< F_c[f(x)] = \int_0^\infty f(x) \cos sx dx = \int_0^\infty \frac{1}{1+x^2} \cos sx dx = I(\text{say})$$

$$\therefore \frac{dI}{ds} = \int_0^\infty \frac{\partial}{\partial s} \left(\frac{\cos sx}{1+x^2} \right) dx = \int_0^\infty \frac{-x \sin sx}{1+x^2} dx = - \int_0^\infty \frac{x^2 \sin sx}{x(1+x^2)} dx = - \int_0^\infty \frac{[1+x^2-1] \sin sx}{x(1+x^2)} dx$$

$$= - \int_0^\infty \frac{\sin sx}{x} dx + \int_0^\infty \frac{\sin sx}{x(1+x^2)} dx = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)} dx \Rightarrow \frac{d^2 I}{ds^2} = \int_0^\infty \frac{x \cos sx}{x(1+x^2)} dx = I$$

$$\therefore \frac{d^2 I}{ds^2} - I = 0 \Rightarrow (D^2 - 1)I = 0, \text{ where } D = \frac{d}{ds}$$

$$\therefore I = c_1 e^s + c_2 e^{-s} \Rightarrow \frac{dI}{ds} = c_1 e^s - c_2 e^{-s}$$

$$\text{When } s = 0, \text{ we get } c_1 + c_2 = I = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2} \text{ \& } c_1 - c_2 = \frac{dI}{ds} = -\frac{\pi}{2} \Rightarrow c_1 = 0 \text{ \& } c_2 = \frac{\pi}{2}$$

$$\therefore F_c[f(x)] = I = \frac{\pi}{2} e^{-s}$$

$$\text{Now } F_s[\phi(x)] = \int_0^\infty \frac{x \sin sx}{1+x^2} dx = -\frac{dI}{ds} = \frac{\pi}{2} e^{-s}$$

5. Solve the integral equation $\int_0^\infty f(\theta) \cos \alpha \theta d\theta = \begin{cases} 1-\alpha, & 0 \leq \alpha \leq 1 \\ 0, & \alpha > 1 \end{cases}$. Hence evaluate $\int_0^\infty \frac{\sin^2 t}{t^2} dt$

$$<< \text{ We have } \int_0^\infty f(\theta) \cos \alpha \theta d\theta = F_c(\alpha)$$

$$\therefore F_c(\alpha) = \begin{cases} 1-\alpha, & 0 \leq \alpha \leq 1 \\ 0, & \alpha > 1 \end{cases} \text{ ----- (1)}$$

∴ by the inverse formula we have

$$f(\theta) = \frac{2}{\pi} \int_0^{\infty} F_c(\alpha) \cos \alpha \theta d\alpha = \frac{2}{\pi} \int_0^1 (1-\alpha) \cos \alpha \theta d\alpha = \frac{2}{\pi} \left[(1-\alpha) \frac{\sin \alpha \theta}{\theta} - (-1) \left(-\frac{\cos \alpha \theta}{\theta^2} \right) \right]_0^1$$

$$= \frac{2(1-\cos \theta)}{\pi \theta^2}$$

$$\text{Now } F_c(\alpha) = \int_0^{\infty} f(\theta) \cos \alpha \theta d\theta = \int_0^{\infty} \frac{2(1-\cos \theta)}{\pi \theta^2} \cos \alpha \theta d\theta \text{----- (2)}$$

$$\text{From (1) \& (2) we have } \frac{2}{\pi} \int_0^{\infty} \frac{1-\cos \theta}{\theta^2} \cos \alpha \theta d\theta = \begin{cases} 1-\alpha, & 0 \leq \alpha \leq 1 \\ 0, & \alpha > 1 \end{cases}$$

Now take limit as $\alpha \rightarrow 0$, we get

$$\frac{2}{\pi} \int_0^{\infty} \frac{1-\cos \theta}{\theta^2} d\theta = 1 \Rightarrow \int_0^{\infty} \frac{1-\cos \theta}{\theta^2} d\theta = \frac{\pi}{2} \Rightarrow \int_0^{\infty} \frac{2 \sin^2 \theta / 2}{\theta^2} d\theta = \frac{\pi}{2}, \text{ put } \frac{\theta}{2} = t, \text{ then } d\theta = 2dt$$

$$\Rightarrow \int_0^{\infty} \frac{2 \sin^2 t}{(t/2)^2} 2dt = \frac{\pi}{2} \Rightarrow \int_0^{\infty} \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$$

$$(6) \text{ Solve the integral equation } \int_0^{\infty} f(x) \cos \alpha x dx = e^{-a\alpha}$$

Let $\phi(\alpha)$ be defined by $\phi(\alpha) = e^{-a\alpha}$

$$\text{Given } \phi(\alpha) = \int_0^{\infty} f(x) \cos \alpha x dx = \hat{f}_c(\alpha)$$

Using this in the inversion formula, we get

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \phi(\alpha) \cos \alpha x d\alpha = \frac{2}{\pi} \int_0^{\infty} e^{-a\alpha} \cos \alpha x d\alpha$$

$$= \frac{2}{\pi} \left[\frac{e^{-a\alpha}}{a^2 + x^2} \{ -a \cos \alpha x + x \sin \alpha x \} \right]_0^{\infty} = \frac{2}{\pi} \left[\frac{a}{a^2 + x^2} \right] = \frac{2a}{\pi(a^2 + x^2)}$$

Exercises:

$$1. \text{ Find the Fourier sine transform of } f(x) = \begin{cases} 1, & 0 \leq x \leq a \\ 0, & x > a \end{cases} \quad \left[\text{Ans : } \frac{1 - \cos sa}{s} \right]$$

$$2. \text{ Find } f(x) \text{ from the integral equation } \int_0^{\infty} f(x) \sin \alpha x dx = \begin{cases} 1, & 0 \leq \alpha \leq 1 \\ 2, & 1 \leq \alpha < 2 \\ 0, & \alpha \geq 2 \end{cases}$$

$$\left[\text{Ans : } \frac{2}{\pi x} [1 + \cos x - 2 \cos 2x] \right]$$

3. Find the sine transforms of the following functions

$$(i) f(x) = \begin{cases} x, & 0 < x < 1 \\ a - x, & 1 < x < a \\ 0, & x > a \end{cases} \quad (ii) f(x) = xe^{-ax}, a > 0 \quad (iii) f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x > a \end{cases}$$

$$(4) \text{ Solve for } f(x) \text{ given } \int_0^{\infty} f(x) \sin \alpha x dx = \begin{cases} 1 - \alpha, & 0 \leq \alpha \leq 1 \\ 0, & \alpha > 1 \end{cases}$$

5) Find the inverse sine transforms of the following functions:

$$(i) \hat{f}_s(s) = \frac{e^{-as}}{s}, a > 0 \quad (ii) \hat{f}_s(s) = \frac{\pi}{2}$$

6. Find the Cosine transform of $f(x) = e^{-ax}, a > 0$. Hence evaluate $\int_0^{\infty} \frac{\cos kx}{x^2 + a^2} dx$

$$\left(\text{Ans: } \frac{a}{a^2 + \alpha^2} \& \int_0^{\infty} \frac{\cos kx}{x^2 + a^2} dx = \frac{\pi e^{-ax}}{2a} \right)$$

7. Find the Fourier Cosine Transforms of the following functions:

$$(i) f(x) = \begin{cases} 4x, & 0 < x < 1 \\ 4 - x, & 1 < x < 4 \\ 0, & x > 4 \end{cases} \quad (ii) f(x) = e^{-ax^2}, a > 0 \quad (iii) f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x > a \end{cases}$$

$$(iv) f(x) = xe^{-ax}, a > 0 \quad (v) f(x) = \frac{1}{1+x^2} \quad (vi) f(x) = \frac{\cos 2x}{1+x^2}$$

$$8. \text{ Solve for } f(x) \text{ given } \int_0^{\infty} f(x) \cos \alpha x dx = \begin{cases} 1 - \alpha, & 0 \leq \alpha \leq 1 \\ 0, & \alpha > 1 \end{cases}$$

Z – TRANSFORMS

Introduction

The Z-transform plays an important role in the study of communications, sample data control systems, discrete signal processing, solutions of difference equations etc.

Definition:

A difference equation is a relation between the differences of an unknown function at one or more general values of the argument.

Eg: $\Delta y_{(n+1)} + y_{(n)} = 2$ & $\Delta^2 y_{(n+1)} + \Delta y_{(n-1)} = 0$ are difference equations.

Another way of writing a difference equation is as follows:

We know that $\Delta y_{(n+1)} = y_{(n+2)} - y_{(n+1)}$

Hence the first equation can be written as $y_{(n+2)} - y_{(n+1)} + y_{(n)} = 2$ ----- (1)

Also $\Delta^2 y_{(n+1)} = y_{(n+3)} - 2y_{(n+2)} + y_{(n+1)}$

Hence the second equation can be written as

$y_{(n+3)} - 2y_{(n+2)} + y_{(n+1)} + y_{(n)} - y_{(n-1)} = 0$ ----- (2)

Order of a difference equation:

Order of a difference equation is the difference between the largest and the smallest arguments occurring in the difference equation divided by the unit of increment.

Thus the order of the equation (1) is 2. Since

$$\frac{\text{Largest argument} - \text{smallest argument}}{\text{Unit of increment}} = \frac{(n+2) - n}{1} = 2$$

& that of (2) is $\frac{(n+3) - (n-1)}{1} = 4$

Solution: Solution of a difference equation is an expression for $y_{(n)}$ which satisfies the given difference equation.

General solution: The general solution of a difference equation is that in which the number of arbitrary constants is equal to the order of the difference equation.

A Particular solution or Particular integral is that solution which is obtained from the general solution by giving particular values to the constants.

Z – TRANSFORMS:

Definition: Let $u_n = f(n)$ be a real-valued function defined for $n = 0, 1, 2, 3, \dots$ and $u_n = 0$ for $n < 0$. Then the Z-transform of u_n denoted by $Z(u_n)$ is defined by

$$U(z) = Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n} \text{ ----- (1) whenever the infinite series converges}$$

We can also write it as $Z^{-1}[U(z)] = u_n$ and is called the inverse Z – Transform.

Properties of Z-transform:

1. Linearity property:-

Consider the sequences $\{u_n\}$ & $\{v_n\}$ and constants a and b. Then

$$Z[au_n + bv_n] = aZ[u_n] + bZ[v_n]$$

Proof: By definition, we have $Z[au_n + bv_n] = \sum_{n=0}^{\infty} [au_n + bv_n] z^{-n}$

$$= a \sum_{n=0}^{\infty} u_n z^{-n} + b \sum_{n=0}^{\infty} v_n z^{-n} = aZ(u_n) + bZ(v_n)$$

In particular, for $a = b = 1$, we get $Z[u_n + v_n] = Z[u_n] + Z[v_n]$ and for $a = -b = 1$, we get

$$Z[u_n - v_n] = Z[u_n] - Z[v_n]$$

2. Damping property:-

Let $Z(u_n) = U(z)$. Then (i) $Z(a^n u_n) = U\left(\frac{z}{a}\right)$ (ii) $Z(a^{-n} u_n) = U(az)$

Proof: By definition, we have $Z(a^n u_n) = \sum_{n=0}^{\infty} (a^n u_n) z^{-n} = \sum_{n=0}^{\infty} u_n \left(\frac{z}{a}\right)^{-n} = U\left(\frac{z}{a}\right)$

$$\text{Thus } Z(a^n u_n) = U\left(\frac{z}{a}\right)$$

This is the result as desired. Here, we note that that if $Z(u_n) = U(z)$, then

$$Z(a^n u_n) = [U(z)]_{Z \rightarrow Z/a} = U\left(\frac{z}{a}\right)$$

Next, $Z(a^{-n} u_n) = \sum_{n=0}^{\infty} (a^{-n} u_n) z^{-n} = \sum_{n=0}^{\infty} u_n (az)^{-n} = U(az)$

Thus $Z(a^{-n} u_n) = U(az)$. This is the result as desired.

3. Shifting property:

(a) Right shifting rule:

If $Z(u_n) = U(z)$, then $Z(u_{n-k}) = z^{-k} U(z)$ where $k > 0$

Proof: By definition, we have $Z(u_{n-k}) = \sum_{n=0}^{\infty} u_{n-k} z^{-n}$

Since $u_n = 0$ for $n < 0$, we have $u_{n-k} = 0$ for $n = 0, 1, 2, \dots, (k-1)$

$$\begin{aligned}\text{Hence } Z(u_{n-k}) &= \sum_{n=k}^{\infty} u_{n-k} z^{-n} = u_0 z^{-k} + u_1 z^{-(k+1)} + \dots + \infty = z^{-k} [u_0 + u_1 z^{-1} + \dots + \infty] \\ &= z^{-k} \sum_{n=0}^{\infty} u_n z^{-n} = z^{-k} U(z)\end{aligned}$$

Thus $Z(u_{n-k}) = z^{-k} U(z)$

(b) Left shifting rule: $Z(u_{n+k}) = z^k [U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \dots - u_{k-1} z^{-(k-1)}]$

$$\begin{aligned}\text{Proof: } Z(u_{n+k}) &= \sum_{n=0}^{\infty} u_{n+k} z^{-n} = z^k \sum_{n=0}^{\infty} u_{n+k} z^{-(k+n)} = z^k \left\{ \sum_{n=0}^{\infty} u_{n+k} z^{-(k+n)} \right\} \\ &= z^k \left[\sum_{m=k}^{\infty} u_m z^{-m} \right], \text{ where } m = n+k \\ &= z^k \left[\sum_{n=k}^{\infty} u_n z^{-n} \right] = z^k \left[\sum_{n=0}^{\infty} u_n z^{-n} - \sum_{n=0}^{k-1} u_n z^{-n} \right] \\ &= z^k [U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \dots - u_{k-1} z^{-(k-1)}]\end{aligned}$$

Particular cases:

In particular, we have the following standard results:

1. $Z(u_{n+1}) = z[U(z) - u_0]$
2. $Z(u_{n+2}) = z^2[U(z) - u_0 - u_1 z^{-1}]$
3. $Z(u_{n+3}) = z^3[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}]$ etc.

Some Standard Z-Transforms:

1. Transform of a^n :

By definition, we have $Z(a^n) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \dots + \infty$

The series on the RHS is a Geometric series. Sum to infinity of the series is $\frac{1}{1 - \frac{a}{z}}$

$$\text{or } \frac{z}{z-a} \quad \text{Thus, } Z(a^n) = \frac{z}{z-a}$$

In particular, when $a=1$, we get $Z(1) = \frac{z}{z-1}$

2. Transform of e^{an} :

Here $Z(e^{an}) = Z(k^n)$ where $k = e^a$

$$= \frac{z}{z-k} = \frac{z}{z-e^a}$$

$$\text{Thus } Z(e^{an}) = \frac{z}{z - e^a}$$

3. Transform of n^p , p being a positive integer:

$$\text{We have, } Z(n^p) = \sum_{n=0}^{\infty} n^p z^{-n} = z \sum_{n=0}^{\infty} n^{p-1} z^{-(n+1)} \quad \text{----- (1)}$$

$$\text{By changing } p \text{ to } p-1, \text{ we get } Z(n^{p-1}) = \sum_{n=0}^{\infty} n^{p-1} z^{-n}$$

Differentiating with respect to z , we get

$$\frac{d}{dz} [Z(n^{p-1})] = \frac{d}{dz} \sum_{n=0}^{\infty} n^{p-1} z^{-n} = \sum_{n=0}^{\infty} n^{p-1} (-n) z^{-(n+1)}$$

$$\text{Using this in (1), we get } Z(n^p) = -z \frac{d}{dz} [Z(n^{p-1})]$$

Particular cases of $Z(n^p)$:-

$$1. \text{ For } p = 1, \text{ we get } Z(n) = -z \frac{d}{dz} [Z(1)] = -z \frac{d}{dz} \left(\frac{z}{z-1} \right) = \frac{z}{(z-1)^2}$$

$$\text{Thus, } Z(n) = \frac{z}{(z-1)^2}$$

$$2. \text{ For } p = 2, \text{ we get } Z(n^2) = -z \frac{d}{dz} [Z(n)] = -z \frac{d}{dz} \left(\frac{z}{(z-1)^2} \right) = \frac{z^2 + z}{(z-1)^3}$$

$$\text{Thus, } Z(n^2) = \frac{z^2 + z}{(z-1)^3}$$

$$3. \text{ For } p = 3, \text{ we get } Z(n^3) = \frac{z^3 + 4z^2 + z}{(z-1)^4} \text{ \& soon}$$

4. Transform of na^n

By damping property, we have

$$Z(na^n) = [Z(n)]_{Z \rightarrow Z/a} = \left[\frac{z}{(z-1)^2} \right]_{Z \rightarrow Z/a} = \frac{z/a}{\left(\frac{z}{a} - 1 \right)^2} = \frac{az}{(z-a)^2}$$

$$\text{Thus, } Z(na^n) = \frac{az}{(z-a)^2}$$

5. Transform of $n^2 a^n$:

$$\text{We have, } Z(n^2 a^n) = [Z(n^2)]_{Z \rightarrow Z/a} = \left[\frac{z^2 + z}{(z-1)^3} \right]_{Z \rightarrow Z/a} = \frac{(z/a)^2 + z/a}{(z/a - 1)^3} = \frac{(z^2 + az)a^3}{a^2(z-a)^3}$$

Thus, $Z(n^2 a^n) = \frac{az^2 + a^2 z}{(z-a)^3}$

6. Transforms of $\cosh n\theta$ and $\sinh n\theta$:

We have $\cosh n\theta = \frac{e^{n\theta} + e^{-n\theta}}{2} \Rightarrow Z(\cosh n\theta) = \frac{1}{2} Z(e^{n\theta} + e^{-n\theta})$
 $= \frac{1}{2} [Z(e^{n\theta}) + Z(e^{-n\theta})]$, by using the linearity property

$$= \frac{1}{2} \left[\frac{z}{z - e^\theta} + \frac{z}{z - e^{-\theta}} \right] = \frac{z}{2} \left[\frac{z - e^{-\theta} + z - e^\theta}{z^2 - z(e^\theta + e^{-\theta}) + 1} \right] = z \left[\frac{z - \frac{(e^\theta + e^{-\theta})}{2}}{z^2 - z(e^\theta + e^{-\theta}) + 1} \right]$$

$$= \frac{z[z - \cosh \theta]}{z^2 - 2z \cosh \theta + 1}$$

Next, $\sinh n\theta = \frac{e^{n\theta} - e^{-n\theta}}{2} \Rightarrow Z(\sinh n\theta) = \frac{z}{2} \left[\frac{1}{z - e^\theta} - \frac{1}{z - e^{-\theta}} \right]$

$$= \frac{z}{2} \left[\frac{e^\theta - e^{-\theta}}{z^2 - 2z \cosh \theta + 1} \right] = \frac{z \sinh \theta}{z^2 - 2z \cosh \theta + 1}$$

7. Transforms of $\cos n\theta$ and $\sin n\theta$

We have $\cos n\theta = \frac{e^{in\theta} + e^{-in\theta}}{2}$

$$Z(\cos n\theta) = \frac{1}{2} Z(e^{in\theta} + e^{-in\theta}) = \frac{1}{2} \left[\frac{z}{z - e^{i\theta}} + \frac{z}{z - e^{-i\theta}} \right] = \frac{z}{2} \left[\frac{2z - (e^{i\theta} + e^{-i\theta})}{z^2 - z(e^{i\theta} + e^{-i\theta}) + 1} \right]$$

$$= \frac{z[z - \cos \theta]}{z^2 - 2z \cos \theta + 1}$$

Next, $\sin n\theta = \frac{e^{in\theta} - e^{-in\theta}}{2i}$

$$Z(\sin n\theta) = \frac{1}{2i} \left[\frac{z}{z - e^{i\theta}} - \frac{z}{z - e^{-i\theta}} \right] = \frac{z}{2i} \left[\frac{e^{i\theta} - e^{-i\theta}}{z^2 - 2z \cos \theta + 1} \right] = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$$

8. Transforms of $a^n \cos n\theta$ and $a^n \sin n\theta$

We know that $Z(\cos n\theta) = \frac{z[z - \cos \theta]}{z^2 - 2z \cos \theta + 1} = U(z)$. Hence by using damping rule we get

$$Z(a^n \cos n\theta) = U(z/a) = \frac{z/a [z/a - \cos \theta]}{(z/a)^2 - 2(z/a) \cos \theta + 1} = \frac{z(z - a \cos \theta)}{z^2 - 2az \cos \theta + a^2}$$

Similarly we know that $Z(\sin n\theta) = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1} = U(z)$. Hence by using damping rule we

get

$$Z(a^n \sin n\theta) = U(z/a) = \frac{(z/a) \sin \theta}{(z/a)^2 - 2(z/a) \cos \theta + 1} = \frac{az \sin \theta}{z^2 - 2az \cos \theta + a^2}$$

Examples: Find the Z-transforms of the following:

1. $3n - 4 \sin(n\pi/4) + 5a$

$$\ll Z(3n - 4 \sin(n\pi/4) + 5a) = 3Z(n) - 4Z(\sin(n\pi/4)) + 5aZ(1)$$

$$\begin{aligned} &= 3 \cdot \frac{z}{(z-1)^2} - 4 \cdot \frac{z \sin(\pi/4)}{z^2 - 2z \cos(\pi/4) + 1} + 5a \cdot \frac{z}{z-1} = \frac{3z}{(z-1)^2} - \frac{4z \left(\frac{1}{\sqrt{2}} \right)}{z^2 - 2z \left(\frac{1}{\sqrt{2}} \right) + 1} + \frac{5az}{z-1} \\ &= \frac{3z + 5az(z-1)}{(z-1)^2} - \frac{2\sqrt{2}z}{z^2 - \sqrt{2}z + 1} = \frac{(3-5a)z + 5az^2}{(z-1)^2} - \frac{2\sqrt{2}z}{z^2 - \sqrt{2}z + 1} \end{aligned}$$

2. $(n+1)^2$

$$\begin{aligned} \ll Z[(n+1)^2] &= Z(n^2 + 2n + 1) = Z(n^2) + 2Z(n) + Z(1) = \frac{z^2 + z}{(z-1)^3} + 2 \cdot \frac{z}{(z-1)^2} + \frac{z}{z-1} \\ &= \frac{z^2 + z + 2z(z-1) + z(z-1)^2}{(z-1)^3} = \frac{z^2 + z + 2z^2 - 2z + z^3 - 2z^2 + z}{(z-1)^3} = \frac{z^3 + z^2}{(z-1)^3} \end{aligned}$$

3. $\sin(3n+5)$

$$\begin{aligned} \ll Z[\sin(3n+5)] &= Z(\sin 3n \cos 5 + \cos 3n \sin 5) = \cos 5 Z(\sin 3n) + \sin 5 Z(\cos 3n) \\ &= \cos 5 \cdot \frac{z \sin 3}{z^2 - 2z \cos 3 + 1} + \sin 5 \cdot \frac{z(z - \cos 3)}{z^2 - 2z \cos 3 + 1} = \frac{z \sin 3 \cos 5 + z^2 \sin 5 - z \sin 5 \cos 3}{z^2 - 2z \cos 3 + 1} \\ &= \frac{z(\sin 3 \cos 5 - \cos 3 \sin 5) + z^2 \sin 5}{z^2 - 2z \cos 3 + 1} = \frac{z \sin(-2) + z^2 \sin 5}{z^2 - 2z \cos 3 + 1} = \frac{z(z \sin 5 - \sin 2)}{z^2 - 2z \cos 3 + 1} \end{aligned}$$

4. ne^{an}

Let $u_n = n$, $e^{an} = k^n$ where $k = e^a$

Therefore $Z(ne^{an}) = Z(nk^n) = U(z/k)$, By damping rule.

Where $U(z) = Z(n) = \frac{z}{(z-1)^2} \Rightarrow U(z/k) = \frac{z/k}{(z/k-1)^2} = \frac{kz}{(z-k)^2}$, where $k = e^a$

Hence $Z(ne^{an}) = U(z/k) = \frac{e^a z}{(z - e^a)^2}$

5. $n^2 e^{an}$

Let $u_n = n^2$, $e^{an} = k^n$ where $k = e^a$

Therefore $Z(n^2 e^{an}) = Z(n^2 k^n) = U(z/k)$, By damping rule.

Where $U(z) = Z(n^2) = \frac{z^2 + z}{(z-1)^3} \Rightarrow U(z/k) = \frac{(z/k)^2 + z/k}{(z/k-1)^3} = \frac{kz^2 + k^2z}{(z-k)^3}$, where $k = e^a$

Hence $Z(n^2 e^{an}) = U(z/k) = \frac{e^a z^2 + (e^a)^2 z}{(z-e^a)^3} = \frac{e^a z(z+e^a)}{(z-e^a)^3}$

6. i) $a^n \cosh n\theta$

ii) $a^n \sinh n\theta$

i) We know that $Z(\cosh n\theta) = \frac{z[z - \cosh \theta]}{z^2 - 2z \cosh \theta + 1} = U(z)$, Hence by using damping rule we

get $Z(a^n \cosh n\theta) = U(z/a) = \frac{(z/a)[(z/a) - \cosh \theta]}{(z/a)^2 - 2(z/a) \cosh \theta + 1} = \frac{z(z - a \cosh \theta)}{z^2 - 2az \cosh \theta + a^2}$

ii) We know that $Z(\sinh n\theta) = \frac{z \sinh \theta}{z^2 - 2z \cosh \theta + 1} = U(z)$, Hence by using damping rule we

get $Z(a^n \sinh n\theta) = U(z/a) = \frac{(z/a) \cosh \theta}{(z/a)^2 - 2(z/a) \cosh \theta + 1} = \frac{az \cosh \theta}{z^2 - 2az \cosh \theta + a^2}$

7. $e^t \sin 2t$

We know that $\sin 2t = \frac{z \sin 2}{z^2 - 2z \cos 2 + 1} = U(z)$

$\therefore Z(e^t \sin 2t) = U(z/e)$, By damping rule.

$$= \frac{(z/e) \sin 2}{(z/e)^2 - 2(z/e) \cos 2 + 1} = \frac{ez \sin 2}{z^2 - 2ez \cos 2 + e^2}$$

8. $e^k \cos k\alpha$ ($k \geq 0$)

We know that $Z(\cos k\alpha) = \frac{z(z - \cos \alpha)}{z^2 - 2z \cos \alpha + 1} = U(z)$

$\therefore Z(e^k \cos k\alpha) = Z((e^1)^k \cos k\alpha) = U(z/e) = \frac{(z/e)[(z/e) - \cos \alpha]}{(z/e)^2 - 2(z/e) \cos \alpha + 1} = \frac{z(z - e \cos \alpha)}{z^2 - 2ez \cos \alpha + e^2}$

9. $\cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)$

$$\begin{aligned} << Z\left(\cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)\right) &= Z\left(\cos \frac{n\pi}{2} \cos \frac{\pi}{4} - \sin \frac{n\pi}{2} \sin \frac{\pi}{4}\right) = \cos \frac{\pi}{4} Z\left(\cos \frac{n\pi}{2}\right) - \sin \frac{\pi}{4} Z\left(\sin \frac{n\pi}{2}\right) \\ &= \frac{1}{\sqrt{2}} \left(\frac{z(z - \cos \frac{\pi}{2})}{z^2 - 2z \cos \frac{\pi}{2} + 1} \right) - \frac{1}{\sqrt{2}} \left(\frac{z \sin \frac{\pi}{2}}{z^2 - 2z \cos \frac{\pi}{2} + 1} \right) = \frac{1}{\sqrt{2}} \left(\frac{z^2}{z^2 + 1} - \frac{z}{z^2 + 1} \right) = \frac{z(z-1)}{\sqrt{2}(z^2+1)} \end{aligned}$$

10. $\cosh\left(\frac{n\pi}{2} + \theta\right)$

$$\begin{aligned}
\ll Z\left[\cosh\left(\frac{n\pi}{2} + \theta\right)\right] &= Z\left[\frac{e^{n\pi/2+\theta} + e^{-(n\pi/2+\theta)}}{2}\right] = \frac{1}{2}\left[e^{\theta}Z\left(e^{n\pi/2}\right) + e^{-\theta}Z\left(e^{-(n\pi/2)}\right)\right] \\
&= \frac{1}{2}\left[e^{\theta}\frac{z}{z-e^{\pi/2}} + e^{-\theta}\frac{z}{z-e^{-\pi/2}}\right] \text{ Using } Z(e^{ax}) = \frac{z}{z-e^a} \\
&= \frac{z}{2}\left[\frac{e^{\theta}(z-e^{-\pi/2}) + e^{-\theta}(z-e^{\pi/2})}{z^2 - z(e^{\pi/2} + e^{-\pi/2}) + 1}\right] = \frac{z}{2}\left[\frac{z(e^{\theta} + e^{-\theta}) - e^{-(\pi/2-\theta)} - e^{\pi/2-\theta}}{z^2 - z(e^{\pi/2} + e^{-\pi/2}) + 1}\right] \\
&= \frac{z}{2}\left[\frac{z \cdot 2 \cosh \theta - [e^{(\pi/2-\theta)} + e^{-(\pi/2-\theta)}]}{z^2 - 2z \cosh(\pi/2) + 1}\right] = z\left[\frac{z \cosh \theta - \cosh(\pi/2 - \theta)}{z^2 - 2z \cosh(\pi/2) + 1}\right] \\
&= \frac{z^2 \cosh \theta - z \cosh(\pi/2 - \theta)}{z^2 - 2z \cosh(\pi/2) + 1}
\end{aligned}$$

11. ${}^nC_p$ ($0 \leq p \leq n$)

$$\begin{aligned}
\ll Z\left({}^nC_p\right) &= \sum_{p=0}^n \left({}^nC_p z^{-p}\right) = 1 + {}^nC_1 z^{-1} + {}^nC_2 z^{-2} + \dots + {}^nC_n z^{-n} = (1 + z^{-1})^n \\
&= \left(1 + \frac{1}{z}\right)^n = \left(\frac{z+1}{z}\right)^n
\end{aligned}$$

12. ${}^{n+p}C_n$

$$\begin{aligned}
Z\left({}^{n+p}C_n\right) &= \sum_{p=0}^{\infty} {}^{n+p}C_n z^{-p} = 1 + {}^{n+1}C_n z^{-1} + {}^{n+2}C_n z^{-2} + {}^{n+3}C_n z^{-3} + \dots \\
&= 1 + {}^{n+1}C_1 z^{-1} + {}^{n+2}C_2 z^{-2} + {}^{n+3}C_3 z^{-3} + \dots, \quad \text{Using } {}^nC_{n-r} = {}^nC_r \\
&= 1 + (n+1)z^{-1} + \frac{(n+2)(n+1)}{2!} z^{-2} + \frac{(n+3)(n+2)(n+1)}{3!} z^{-3} + \dots \\
&= 1 + (-n-1)(-z)^{-1} + \frac{(-n-2)(-n-1)}{2!} (-z^{-1})^2 + \frac{(-n-3)(-n-2)(-n-1)}{3!} (-z^{-1})^3 + \dots \\
&= (1 - z^{-1})^{-n-1} = \left(1 - \frac{1}{z}\right)^{-(n+1)} = \left(\frac{z-1}{z}\right)^{-(n+1)} = \left(\frac{z}{z-1}\right)^{(n+1)}
\end{aligned}$$

13. Unit impulse sequence $\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$

$$\ll Z[\delta(n)] = \sum_{n=0}^{\infty} \delta(n) z^{-n} = 1 + 0 + 0 + 0 + \dots = 1$$

14. Unit step sequence $u(n) = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$

$$\ll Z[u(n)] = \sum_{n=0}^{\infty} u(n) z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + \dots = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

15. Show that $Z\left(\frac{1}{n!}\right) = e^{1/z}$. Hence evaluate $Z\left(\frac{1}{(n+1)!}\right)$ & $Z\left(\frac{1}{(n+2)!}\right)$

$$<< Z\left(\frac{1}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \dots = e^{z^{-1}} = e^{1/z}$$

Shifting $\frac{1}{n!}$ one unit to the left gives $Z\left(\frac{1}{(n+1)!}\right) = z[U(z) - u_0] = z\left[Z\left(\frac{1}{n!}\right) - 1\right] = z(e^{1/z} - 1)$

Similarly $Z\left(\frac{1}{(n+2)!}\right) = z^2[U(z) - u_0 - u_1 z^{-1}] = z^2[e^{1/z} - 1 - z^{-1}]$

16. $u_n = \left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n$

We have, $Z(u_n) = Z\left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right] = Z\left(\frac{1}{2}\right)^n + Z\left(\frac{1}{4}\right)^n = \frac{z}{z - \frac{1}{2}} + \frac{z}{z - \frac{1}{4}} = \frac{2z}{2z - 1} + \frac{4z}{4z - 1}$

$$= \frac{2z(8z - 3)}{(2z - 1)(4z - 1)}$$

Multiplication by n: If $Z(u_n) = U(z)$, then $Z(nu_n) = -z \frac{d(U(z))}{dz}$

Proof: $Z(nu_n) = \sum_{n=0}^{\infty} nu_n z^{-n} = -z \sum_{n=0}^{\infty} u_n (-n) z^{-n-1} = -z \sum_{n=0}^{\infty} u_n \frac{d(z^{-n})}{dz} = -z \sum_{n=0}^{\infty} \frac{d(u_n z^{-n})}{dz}$

$$= -z \frac{d}{dz} \sum_{n=0}^{\infty} u_n z^{-n} = -z \frac{d(U(z))}{dz}$$

Note: In general $Z(n^m u_n) = (-z)^m \frac{d^m[U(z)]}{dz^m}$

Example: Find the z - transform of i) $n \sin n\theta$ ii) $n^2 e^{n\theta}$

i) We know that $Z(\sin n\theta) = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$, using $Z(nu_n) = -z \frac{d}{dz} U(z)$, we get

$$Z(n \sin n\theta) = -z \frac{d}{dz} \left(\frac{z \sin \theta}{z^2 - 2z \cos \theta + 1} \right) = -z \left(\frac{(z^2 - 2z \cos \theta + 1) \sin \theta - z \sin \theta (2z - 2 \cos \theta)}{(z^2 - 2z \cos \theta + 1)^2} \right)$$

$$= -z \left(\frac{z^2 \sin \theta - 2z \cos \theta \sin \theta + 1 \sin \theta - 2z^2 \sin \theta + 2z \sin \theta \cos \theta}{(z^2 - 2z \cos \theta + 1)^2} \right)$$

$$= -z \left(\frac{\sin \theta - z^2 \sin \theta}{(z^2 - 2z \cos \theta + 1)^2} \right) = \frac{z(z^2 - 1) \sin \theta}{(z^2 - 2z \cos \theta + 1)^2}$$

ii) We know that $Z(e^{n\theta}) = \frac{z}{z - e^\theta}$ & $Z(n^2 u_n) = (-z)^2 \frac{d^2}{dz^2} U(z)$

$$\begin{aligned}\therefore Z(n^2 e^{n\theta}) &= (-z)^2 \frac{d^2}{dz^2} \left(\frac{z}{z - e^\theta} \right) = z^2 \frac{d}{dz} \left[\frac{(z - e^\theta) \cdot 1 - z(1)}{(z - e^\theta)^2} \right] = z^2 \frac{d}{dz} \left[\frac{-e^\theta}{(z - e^\theta)} \right] \\ &= -z^2 e^\theta \left(\frac{-2}{(z - e^\theta)^3} \right) = \frac{2e^\theta z^2}{(z - e^\theta)^3}\end{aligned}$$

I. Initial value theorem: If $Z(u_n) = U(z)$, then $u_0 = \lim_{z \rightarrow \infty} z U(z)$

Proof: we know that $U(z) = Z(u_n) = u_0 + u_1 z^{-1} + u_2 z^{-2} + \dots$

Taking limit as $z \rightarrow \infty$, we get $\lim_{z \rightarrow \infty} z U(z) = u_0$

Similarly $u_1 = \lim_{z \rightarrow \infty} \{z[U(z) - u_0]\}$, $u_2 = \lim_{z \rightarrow \infty} \{z^2[U(z) - u_0 - u_1 z^{-1}]\}$ & so on

II. Final value theorem: If $Z(u_n) = U(z)$, then $\lim_{n \rightarrow \infty} u_n = \lim_{z \rightarrow 1} (z - 1)U(z)$

Proof: By definition $Z(u_{n+1} - u_n) = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$ Or

$$Z(u_{n+1}) - Z(u_n) = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$$

$$\text{Or } Z[U(z) - u_0] - U(z) = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n} \quad \text{Or } U(z)(z - 1) - u_0 z = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$$

Taking limits of both sides as $z \rightarrow 1$, we get

$$\lim_{z \rightarrow 1} [U(z)(z - 1) - u_0 z] = \lim_{n \rightarrow \infty} \sum_{n=0}^{\infty} (u_{n+1} - u_n) = \lim_{n \rightarrow \infty} [(u_1 - u_0) + (u_2 - u_1) + (u_3 - u_2) + \dots + (u_{n+1} - u_n)]$$

$$= \lim_{n \rightarrow \infty} u_{n+1} - u_0 = u_{\infty} - u_0$$

$$\text{Hence } u_{\infty} = \lim_{z \rightarrow 1} [U(z)(z - 1)] \quad \text{Or } \lim_{n \rightarrow \infty} u_n = \lim_{z \rightarrow 1} [(z - 1)U(z)]$$

Example: If $U(z) = \frac{2z^2 + 5z + 14}{(z - 1)^4}$, evaluate u_2 & u_3

$$<< U(z) = \frac{2z^2 + 5z + 14}{(z - 1)^4} = \frac{1}{z^2} \cdot \frac{2 + 5z^{-1} + 14z^{-2}}{(1 - z^{-1})^4},$$

$$\text{By Initial value theorem, } u_0 = \lim_{z \rightarrow \infty} z U(z) = \lim_{z \rightarrow \infty} \left[\frac{1}{z^2} \cdot \frac{2 + 5z^{-1} + 14z^{-2}}{(1 - z^{-1})^4} \right] = 0$$

$$\begin{aligned}\text{Similarly } u_1 &= \lim_{z \rightarrow \infty} \{z[U(z) - u_0]\} = \lim_{z \rightarrow \infty} z \left[\frac{1}{z^2} \cdot \frac{2 + 5z^{-1} + 14z^{-2}}{(1 - z^{-1})^4} \right] \\ &= \lim_{z \rightarrow \infty} \left[\frac{1}{z} \cdot \frac{2 + 5z^{-1} + 14z^{-2}}{(1 - z^{-1})^4} \right] = 0\end{aligned}$$

$$\begin{aligned}
\text{Now } u_2 &= \lim_{z \rightarrow \infty} Lt \left\{ z^2 [U(z) - u_0 - u_1 z^{-1}] \right\} = \lim_{z \rightarrow \infty} Lt \left\{ z^2 \left[\frac{1}{z^2} \cdot \frac{2 + 5z^{-1} + 14z^{-2}}{(1 - z^{-1})^4} \right] \right\} \\
&= \lim_{z \rightarrow \infty} Lt \left[\frac{2 + 5z^{-1} + 14z^{-2}}{(1 - z^{-1})^4} \right] = 2 \\
&\& u_3 = \lim_{z \rightarrow \infty} Lt \left\{ z^3 [U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}] \right\} = \lim_{z \rightarrow \infty} Lt \left\{ z^3 \left[\frac{2z^2 + 5z + 14}{(z - 1)^4} - 0 - 0 - 2z^{-2} \right] \right\} \\
&= \lim_{z \rightarrow \infty} Lt \left\{ z^3 \left[\frac{2z^2 + 5z + 14}{(z - 1)^4} - \frac{2}{z^2} \right] \right\} = \lim_{z \rightarrow \infty} Lt \left\{ z^3 \left[\frac{z^2(2z^2 + 5z + 14) - 2(z - 1)^4}{z^2(z - 1)^4} \right] \right\} \\
&= \lim_{z \rightarrow \infty} Lt \left\{ z^3 \left[\frac{2z^4 + 5z^3 + 14z^2 - 2z^4 - 8z^2 - 2 + 8z^3 + 8z - 4z^2}{z^2(z - 1)^4} \right] \right\} \\
&= \lim_{z \rightarrow \infty} Lt \left\{ z^3 \left[\frac{13z^3 + 2z^2 + 8z - 2}{z^2(z - 1)^4} \right] \right\} = \lim_{z \rightarrow \infty} Lt \left\{ z^3 \left[\frac{z^3[13 + 2z^{-1} + 8z^{-2} - 2z^{-3}]}{z^6(1 - z^{-1})^4} \right] \right\} = 13
\end{aligned}$$

Exercises: Find the Z – Transforms of the following:

1) $\frac{1}{(n+1)!}$ 2) $(\cos \theta + i \sin \theta)^n$ 3) $2n + 5 \sin \frac{n\pi}{4} - 3a^4$ 4) $(n-1)^2$ 5) a^{n+3}

6) Show that i) $Z(e^{-an} \cos n\theta) = \frac{ze^a(ze^a - \cos \theta)}{z^2 e^{2a} - 2ze^a \cos \theta + 1}$

ii) $Z(e^{-an} \sin n\theta) = \frac{ze^a \sin \theta}{z^2 e^{2a} - 2ze^a \cos \theta + 1}$

7) Using $Z(n^2) = \frac{z^2 + z}{(z-1)^3}$, Show that $Z(n+1)^2 = \frac{z^3 + z^2}{(z-1)^3}$

8) If $Z(u_n) = \frac{z}{z-1} + \frac{z}{z^2+1}$, find the Z – Transform of u_{n+2}

9) If $U(z) = \frac{2z^2 + 3z + 12}{(z-1)^4}$, find the value of u_2 & u_3

10) Given that $Z(u_n) = \frac{2z^2 + 3z + 4}{(z-3)^3}$, $|z| > 3$, show that $u_1 = 2, u_2 = 21, u_3 = 139$

11) Show that $Z\left(\frac{1}{n}\right) = z \log \frac{z}{z-1}$

12) Using $Z(n) = \frac{z}{(z-1)^2}$, show that $Z(n \cos \theta) = \frac{(z^3 + z) \cos \theta - 2z^2}{(z^2 - 2z \cos \theta + 1)^2}$

Answers: 1. $z(e^{1/z} - 1)$ 2. $\frac{z}{z - e^{i\theta}}$ 3. $\frac{2z}{(z-1)^2} + \frac{2/\sqrt{2}}{z^2 - \sqrt{2}z + 1} - \frac{3a^4 z}{z-1}$

4. $\frac{z^3 - 3z^2 + 4z}{(z-1)^3}$ 5) $\frac{za^3}{z-a}$ 8) $\frac{z^2(1+3z^2)}{(1-z)(1+z^2)}$

9) $u_2 = 2, u_3 = 11$

Inverse Z – Transform: If $U(z)$ is Z – Transform of u_n , then u_n is called Inverse Z – Transform of $U(z)$, Denoted by $u_n = Z^{-1}[U(z)]$.

Some Useful Inverse Z – transforms:

1. $Z^{-1}\left[\frac{z}{z-a}\right] = a^n$	2. $Z^{-1}\left[\frac{z}{z-1}\right] = 1$
3. $Z^{-1}\left[\frac{az}{(z-a)^2}\right] = a^n \cdot n$	4. $Z^{-1}\left[\frac{z}{(z-1)^2}\right] = n$
5. $Z^{-1}\left[\frac{az^2 + a^2 z}{(z-a)^3}\right] = a^n \cdot n^2$	6. $Z^{-1}\left[\frac{z^2 + z}{(z-1)^3}\right] = n^2$
7. $Z^{-1}\left[\frac{az^3 + 4a^2 z^2 + a^3 z}{(z-a)^4}\right] = a^n \cdot n^3$	8. $Z^{-1}\left[\frac{z^3 + 4z^2 + z}{(z-1)^4}\right] = n^3$
9. $Z^{-1}\left[\frac{z}{z^2 + 1}\right] = \sin(n\pi/2)$	10. $Z^{-1}\left[\frac{z^2}{z^2 + 1}\right] = \cos(n\pi/2)$

Evaluation of inverse Z – Transforms: We have the following 3 methods –

1. Power series method:

Example: 1) Find the inverse Z – Transform of $\log \frac{z}{z+1}$, by power series method.

<< Putting $z = 1/y$, we get $U(z) = \log \frac{z}{z+1} = \log \frac{1/y}{1/y+1} = \log \frac{1}{1+y} = -\log(1+y)$

$$= -y + \frac{1}{2}y^2 - \frac{1}{3}y^3 + \dots$$

$$= -z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{3}z^{-3} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{-n}$$

$$\therefore u_n = \begin{cases} 0 & \text{for } n=0 \\ (-1)^n / n & \text{other wise} \end{cases}$$

2) Find the inverse Z – Transform of $\frac{z}{(z+1)^2}$, by division method

$$z^{-1} - 2z^{-2} + 3z^{-3} - \dots$$

<< By Dividing we get
$$\begin{array}{r} z^2 + 2z + 1 \overline{) z} \\ \underline{z + 2 + z^{-1}} \\ -2 - z^{-1} \\ \underline{-2 - 4z^{-1} - 2z^{-3}} \\ 3z^{-1} + 2z^{-2} \\ \underline{3z^{-1} + 6z^{-2} + 3z^{-3}} \\ -4z^{-2} - 3z^{-3} \quad \& \text{so on} \end{array}$$

Hence $\frac{z}{(z+1)^2} = z^{-1} - 2z^{-2} + 3z^{-3} - 4z^{-4} + \dots$, which is an infinite series

i.e. $U(z) = \sum_{n=0}^{\infty} (-1)^{n-1} n z^{-n} \Rightarrow u_n = (-1)^n n$

II. Partial fraction method:

1. Find the inverse Z – Transform of i) $\frac{2z^2 + 3z}{(z+2)(z-4)}$ ii) $\frac{z^3 - 20z}{(z-2)^3(z-4)}$

<< i) Write $U(z) = \frac{2z^2 + 3z}{(z+2)(z-4)}$ as $\frac{U(z)}{z} = \frac{2z+3}{(z+2)(z-4)} = \frac{A}{z+2} + \frac{B}{z-4}$

Then $2z+3 = A(z-4) + B(z+2) \Rightarrow 11 = 6B \Rightarrow B = 11/6$ & $-1 = -6A \Rightarrow A = 1/6$

$\frac{U(z)}{z} = \frac{1}{6} \frac{1}{z+2} + \frac{11}{6} \frac{1}{z-4} \Rightarrow U(z) = \frac{1}{6} \frac{z}{z+2} + \frac{11}{6} \frac{z}{z-4}$

On inversion we get $u_n = \frac{1}{6}(-2)^n + \frac{11}{6}(4)^n$

ii) Write $U(z) = \frac{z^3 - 20z}{(z-2)^3(z-4)}$ as $\frac{U(z)}{z} = \frac{z^2 - 20}{(z-2)^3(z-4)} = \frac{A+Bz+Cz^2}{(z-2)^3} + \frac{D}{z-4}$

Then $(A+Bz+Cz^2)(z-4) + D(z-2)^3 = z^2 - 20$

If $z = 4$, then $8D = 16 - 20 \Rightarrow D = -1/2$, By putting $z = 0, 1$ & -1 , we get

$-4A - 8D = -20 \Rightarrow A + 2D = 5 \Rightarrow A - 1 = 5 \Rightarrow A = 6$

$(A+B+C)(-3) - D = 1 - 20 \Rightarrow -3A - 3B - 3C - D = -19 \Rightarrow 3A + 3B + 3C - 1/2 = 19$

$6 + B + C = \frac{19 + 1/2}{3} = \frac{39}{6} = \frac{13}{2} \Rightarrow B + C = \frac{13}{2} - 6 = \frac{1}{2} \dots \dots \dots (1)$

$$\& (A - B + C)(-5) - 27D = -19 \Rightarrow -5(A - B + C) = -19 + 27(-1/2) = \frac{-38 - 27}{2} = \frac{-65}{2}$$

$$\Rightarrow A - B + C = \frac{-65}{2(-5)} = \frac{13}{2} \Rightarrow -B + C = \frac{13}{2} - 6 = \frac{1}{2} \text{----- (2)}$$

By adding (1) & (2) we get

$$2C = 1 \Rightarrow C = 1/2 \Rightarrow B = 0$$

$$\begin{aligned} \therefore \frac{U(z)}{z} &= \frac{6 + 1/2z^2}{(z-2)^3} - \frac{1}{2} \frac{1}{z-4} \Rightarrow U(z) = \frac{6z + 1/2z^3}{(z-2)^3} - \frac{1}{2} \frac{z}{z-4} = \frac{1}{2} \left[\frac{12z + z^3}{(z-2)^3} - \frac{z}{z-4} \right] \\ &= \frac{1}{2} \left[\frac{z(z-2)^2 + 4z^2 + 8z}{(z-2)^3} - \frac{z}{z-4} \right] = \frac{1}{2} \left[\frac{z}{z-2} + 2 \cdot \frac{2z^2 + 2^2z}{(z-2)^3} - \frac{z}{z-4} \right] \end{aligned}$$

$$\text{On inversion, we get } u_n = \frac{1}{2} \left[2^n + 2 \cdot 2^n \cdot n^2 - 4^n \right] = 2^{n-1} + n^2 2^n + 2^{2n-1}$$

2. Find the inverse Z – Transform of $\frac{2(z^2 - 5z + 6.5)}{(z-2)(z-3)^2}$, for $2 < |z| < 3$

$$<< U(z) = \frac{2(z^2 - 5z + 6.5)}{(z-2)(z-3)^2} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{(z-3)^2}, \text{ Then}$$

$$2(z^2 - 5z + 6.5) = A(z-3)^2 + B(z-2)(z-3) + C(z-2)$$

$$z = 2 \Rightarrow 2(4 - 10 + 6.5) = A \Rightarrow A = 1, \quad z = 3 \Rightarrow 2(9 - 15 + 6.5) = C \Rightarrow C = 1$$

$$z = 0 \Rightarrow 13 = 9A + 6B - 2C, \text{ i.e. } 13 = 9 + 6B - 2 \Rightarrow 6B = 6 \Rightarrow B = 1$$

$$\therefore U(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{(z-3)^2} = \frac{1}{z} \left(1 - \frac{2}{z} \right)^{-1} - \frac{1}{3} \left(1 - \frac{z}{3} \right)^{-1} + \frac{1}{9} \left(1 - \frac{z}{3} \right)^{-2}$$

(so that $2/z < 1$ & $z/3 < 1$)

$$= \frac{1}{z} \left[1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \dots \right] - \frac{1}{3} \left[1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \dots \right] + \frac{1}{9} \left[1 + \frac{2z}{3} + \frac{3z^2}{9} + \frac{4z^3}{27} + \dots \right]$$

Where $2 < |z| < 3$

$$= \left(\frac{1}{z} + \frac{2}{z^2} + \frac{2^2}{z^3} + \frac{2^3}{z^4} + \dots \right) - \left(\frac{1}{3} + \frac{z}{3^2} + \frac{z^2}{3^3} + \frac{z^3}{3^4} + \dots \right) + \left(\frac{1}{3^2} + \frac{2z}{3^3} + \frac{3z^2}{3^4} + \frac{4z^3}{3^5} + \dots \right)$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} \left(\frac{1}{3} \right)^{n+1} z^n + \sum_{n=0}^{\infty} (n+1) \left(\frac{1}{3} \right)^{n+2} z^n$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} \left[(n+1) \left(\frac{1}{3} \right)^{n+2} - \left(\frac{1}{3} \right)^{n+1} \right] z^n$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} (n-2) \left(\frac{1}{3} \right)^{n+2} z^n$$

On inversion, we get

$$u_n = 2^{n-1}, n \geq 1 \text{ \& } u_n = -(n+2)3^{n-2}, n \leq 0$$

Exercises: Find the inverse Z – Transform of the following:

1. $\frac{z^2}{(z-1)(z-3)}$
2. $\frac{4z}{z-a}, |z| > |a|$
3. $\frac{5z}{(2-z)(3z-1)}$
4. $\frac{z}{(z-1)^2}$
5. $\frac{3z^2+z}{(5z-1)(5z+2)}$
6. $\frac{8z-z^3}{(4-z)^3}$
7. $\frac{4z^2-2z}{z^3-5z^2+8z-4}$
8. $\frac{z+3}{(z+1)(z-2)}$

Answers:

1. $\frac{1}{2}(3^{n+1}-1)$
2. $4a^n$
3. $\left(\frac{1}{3}\right)^n - 2^n$
4. n
5. $\frac{13}{75}(0.2)^n + \frac{4}{75}(-0.4)^n$
6. $(n^2+7n+4)(4)^{n-1}$
7. $2+2^n+3(n-1)2^n, (n \geq 1)$
8. $2(-i)^{n-1} - (-2)^{n-1}$

Application to difference equations:

Procedure to solve a linear difference equation with constant coefficients by Z 0

Transforms:

1. Take the Z – transforms of both sides of the difference equation.
2. Transpose all terms without $U(z)$ to the right.
3. Divide by the coefficient of $U(z)$, getting $U(z)$ as a function of z .
4. Find inverse Z – Transform to get u_n .

Example: 1) Solve $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$ with $u_0 = 0, u_1 = 1$

Solution: If $Z(u_n) = U(z)$, Then $Z(u_{n+1}) = z[U(z) - u_0]$ & $Z(u_{n+2}) = z^2[U(z) - u_0 - u_1z^{-1}]$

$$\text{Also } Z(3^n) = \frac{z}{z-3}$$

$\therefore$ Taking the Z – Transform on both sides, we get

$$Z[u_{n+2}] + 4Z[u_{n+1}] + 3Z[u_n] = Z[3^n]$$

$$\Rightarrow z^2[U(z) - u_0 - u_1z^{-1}] + 4z[U(z) - u_0] + 3U(z) = \frac{z}{z-3}$$

$$\Rightarrow z^2[U(z) - 0 - 1.z^{-1}] + 4z[U(z) - 0] + 3U(z) = \frac{z}{z-3}, (\text{Using the given conditions}).$$

$$\text{i.e. } (z^2 + 4z + 3)U(z) = z + \frac{z}{z-3} = \frac{z^2 - 3z + z}{z-3} = \frac{z^2 - 2z}{z-3}$$

$$\therefore U(z) = \frac{z^2 - 2z}{(z^2 + 4z + 3)(z - 3)} = \frac{z^2 - 2z}{(z + 1)(z + 3)(z - 3)}$$

$$\text{Let } \frac{U(z)}{z} = \frac{z - 2}{(z + 1)(z + 3)(z - 3)} = \frac{A}{z + 1} + \frac{B}{z + 3} + \frac{C}{z - 3}$$

$$\Rightarrow z - 2 = A(z + 3)(z - 3) + B(z + 1)(z - 3) + C(z + 1)(z + 3)$$

$$z = -3 \Rightarrow -5 = B(-2)(-6) \Rightarrow B = -\frac{5}{12}, \quad z = 3 \Rightarrow 1 = C(4)(6) \Rightarrow C = \frac{1}{24},$$

$$z = -1 \Rightarrow -3 = A(2)(-4) \Rightarrow A = \frac{3}{8}.$$

$$\therefore \frac{U(z)}{z} = \frac{3}{8} \frac{1}{z + 1} - \frac{5}{12} \frac{1}{z + 3} + \frac{1}{24} \frac{1}{z - 3} \Rightarrow U(z) = \frac{3}{8} \frac{z}{z + 1} - \frac{5}{12} \frac{z}{z + 3} + \frac{1}{24} \frac{z}{z - 3},$$

On inversion we get,

$$u_n = \frac{3}{8} Z^{-1} \left[\frac{z}{z + 1} \right] - \frac{5}{12} Z^{-1} \left[\frac{z}{z + 3} \right] + \frac{1}{24} Z^{-1} \left[\frac{z}{z - 3} \right] = \frac{3}{8} (-1)^n - \frac{5}{12} (-3)^n + \frac{1}{24} 3^n$$

2) Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ with $y_0 = y_1 = 0$, using Z - Transform.

Solution: Let $Z(y_n) = U(z)$, then $Z(y_{n+1}) = z[U(z) - y_0]$ & $Z(y_{n+2}) = z^2[U(z) - y_0 - y_1 z^{-1}]$

Also $Z(2^n) = \frac{z}{z - 2}$. $\therefore$ by taking Z - Transform we get,

$$Z[y_{n+2}] + 6Z[y_{n+1}] + 9Z[y_n] = Z[2^n] \Rightarrow z^2[U(z) - y_0 - y_1 z^{-1}] + 6z[U(z) - y_0] + 9U(z) = \frac{z}{z - 2}$$

$$\text{i.e. } z^2[U(z) - 0 - 0 \cdot z^{-1}] + 6z[U(z) - 0] + 9U(z) = \frac{z}{z - 2} \quad (\text{Using the condition } y_0 = y_1 = 0)$$

$$\Rightarrow (z^2 + 6z + 9)U(z) = \frac{z}{z - 2} \Rightarrow U(z) = \frac{z}{(z^2 + 6z + 9)(z - 2)}$$

$$\text{Let } \frac{U(z)}{z} = \frac{1}{(z^2 + 6z + 9)(z - 2)} = \frac{1}{(z + 3)^2(z - 2)} = \frac{A}{z + 3} + \frac{B}{(z + 3)^2} + \frac{C}{z - 2}$$

$$\therefore 1 = A(z + 3)(z - 2) + B(z - 2) + C(z + 3)^2$$

$$z = 2 \Rightarrow 1 = C(25) \Rightarrow C = \frac{1}{25}, \quad z = -3 \Rightarrow 1 = B(-5) \Rightarrow B = -\frac{1}{5} \quad \& \quad z = 0 \Rightarrow 1 = -6A - 2B + 9C$$

$$\Rightarrow 1 = -6A - 2\left(-\frac{1}{5}\right) + 9\left(\frac{1}{25}\right) \Rightarrow 6A = \frac{9}{25} + \frac{2}{5} - 1 = \frac{9 + 10 - 25}{25} = -\frac{6}{25} \Rightarrow A = -\frac{1}{25}$$

$$\therefore \frac{U(z)}{z} = -\frac{1}{25} \frac{1}{z + 3} - \frac{1}{5} \frac{1}{(z + 3)^2} + \frac{1}{25} \frac{1}{z - 2} = \frac{1}{25} \left[\frac{1}{z - 2} - \frac{1}{z + 3} - \frac{5}{(z + 3)^2} \right]$$

$$\Rightarrow U(z) = \frac{1}{25} \left[\frac{z}{z - 2} - \frac{z}{z + 3} - \frac{5z}{(z + 3)^2} \right], \text{ On taking inverse Z - transform we get,}$$

$$y_n = \frac{1}{25} \left[Z^{-1} \left[\frac{z}{z-2} \right] - Z^{-1} \left[\frac{z}{z+3} \right] + \frac{5}{3} Z^{-1} \left[\frac{-3z}{(z+3)^2} \right] \right]$$

$$= \frac{1}{25} \left[2^n - (-3)^n + \frac{5}{3} (n(-3)^n) \right] \left[\because Z^{-1} \left[\frac{az}{(z-a)^2} \right] = na^n \right]$$

3) Find the response of the system $y_{n+2} - 5y_{n+1} + 6y_n = u_n$ with $y_0 = 0$, $y_1 = 1$ & $u_n = 1$ for $n = 0, 1, 2, \dots$ by Z – Transform method.

Solution: Taking Z – Transform of both sides we get,

$$Z[y_{n+2}] - 5Z[y_{n+1}] + 6Z[y_n] = Z[u_n] = Z[1]$$

i.e. $z^2[U(z) - y_0 - y_1z^{-1}] - 5z[U(z) - y_0] + 6U(z) = \frac{z}{z-1}$, Using $y_0 = 0$ & $y_1 = 1$ we get,

$$z^2[U(z) - 0 - 1 \cdot z^{-1}] - 5z[U(z) - 0] + 6U(z) = \frac{z}{z-1}$$

$$\Rightarrow (z^2 - 5z + 6)U(z) - z = \frac{z}{z-1} \Rightarrow (z^2 - 5z + 6)U(z) = z + \frac{z}{z-1} = \frac{z^2 - z + z}{z-1} = \frac{z^2}{z-1}$$

$$\Rightarrow U(z) = \frac{z^2}{(z-1)(z^2 - 5z + 6)} = \frac{z^2}{(z-1)(z-2)(z-3)}$$

$$\text{Let } \frac{U(z)}{z} = \frac{z}{(z-1)(z-2)(z-3)} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3}$$

$$\Rightarrow z = A(z-2)(z-3) + B(z-1)(z-3) + C(z-1)(z-2)$$

$$z = 1 \Rightarrow 1 = A(-1)(-2) \Rightarrow A = \frac{1}{2}, \quad z = 2 \Rightarrow 2 = B(1)(-1) \Rightarrow B = -2,$$

$$z = 3 \Rightarrow 3 = C(2)(1) \Rightarrow C = \frac{3}{2}$$

$$\therefore \frac{U(z)}{z} = \frac{1}{2} \frac{1}{z-1} - \frac{2}{z-2} + \frac{3}{2} \frac{1}{z-3} \Rightarrow U(z) = \frac{1}{2} \frac{z}{z-1} - 2 \frac{z}{z-2} + \frac{3}{2} \frac{z}{z-3}, \text{ inversion we get}$$

$$y_n = \frac{1}{2} Z^{-1} \left[\frac{z}{z-1} \right] - 2 Z^{-1} \left[\frac{z}{z-2} \right] + \frac{3}{2} Z^{-1} \left[\frac{z}{z-3} \right] = \frac{1}{2} (1)^n - 2 \cdot 2^n + \frac{3}{2} 3^n = 2^{-1} - 2^{n+1} + 3^{n+1} 2^{-1}$$

4. Using the Z – Transform, solve $u_{n+2} - 2u_{n+1} + u_n = 3n + 5$

Solution: Taking Z – Transform of both sides we get,

$$Z[u_{n+2}] - 2Z[u_{n+1}] + Z[u_n] = Z[3n + 5]$$

$$\Rightarrow z^2[U(z) - u_0 - u_1z^{-1}] - 2z[U(z) - u_0] + U(z) = 3 \frac{z}{(z-1)^2} + 5 \frac{z}{z-1}$$

$$\Rightarrow (z^2 - 2z + 1)U(z) - z^2u_0 - u_1z + 2zu_0 = \frac{3z}{(z-1)^2} + \frac{5z}{z-1}$$

$$\Rightarrow (z^2 - 2z + 1)U(z) = \frac{3z + 5z^2 - 5z}{(z-1)^2} + (z^2 - 2z)u_0 + u_1z = \frac{5z^2 - 2z}{(z-1)^2} + (z^2 - 2z)u_0 + u_1z$$

$$\Rightarrow U(z) = \frac{5z^2 - 2z}{(z-1)^4} + \frac{(z^2 - 2z)}{(z-1)^2}u_0 + u_1 \frac{z}{(z-1)^2}, \text{ On inversion we get,}$$

$$u_n = Z^{-1}\left[\frac{5z^2 - 2z}{(z-1)^4}\right] + u_0 Z^{-1}\left[\frac{(z^2 - 2z)}{(z-1)^2}\right] + u_1 Z^{-1}\left[\frac{z}{(z-1)^2}\right] \text{----- (I)}$$

$$\text{Now we know that } Z^{-1}[1] = \frac{z}{z-1}, Z^{-1}[n] = \frac{z}{(z-1)^2}, Z^{-1}[n^2] = \frac{z^2 + z}{(z-1)^3} \&$$

$$Z^{-1}[n^3] = \frac{z^3 + 4z^2 - z}{(z-1)^4}.$$

$$\therefore \text{ We can write } \frac{5z^2 - 2z}{(z-1)^4} \text{ as}$$

$$\frac{5z^2 - 2z}{(z-1)^4} = A \frac{z^3 + 4z - z}{(z-1)^4} + B \frac{z^2 + z}{(z-1)^3} + C \frac{z}{(z-1)^2} + D \frac{z}{z-1}$$

$$\Rightarrow 5z^2 - 2z = A(z^3 + 4z^2 - z) + B(z^2 + z)(z-1) + Cz(z-1)^2 + Dz(z-1)^3$$

$$\text{i.e. } 5z^2 - 2z = A(z^3 + 4z^2 - z) + B(z^3 - z) + C(z^3 - 2z^2 + z) + D(z^4 - 3z^3 + 3z^2 - z)$$

By equating coefficient of like powers of z, we get

$$D = 0, \quad A + B + C - 3D = 0, \quad 4A - 2C + 3D = 5, \quad -A - B + C - D = -2$$

$$D = 0 \Rightarrow A + B + C = 0 \text{----- (1)}$$

$$4A - 2C = 5 \text{----- (2)}$$

$$-A - B + C = -2 \text{----- (3)}$$

$$\text{Adding (1) \& (3) we get } 2C = -2 \Rightarrow C = -1$$

$$\text{From (2) we get } 4A - 2(-1) = 5 \Rightarrow 4A = 5 - 2 = 3 \Rightarrow A = \frac{3}{4}.$$

$$\text{From (3) we get } B = -A + C + 2 = -\frac{3}{4} - 1 + 2 = \frac{-3 - 4 + 8}{4} = \frac{1}{4}.$$

$$\therefore \frac{5z^2 - 2z}{(z-1)^4} = \frac{3}{4} \frac{z^3 + 4z - z}{(z-1)^4} + \frac{1}{4} \frac{z^2 + z}{(z-1)^3} - \frac{z}{(z-1)^2}$$

$$\text{Hence } Z^{-1}\left[\frac{5z^2 - 2z}{(z-1)^4}\right] = \frac{3}{4} Z^{-1}\left[\frac{z^3 + 4z - z}{(z-1)^4}\right] + \frac{1}{4} Z^{-1}\left[\frac{z^2 + z}{(z-1)^3}\right] - Z^{-1}\left[\frac{z}{(z-1)^2}\right]$$

$$= \frac{3}{4}n^3 + \frac{1}{4}n^2 - n = \frac{1}{4}(3n^3 + n^2 - 4n) = \frac{n}{4}(3n^2 + n - 4) = \frac{1}{4}n(n-1)(3n+4)$$

$$\frac{z-2}{(z-1)^2} = \frac{A}{z-1} + \frac{B}{(z-1)^2} \Rightarrow z-2 = A(z-1) + B$$

$$z=1 \Rightarrow -1 = B \Rightarrow B = -1 \text{ \& } z=0 \Rightarrow -2 = -A+B \Rightarrow A = B+2 = -1+2 = 1$$

$$\therefore \frac{z-2}{(z-1)^2} = \frac{1}{z-1} - \frac{1}{(z-1)^2} \Rightarrow \frac{z^2-2z}{(z-1)^2} = \frac{z}{z-1} - \frac{z}{(z-1)^2}$$

$$\therefore Z^{-1}\left[\frac{z^2-2z}{(z-1)^2}\right] = Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z}{(z-1)^2}\right] = 1-n \text{ \& } Z^{-1}\left[\frac{z}{(z-1)^2}\right] = n$$

Substituting these values in (I) we get

$$u_n = \frac{1}{4}n(n-1)(3n+4) + u_0(1-n) + u_1n = \frac{1}{4}n(n-1)(3n+4) + u_0 + (u_1 - u_0)n$$

$$\frac{1}{4}n(n-1)(3n+4) + c_0 + c_1n, \text{ Where } c_0 = u_0 \text{ \& } c_1 = u_1 - u_0$$

Exercises: Solve the following difference equations using Z – Transforms

$$1. y_{n+2} - 4y_n = 0 \text{ given that } y_0 = 0, y_1 = 2 \quad 2. u_{n+2} - 5u_{n+1} + 6u_n = 0$$

$$3. x(n+2) - 3x(n+1) + 2x(n) = 0, \text{ given that } x(0) = 0, x(1) = 1$$

$$4. f(n) = 3f(n-1) - 4f(n-2) = 0, n \geq 2, \text{ given that } f(0) = 3, f(1) = -2$$

$$5. u_{n+2} - 5u_{n+1} - 6u_n = 2^n$$

$$6. y_{n+2} - 6y_{n+1} + 9y_n = 3^n$$

$$7. y_{n+1} + \frac{1}{4}y_n = \left(\frac{1}{4}\right)^n \quad (n \geq 0), y_0 = 0 \quad 8. y_{n+2} + 2y_{n+1} + y_n = n \text{ with } y_0 = y_1 = 0$$

$$9. u_{k+2} - 2u_{k+1} + u_k = 2^k \text{ with } u_0 = 2, u_1 = 1 \quad 10. y_{n+2} - 5y_{n+1} - 6y_n = 4^n$$

$$\textbf{Answers: } 1. y_n = 2^{n-1} + (-2)^{n-1} \quad 2. u_n = c_1 \cdot 2^n + c_2 \cdot 3^n \quad 3. x(n) = 1 - 2^n$$

$$4. f(n) = 2 + (-4)^n \quad 5. u_n = c_1(-1)^n + c_2(6)^n - \frac{1}{12}(2)^n$$

$$6. y_n = (c_1 + c_2n)3^n + \frac{1}{2}n(n-1)3^{n-2} \quad 7. y_n = 2\left[\left(\frac{1}{4}\right)^n - \left(-\frac{1}{4}\right)^n\right]$$

$$8. y_n = \frac{3}{4}n \quad 9. u_k = 1 - 2k + 2^k \quad 10. y_n = c_1(6)^n + c_2(-1)^n - \frac{1}{10}4^n$$

Linear Programming Problems

Linear programming is a decision making technique under the given constraints on the condition that the relationship among the variables involved is linear. A general relationship among the variables involved is called **objective function**. The variables involved are called **decision variables**.

Canonical and Standard forms of LPP

After the formulation of LPP, the next step is to obtain its solution. But before any method is used to find its solution, the problem must be presented in a suitable form. Two forms are dealt with here, the canonical form and the standard form.

The canonical form : The general linear programming problem can always be expressed in the following form :

$$\text{Maximize } Z = c_1 x_1 + c_2 x_2 + c_3 x_3 + \dots + c_n x_n$$

subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$$

.....

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$$

and the non-negativity restrictions

$$x_1, x_2, x_3, \dots, x_n \geq 0.$$

This form of LPP is called **the canonical form of the LPP**.

In matrix notation the canonical form of LPP can be expressed as :

Maximize $Z = CX$ (objective function)

Subject to $AX \leq b$ (constraints)

and $X \geq 0$ (non-negativity restrictions)

where $C = (c_1 \ c_2 \ \dots \ c_n)$,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

The Standard Form :

The general linear programming problem in the form

Maximize or Minimize

$$Z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

Subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\dots \dots \dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

and $x_1, x_2, \dots, x_n \geq 0$ is known as standard form.

In matrix notation the standard form of LPP can be expressed as :

Maximize or Minimize $Z = CX$ (objective function)

Subject to constraints $AX = b$ and $X \geq 0$. Where, $C = (c_1, c_2, \dots, c_n)$,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

The set of real values $X = (x_1, x_2, \dots, x_n)$ which satisfies the constraint $AX \leq (\geq) B$ is called a **basic solution**.

A set of real values x_i which satisfies the constraints and also satisfy non negativity constraints $x_i \geq 0$ is called **feasible solution**.

A set of real values x_i which satisfies the constraints along with non negativity restrictions and optimizes the objective function is called **optimal solution**.

Simplex Method

It is an iterative method to solve a Linear programming Problem.

Slack Variable: It is a variable introduced to convert " $\leq$ " (less than or equal to) to " $=$ " (equal to).

Surplus Variable: It is a variable introduced to convert " $\geq$ " (greater than or equal to) to " $=$ " (equal to)

PROBLEMS:

1. Use simplex method to maximize $Z = x + 1.5y$ subject to the constraints

$$x + 2y \leq 160, 3x + 2y = 240, x \geq 0, y \geq 0.$$

Solution: Let us introduce slack variables s_1 and s_2 to the inequalities to write them in the following form.

$$x + 2y + 1s_1 + 0s_2 = 160$$

$$3x + 2y + 0s_1 + 1s_2 = 240$$

$$x + 1.5y + 0s_1 + 0s_2 = z \quad \text{is the objective function.}$$

Solution by simplex method is presented in the following table.

NZV	x	y	s_1	s_2	Qty	Ratios	
s_1	1	2	1	0	160	$160/2=80$	-1.5 is least 2 is pivot . s_1 is replaced by y.
s_2	3	2	0	1	240	$240/2=120$	
Indicators(Δ)	-1	-1.5	0	0	0		
y	0.5	1	0.5	0	80		$R_2 \rightarrow -2R_1 + R_2$
s_2	3	2	0	1	240		$R_3 \rightarrow 1.5R_1 + R_3$
Indicators(Δ)	-1	-1.5	0	0	0		
y	0.5	1	0.5	0	80	$\frac{80}{1} = 160$	-0.25 is least. 2 is pivot . s_2 is replaced by $\frac{1}{2}$.
s_2	2	0	-1	1	80	$80/2=40$	
Indicators(Δ)	-0.25	0	0.75	0	120		
y	0.5	1	0.5	0	80		
x	1	0	-0.5	0.5	40		
Indicators(Δ)	-0.25	0	0.75	0	120		
y	0	1	$3/4$	$-1/4$	60	$R_1 \rightarrow -1/2R_2 + R_1$	
x	1	0	$-1/2$	$1/2$	40	$R_3 \rightarrow 0.25R_2 + R_3$	
Indicators(Δ)	0	0	0.625	0.125	130		No negative integer

Since there are no negative indicators , the entries in the last column will give us the optimal solution.

Thus $(Z)_{max} = 130$ at $x = 40$ and $y = 60$.

2. use simplex method to maximize $Z = 2x + 4y$ subject to the constraints
 $3x + y \leq 22$, $2x + 3y = 24$, $x \geq 0$, $y \geq 0$.

Solution: Let us introduce slack variables s_1 and s_2 to the inequalities to write them in the following form.

$$3x + y + 1s_1 + 0s_2 = 22$$

$$2x + 3y + 0s_1 + 1s_2 = 24$$

$$2x + 4y + 0s_1 + 0s_2 = z \quad \text{is the objective function.}$$

Solution by simplex method is presented in the following table.

NZV	x	y	s_1	s_2	Qty	Ratio	
s_1	3	1	1	0	22	22/1=22	8 is least, 3 is pivot, s_2 is replaced by y. Also $1/3.R_2$
s_2	2	3	0	1	24	24/3=8	
Indicators(Δ)	-2	-4	0	0	0		
s_1	3	1	1	0	22		$R_2 \rightarrow -2R_1 + R_2$
y	2/3	1	0	1/3	8		
Indicators(Δ)	-2	-4	0	0	0		$R_2 \rightarrow -2R_1 + R_2$
s_1	7/3	0	1	-1/3	14		
y	2/3	0	0	1/3	8		
Indicators(Δ)	2/3	0	0	4/3	32		No negative indicators.

Thus $(Z)_{max} = 32$ at $x = 0$ and $y = 8$.

3. Maximize $P = 2x + 3y + z$ subject to the constraints .

$$x + 3y + 2z \leq 11, \quad x + 2y + 5z \leq 19, \quad 3x + y + 4z \leq 25, \quad x \geq 0, y, z \geq 0.$$

Solution: Let us introduce slack variables s_1, s_2 and s_3 to the inequalities to write them in the following form.

$$x + 3y + 2z + 1s_1 + 0s_2 + 0s_3 = 11$$

$$x + 2y + 5z + 0s_1 + 1s_2 + 0s_3 = 19$$

$$3x + y + 4z + 0s_1 + 0s_2 + 1s_3 = 25$$

$$2x + 3y + z + 0s_1 + 0s_2 + 0s_3 = P \text{ is the objective function.}$$

Solution by simplex method is presented in the following table.

NZV	x	y	z	s_1	s_2	s_3	Qty	Ratio	
s_1	1	3	2	1	0	0	11	11/3=3.33	3.33 is least, 3 is pivot, s_1 is replaced by y. Also $1/3.R_1$
s_2	1	2	5	0	1	0	19	19/2=9.5	
s_3	3	1	4	0	0	1	25	25/1=25	
Indicators(Δ)	-2	-3	-1	0	0	0	0		
y	1/3	1	2/3	1/3	0	0	11/3		
s_2	1	2	5	0	1	0	19		$R_2 \rightarrow -2R_1 + R_2$
s_3	3	1	4	0	0	1	25		$R_3 \rightarrow -R_1 + R_3$
Indicators(Δ)	-2	-3	-1	0	0	0	0		$R_4 \rightarrow -3R_1 + R_4$
y	1/3	1	2/3	1/3	0	0	11/3	$\frac{11}{3} \div \frac{1}{3} = 11$	8 is least, 8/3 is pivot, s_3 is replaced by x. Also $3/8.R_3$
s_2	1/3	0	11/3	-2/3	1	0	35/3	$\frac{35}{3} \div \frac{1}{3} = 35$	
s_3	8/3	0	10/3	-1/3	0	1	64/3	$\frac{64}{3} \div \frac{8}{8} = 8$	
Indicators(Δ)	-1	0	1	1	0	0	11		
y	1/3	1	2/3	1/3	0	0	11/3		$R_1 \rightarrow -\frac{1}{3}R_3 + R_1$
s_2	1/3	0	11/3	-2/3	1	0	35/3		$R_2 \rightarrow -\frac{1}{3}R_3 + R_2$
x	1	0	10/8	-1/8	0	3/8	8		
Indicators(Δ)	-1	0	1	1	0	0	11		$R_4 \rightarrow R_3 + R_4$
y	0	1	1/4	3/8	0	-1/8	1		
s_2	0	0	13/4	-5/8	1	-1/8	9		
x	1	0	10/8	-1/8	0	3/8	8		
Indicators(Δ)	0	0	9/4	7/8	0	3/8	19		No negative indicators.

Thus $(P)_{max} = 19$ at $x = 8, y = 1$ and $z = 0$

4. Use simplex method to minimize $P = x - 3y + 2z$ subject to the constraints .

$$3x - y + 2z \leq 7, -2x + 4y \leq 12, -4x + 3y + 8z \leq 10, x \geq 0, y \geq 0, z \geq 0.$$

Solution: Let us introduce slack variables s_1, s_2 and s_3 to the inequalities to write them in the following form. The given L.P.P is equivalent to maximizing the objective function $-P$ subject to the same constraints.

That is $-P = P' = -x + 3y - 2z$ is to be maximized.

$$3x - y + 2z + 1s_1 + 0s_2 + 0s_3 = 7$$

$$-2x + 4y + 0z + 0s_1 + 1s_2 + 0s_3 = 12$$

$$-4x + 3y + 8z + 0s_1 + 0s_2 + 1s_3 = 10$$

$$-x + 3y - 2z + 0s_1 + 0s_2 + 0s_3 = P' \text{ is the objective function.}$$

Solution by simplex method is presented in the following table.

NZV	x	y	z	s_1	s_2	s_3	Qty	Ratio	
s_1	3	-1	2	1	0	0	7	$\frac{7}{1} - 1 = -7$	3 is least, y is pivot. y replace s_2 . Also $\frac{1}{4}R_2$
s_2	-2	4	0	0	1	0	12	$\frac{12}{4} = 3$	
s_3	-4	3	8	0	0	1	10	$\frac{10}{3} = 3.3$	
Indicators(Δ)	1	-3	2	0	0	0	0		
s_1	3	-1	2	1	0	0	7		$R_1 \rightarrow R_1 + R_2$
y	-1/2	1	0	0	1/4	0	3		
s_3	-4	3	8	0	0	1	10		$R_3 \rightarrow -3R_2 + R_3$
Indicators(Δ)	1	-3	2	0	0	0	0		$R_4 \rightarrow 3R_2 + R_4$
s_1	5/2	0	2	1	1/4	0	10	$10 \div \frac{5}{2} = 4$	4 is least $\frac{5}{2}$ is pivot . x is replaces s_1 . Also $\frac{2}{5}R_1$
y	-1/2	1	0	0	1/4	0	3	$3 \div \frac{-1}{2} = -6$	
s_3	-5/2	0	8	0	-3/4	1	1	$1 \div \frac{-5}{2} = \frac{-2}{5}$	
Indicators(Δ)	-1/2	0	2	0	3/4	0	9		
s_1	1	0	4/5	2/5	1/10	0	4		
y	-1/2	1	0	0	1/4	0	3		$R_2 \rightarrow -\frac{1}{2}R_1 + R_2$
s_3	-5/2	0	8	0	-3/4	1	1		$R_3 \rightarrow \frac{5}{2}R_1 + R_3$
Indicators(Δ)	-1/2	0	2	0	3/4	0	9		$R_4 \rightarrow \frac{1}{2}R_1 + R_4$
x	1	0	4/5	2/5	1/10	0	4		
y	0	1	2/5	1/5	3/10	0	5		
s_3	0	0	10	1	-1/2	1	11		
Indicators(Δ)	0	0	12/5	1/5	4/5	0	11		No negative indicators.

Thus $(P')_{max} = 11$ at $x = 4, y = 5$ and $z = 0$.

5. Show that the L.P.P for minimizing the objective function $Z = -2u - 3v - w$ subject to the constraints $u + 2v - 2w \leq 5, 3u - v - w \leq 2, u, v, w \geq 0$ has unbounded solution.

Solution: Let $Z' = -Z = 2u + 3v + w$ for miximization to the objective function. Let us introduce slack variables s_1 and s_2 to the inequalities to write them in the following form.

$$u + 2v - 2w + 1s_1 + 0s_2 = 5$$

$$3u - v - w + 0s_1 + 1s_2 = 2$$

$$2u + 3v + w + 0s_1 + 0s_2 = Z' \quad \text{is the objective function.}$$

Solution by simplex method is presented in the following table.

NZV	u	v	w	s_1	s_2	Qty	Ratio	
s_1	1	2	-2	1	0	5	$\frac{5}{2} = 2.5$	2.5 is least 2 is pivot v replace s_1 . Also $\frac{1}{2} \cdot R_1$.
s_2	3	-1	-1	0	1	2	Negative	
Indicators(Δ)	-2	-3	-1	0	0	0		
v	1/2	1	-1	1/2	0	5/2		
s_2	3	-1	-1	0	1	2		$R_2 \rightarrow R_1 + R_2$
Indicators(Δ)	-2	-3	-1	0	0	0		$R_3 \rightarrow 3R_1 + R_3$
v	1/2	1	-1	1/2	0	5/2	$\frac{5}{2} = \frac{-5}{-1}$	
s_2	7/2	0	-2	1/2	1	9/2	$\frac{9/2}{-2} = \frac{-9}{4}$	
Indicators(Δ)	-1/2	0	-4	3/2	0	15/2		No negative indicators.

Since both the ratios are negative we cannot proceed further.

Thus we conclude the L.P.P has unbounded solution.

Graphical Method in Linear Programming Problems (LPP)

The **Graphical Method** is a technique used to solve a **Linear Programming Problem (LPP)** when there are **only two decision variables**. It helps determine the optimal value (maximum or minimum) of a linear objective function while satisfying a set of linear constraints

1. Maximize $z = x + 1.5y$ given $x \geq 0, y \geq 0$ subject to the constraints:
 $x + 2y \leq 160, 3x + 2y \leq 240$ by graphical method.

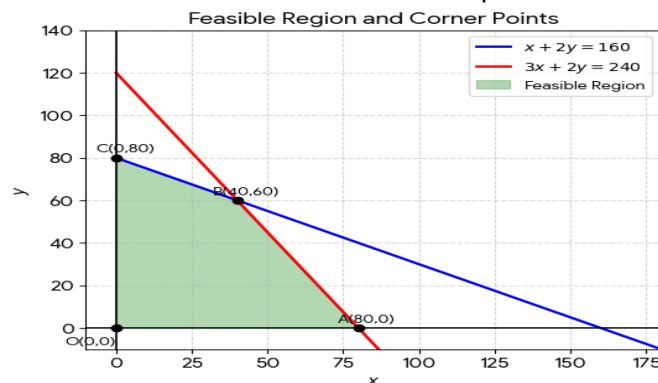
Solution: Let us consider the equations,

$$\begin{aligned} x + 2y &= 160, & 3x + 2y &= 240 \\ \text{Implies } \frac{x}{160} + \frac{y}{80} &= 1 \dots eqn1, & \frac{x}{80} + \frac{y}{120} &= 1 \dots eqn2 \end{aligned}$$

Let *eqn1* represent the straight line joining the points $E(160, 0), C(0, 80)$ and *eqn2*

Represent the straight line CD joining $A(80, 0), F(0, 120)$.

We draw the lines AB and CD in the XOY plane.



Shaded portion is the feasible and $OACB$ is the convex polygon. The point B is the point of intersection of the lines AB and CD . We can obtain this point by solving simultaneously the system of equations:
 $x + 2y = 160$ and $3x + 2y = 240$ on solving we obtain $B(X, Y) = (40, 60)$.

The value of the objective function at the corners of the convex polygon $OCEB$ are tabulated.

corner	Value of $z = x + 1.5y$
$O(0,0)$	0
$A(80,0)$	80
$B(40,60)$	130
$C(0,80)$	120

Thus Z is maximum at $B(x, y) = (40, 60)$ and has the maximum value equal to 130.

2. Minimize $z = 5x + 4y$ subject to the constraints:

$x + 2y \geq 10, x + y \geq 8, 2x + y \geq 12, x \geq 0, y \geq 0$ by graphical method.

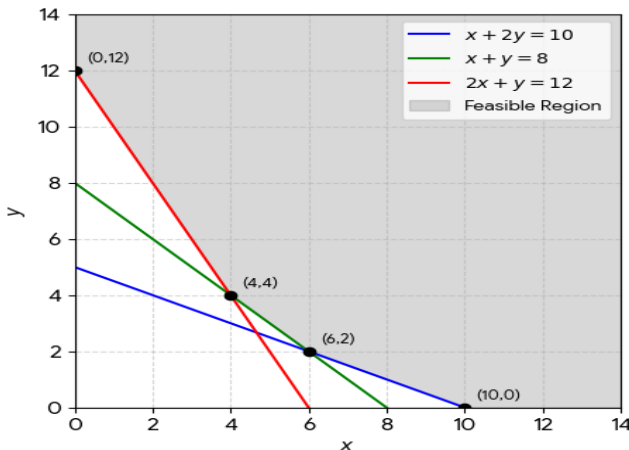
Solution: Let us consider the equations,

$$x + 2y = 10, \quad x + y = 8, \quad 2x + y = 12$$

Implies $\frac{x}{10} + \frac{y}{5} = 1 \dots eqn1$, $\frac{x}{8} + \frac{y}{8} = 1 \dots eqn2$, $\frac{x}{6} + \frac{y}{12} = 1 \dots eqn3$

Let $eqn1, eqn2, eqn3$ respectively represent the straight line AB, CD, EF joining the points $A(10, 0), B(0, 5), C(8, 0), D(0, 8), E(6, 0)$ and $F(0, 12)$.

We draw the lines in the XOY plane.



Shaded portion is the feasible region. The point 'G' is the point of intersection of the lines AB and CD. H is the point of intersection of the lines CD and EF.

We can obtain this point by solving simultaneously the system of equations:

$x + 2y = 160$ and $3x + 2y = 240$ on solving we obtain $H(X, Y) = (4, 4)$.

Here the feasible region is unbounded and we tabulate the value of the objective function corresponding to the points A, G, H, F.

The value of the objective function at the corners of the convex polygon OCEB are tabulated.

corner	Value of $z = 5x + 4y$
A(10,0)	50
G(6,2)	38
H(4,4)	36
F(0,12)	48

Thus Z is minimum for $x=4, y=4$ and the minimum value is 36.

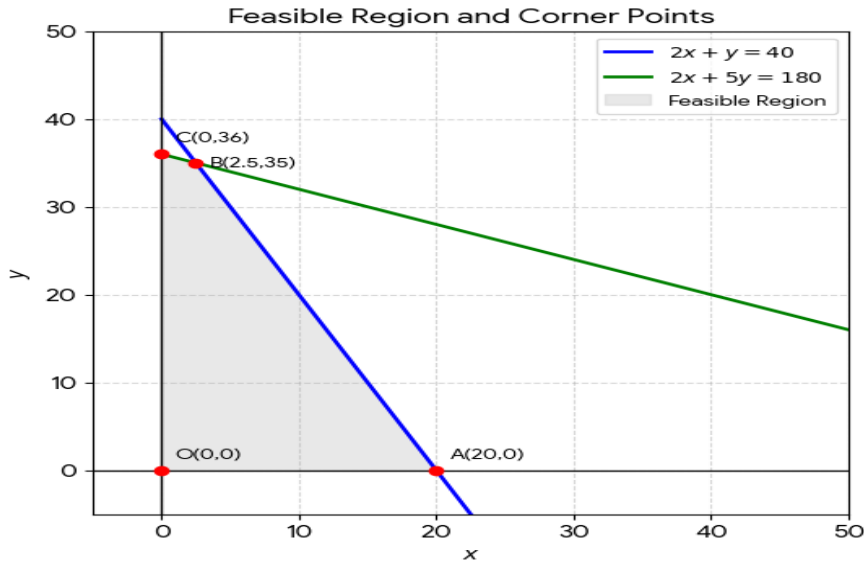
3. Use the graphical method to maximize $z = 3x + 4y$ subject to the constraints,
 $2x + y \leq 40$, $2x + 5y \leq 180$, $x \geq 0$, $y \geq 0$.

Solution: Let us consider the equations,

$$2x + y = 40, \quad 2x + 5y = 180,$$

Implies $\frac{x}{20} + \frac{y}{40} = 1 \dots eqn1$, $\frac{x}{90} + \frac{y}{36} = 1 \dots eqn2$,

Let $eqn1, eqn2$ respectively represent the straight line AB, CD joining the points $A(20, 0), B(0, 40), C(90, 0), D(0, 36)$. We draw the lines in the XOY plane.



Shaded portion is the feasible and OABD is the convex polygon. The point B is the point of intersection.

We can obtain this point by solving simultaneously the system of equations:

$2x + y = 40$ and $2x + 5y = 180$ on solving we obtain $B(X, Y) = (2.5, 35)$.

The value of the objective function at the corners of the convex polygon OABC are tabulated.

corner	Value of $z = 3x + 4y$
O(0,0)	0
A(20,0)	60
D(0,36)	144
B(2.5,35)	147.5

Thus Z is maximum at $B(X, Y) = (2.5, 35)$ and has the maximum value equal to 147.5.

4. Suppose an organisation is manufacturing two products P1 and P2. The profit per tonne of the two products are Rs. 50 and Rs. 60 respectively. Both the products require processing in three types of machine. The following Table indicates the available machine hours per week and the time required on each machine for one tonne of P1 and P2. Formulate this product mix problem in the linear programming form.

Solution: Let x_1 be the number of tonnes of product p_1 manufactured per week.

Let x_2 be the number of tonnes of product p_2 manufactured per week.

Objective Function

The objective is to maximize the total profit per week. The profit contribution is Rs. 50 per tonne for p_1 and Rs. 60 per tonne for p_2 .

Maximize: $Z = 50x_1 + 60x_2$

Constraints

The time required on each machine type and its maximum availability per week lead to the following constraint equations:

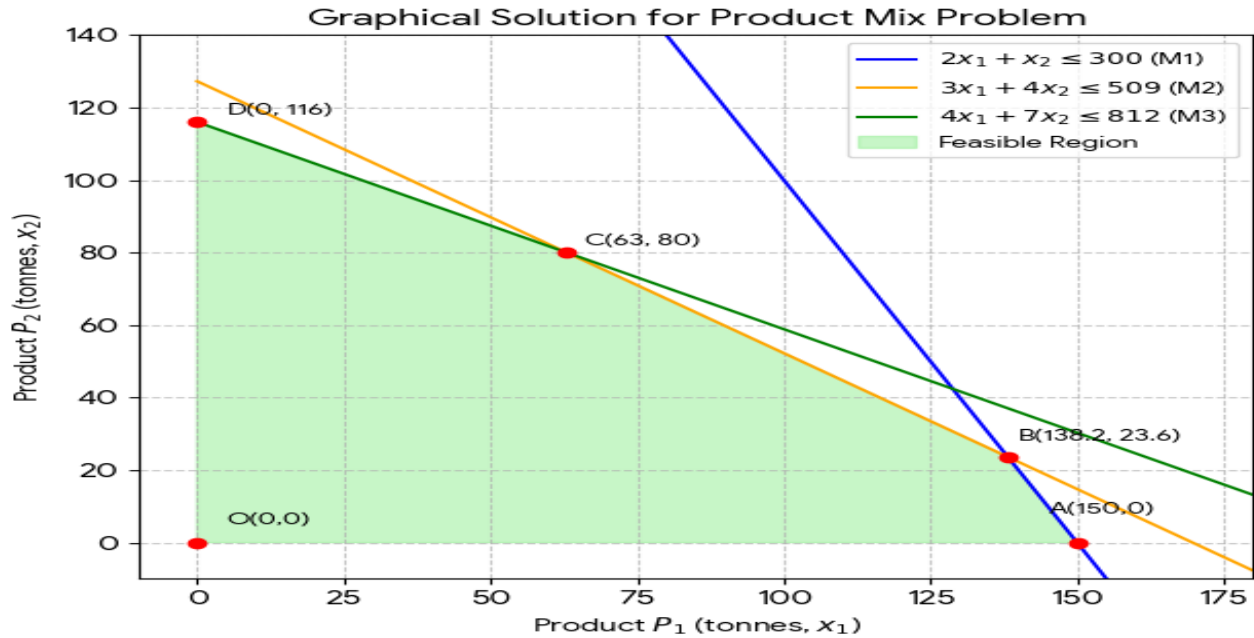
- **Machine 1:** $2x_1 + x_2 \leq 300$
- **Machine 2:** $3x_1 + 4x_2 \leq 509$
- **Machine 3:** $4x_1 + 7x_2 \leq 812$
- **Non-negativity:** $x_1, x_2 \geq 0$

Graphical Solution

To find the solution graphically, we first plot the boundary lines for each constraint inequality by determining their axis intercepts:

Line 1 (Machine 1): $2x_1 + x_2 = 300$

- If $x_1 = 0$ implies $x_2 = 300$; Intercept = (0, 300)
- If $x_2 = 0$ implies $x_1 = 150$; Intercept = (150, 0)
- 2. **Line 2 (Machine 2):** $3x_1 + 4x_2 = 509$
 - If $x_1 = 0$ implies $x_2 = 127.25$; Intercept = (0, 127.25)
 - If $x_2 = 0$ implies $x_1 = 169.67$; Intercept = (169.67, 0)
- 3. **Line 3 (Machine 3):** $4x_1 + 7x_2 = 812$
 - If $x_1 = 0$ implies $x_2 = 116$; Intercept = (0, 116)
 - If $x_2 = 0$ implies $x_1 = 203$; Intercept = (203, 0)



Shaded portion is the feasible and OABCD is the convex polygon. The point C is the point of intersection.

We can obtain this point by solving simultaneously the system of equations:

$2x_1 + x_2 = 300$ and $3x_1 + 4x_2 = 509$ on solving we obtain B(138.2, 23.6). Also $2x_1 + x_2 = 300$ and $4x_1 + 7x_2 = 812$ on solving we obtain C(63, 80).

The value of the objective function at the corners of the convex polygon OABCD are tabulated.

corner	Value of $z = 3x + 4y$
O(0,0)	0
A(150,0)	7500
B(138.2, 23.6)	8326
C(63, 80)	7950
D(0, 116)	6960

Thus Z is maximum at B(138.2, 23.6) and has the maximum value equal to Rs.8326 per week.

Big M method

The Big-M-Method is an alternative method of solving a linear programming problem involving artificial variables. To solve a L.P.P by simplex method, we have to start with the initial basic feasible solution and construct the initial simplex table. In the previous problems we see that the slack variables readily provided the initial basic feasible solution. However, in some problems, the slack variables cannot provide the initial basic feasible solution. In these problems atleast one of the constraint is of $=$ or $\geq$ type. "Big-M-Method is used to solve such L.P.P.

The big M -method or the method of penalties consists of the following basic steps :

Step 1:

Express the linear programming problem in the standard form by introducing slack and/or surplus variables, if any.

Step 2:

Introduce non-negative variables to the left hand side of all the constraints of ($>$ or $=$) type. These variables are called artificial variables. The purpose of introducing artificial variables is just to obtain an initial basic feasible solution. However, addition of these artificial variables causes violation of the corresponding constraints. Therefore we would like to get rid of these variables and would not allow them to appear in the optimum simplex table. To achieve this, we assign a very large penalty ' $-M$ ' to these artificial variables in the objective function, for maximization objective function.

Step 3:

Solve the modified linear programming problem by simplex method.

At any iteration of the usual simplex method there can arise any one of the following three cases :

(a) There is no vector corresponding to some artificial variable, in the basis y_b .

In such a case, we proceed to *step 4*.

(b) There is at least one vector corresponding to some artificial variable, in the basis y_b , at the zero level. That is, the corresponding entry in X_B is zero. Also, the co-efficient of M in each net evaluation $Z_j - C_j$ ($j = 1, 2, \dots, n$) is non-negative.

In such a case, the current basic feasible solution is a degenerate one. This is a case when an optimum solution to the given L.P.P. includes an artificial basic variable and an optimum basic feasible solution still exists.

(c) At least one artificial vector is in the basis y_b , but not at the zero level. That is, the corresponding entry in X_B is non-zero. Also coefficient of M in each net evaluation $Z_j - C_j$ is non-negative,

In this case, the given L.P.P. does not possess any feasible solution.

Step 4:

Application of simplex method is continued until either an optimum basic feasible solution is obtained or there is an indication of the existence of an unbounded solution to the given L.P.P.

Note. While applying simplex method, whenever a vector corresponding to some artificial variable happens to leave the basis, we drop that vector and omit all the entries corresponding to its column from the simplex table.

Example:

Maximize $z = x_1 + 5x_2$

Subject to

$$3x_1 + 4x_2 \leq 6$$

$$x_1 + 3x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

Solution:

Converting inequalities to equalities

By introducing surplus variables, slack variables and artificial variables, the standard form of LPP becomes

Maximize $x_1 + 5x_2 + 0x_3 + 0x_4 - MA_1$

Subject to

$$3x_1 + 4x_2 + x_3 = 6$$

$$x_1 + 3x_2 - x_4 + A_1 = 2$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, A_1 \geq 0$$

where,

x_3 is a slack variable

x_4 is a surplus variable

A_1 is an artificial variable.

Initial basic feasible solution

$$x_1 = x_2 = x_4 = 0, A_1 = 2, x_3 = 6$$

Iteration 1:

	c_j	1	5	0	0	$-M$	
CB	Basic variables B	x_1	x_2	x_3	x_4	A_1	Solution values $b (= X_B)$
0	x_3	3	4	1	0	0	6
$-M$	A_1	1	3	0	-1	1	2
$z_j - c_j$		$-M - 1$	$-3M - 5$	0	M	0	

Calculating values for index row ($z_j - c_j$)

$$z_1 - c_1 = 0 \times 3 + (-M) \times 1 - 1 = -M - 1$$

$$z_2 - c_2 = 0 \times 4 + (-M) \times 3 - 5 = -3M - 5$$

$$z_3 - c_3 = 0 \times 1 + (-M) \times 0 - 0 = 0$$

$$z_4 - c_4 = 0 \times 0 + (-M) \times (-1) - 0 = M$$

$$z_5 - c_5 = 0 \times 0 + (-M) \times 1 - (-M) = 0$$

As M is a large positive number, the coefficient of M in the $z_j - c_j$ row would decide the incoming basic variable.

As $-3M < -M$, x_2 becomes a basic variable in the next iteration.

Key column = x_2 column.

Minimum $(6/4, 2/3) = 2/3$

Key row = A_1 row

Pivot element = 3.

A_1 departs and x_2 enters.

Note: The iteration just completed, artificial variable A_1 was eliminated from the basis. The new solution is shown in the following table.

Iteration 2:

	c_j	1	5	0	0	
c_B	Basic variables B	x_1	x_2	x_3	x_4	Solution values $b (= X_B)$
0	x_3	$5/3$	0	1	$4/3$	$10/3$
5	x_2	$1/3$	1	0	$-1/3$	$2/3$
$z_j - c_j$		$2/3$	0	0	$-5/3$	

Iteration 3:

	c_j	1	5	0	0	
c_B	Basic variables B	x_1	x_2	x_3	x_4	Solution values $b (= X_B)$
0	x_4	$5/4$	0	$3/4$	1	$5/2$
5	x_2	$3/4$	1	$1/4$	0	$3/2$
$z_j - c_j$		$11/4$	0	$5/4$	0	

Result:

The optimal solution is $x_1 = 0$, $x_2 = 3/2$

$$\text{Max } z = 0 + 5 \times 3/2 = 15/2$$

ARTIFICIAL VARIABLE TECHNIQUES

You may recall that while introducing the slack and surplus variables, we had assigned a zero cost to them in the objective function. Moreover, the slack variables readily provided the initial basic feasible solution. There are, however, many linear programming problems where slack variables cannot provide such a solution. To solve such linear programming problems, there are two (closely related) methods, viz.,

- The "big M-method" or the "method of penalties" due to A. Charnes, and
- "two phase method" due to Dantzig, Orden and Wolfe.

SAMPLE PROBLEMS

1. Use penalty (or Big 'M') method to

$$\text{Minimize } z = 4x_1 + 3x_2$$

subject to the constraints :

$$2x_1 + x_2 \geq 10, \quad -3x_1 + 2x_2 \leq 6$$

$$x_1 + x_2 \geq 6, \quad x_1 \geq 0 \text{ and } x_2 \geq 0.$$

Solution. Introducing surplus (negative slack) variables $x_3 \geq 0$, $x_5 \geq 0$ and slack variable $x_4 \geq 0$ in the constraint inequations, the problem becomes

$$\text{Maximize } z^* = -4x_1 - 3x_2 + 0.x_3 + 0.x_4 + 0.x_5$$

subject to the constraints :

$$2x_1 + x_2 - x_3 = 10, \quad -3x_1 + 2x_2 + x_4 = 6$$

$$x_1 + x_2 - x_5 = 6, \quad x_j \geq 0 \quad (j = 1, 2, 3, 4, 5)$$

Clearly, we do not have a ready basic feasible solution. The surplus variables carry negative coefficients (-1). We introduce two new variables $A_1 \geq 0$ and $A_2 \geq 0$ in the first and third equations respectively. These extraneous variables, commonly termed as artificial variables, play the same role as that of slack variables in providing a starting basic feasible solution.

We assign a very high penalty cost (say $-M$, $M \geq 0$) to these variables in the objective function so that they may be driven to zero while reaching optimality.

Now the following initial basic feasible solution is available :

$$A_1 = 10, x_4 = 6 \text{ and } A_2 = 6$$

with $\mathbf{B} = (\mathbf{a}_6, \mathbf{a}_4, \mathbf{a}_7)$ as the basis matrix. The cost matrix corresponding to basic feasible solution is $\mathbf{c}_B = (-M, 0, -M)$

Now, corresponding to the basic variables A_1, x_4 and A_2 , the matrix $\mathbf{Y} = \mathbf{B}^{-1}\mathbf{A}$ and the net evaluations $z_j - c_j$ ($j = 1, 2, \dots, 7$) are computed. The initial basic feasible solution is displayed in the following simplex table :

Initial Iteration.			Introduce y_1 and drop y_6 .			0	0	-M	-M
			-4	-3	0				
$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	y_1	y_2	y_3	y_4	y_5	y_6	y_7
-M	y_6	10	2*	1	-1	0	0	1	0
0	y_4	6	-3	2	0	1	0	0	0
-M	y_7	6	1	1	0	0	-1	0	1
z^*			-16M	-3W+4	-2M+3	M	0	M	0

We observe that the most negative $z_j - c_j$ is $4 - 3M$ ($= z_1 - c_1$). The corresponding column vector y_1 , therefore, enters the basis. Further, since $\min. = 5$; the element y_{11} ($=2$) becomes the leading element for the first iteration

First Iteration: Introduce y_2 and drop y_7 .

$\mathbf{c}_B$	$\mathbf{y}_B$	$\mathbf{x}_B$	y_1	y_2	y_3	y_4	y_5	y_7
-4	y_1	5	1	1/2	-1/2	0	0	0
0	y_4	21	0	7/2	-3/2	1	0	0
-M	y_7	0	0	1/2*	1/2	0	-1	0
z^*			-M-20	0	$\frac{-M}{2} + 1$	$\frac{-M}{2} + 2$	0	M

In the above table, we omitted all entries of column vector y_6 , because the artificial variables A_1 has left the basis and we would not like it to re-enter in any subsequent iterations.

Now since the most negative ($z_j - c_j$) is $z_2 - c_2$; the non-basic vector y_2 enters the basis. Further, since $\min$ is 2 which occurs for the element y_{32} ($= 1/2$), the corresponding basis vector y_3 leaves the basis and the element y_{32} becomes the leading element for the next iteration.

Final Iteration: Optimum Solution,

C_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
-4	y_1	4	1	0	-1	0	1
0	y_4	14	0	0	-5	1	7
-3	y_2	2	0	1	1	0	-2
	z^*	-22	0	0	1	0	2

It is clear from the table that all $z_j - c_j$ are positive. Therefore an optimum basic feasible solution has been attained which is given by

$$x_1 = 4, x_2 = 2, \text{ maximum } z = 22.$$

2. Maximize $z = 3x_1 + 2x_2$ subject to the constraints :

$$2x_1 + x_2 \leq 2, 3x_1 + 4x_2 \geq 12, x_1, x_2 \geq 0.$$

Solution:

Introducing slack variable $x_3 \geq 0$, surplus variable $x_5 \geq 0$ and an artificial variable $A_1 \geq 0$, the reformulated L.P.P. can be written as :

$$\text{Maximize } z = 3x_1 + 2x_2 + 0.x_3 + 0.x_4 - MA_1$$

subject to the constraints :

$$2x_1 + x_2 + x_3 = 2,$$

$$3x_1 + 4x_2 - x_4 + A_1 = 12$$

$$x_1, x_2, x_3, x_4 \geq 0 \text{ and } A_1 \geq 0.$$

An obvious starting basic feasible solution is :

$$x_3 = 2 \text{ and } A_1 = 12.$$

The iterative simplex tables are :

Initial Iteration: Introduce y_2 and drop y_3 .

			3	2	0	0	-M
CB	YB	XB	y1	y2	y3	y4	y5
0	y3	2	2	1*	1	0	0
-M	y5	12	3	4	0	-1	1
	z	-12M-2	-3M-3	-4M	0	M	0

Final Iteration. No solution.

CB	YB	XB	y1	y2	y3	y4	y5
2	y2	2	2	1	1	0	0
-M	y5	4	-5	0	-4	-1	1
	z	4M+4	5M+1	0	4M+2	M	0

Here the coefficient of M in each $z_j - c_j$ is non-negative and an artificial vector appears in the basis, not at the zero level. Thus the given L.P.P. does not possess any feasible solution.

Unrestricted (unconstrained) Variables

Sometimes decision variables are unrestricted in sign (positive, negative or zero). In all such cases, the decision variables can be expressed as the difference between two non-negative variables. For example, if x_1 is unrestricted in sign, then Put $x_1 = x_1' - x_1''$

Example:

Maximize $z = 2x_1 + 3x_2$

Subject to

$$-x_1 + 2x_2 \leq 4$$

$$x_1 + x_2 \leq 6$$

$$x_1 + 3x_2 \leq 9$$

x_1, x_2 are unrestricted in sign

Solution:

Since x_1 and x_2 are unrestricted in sign, we can replace them by non-negative variables x_1' , x_1'' , x_2' , x_2'' .

$$\text{Put } x_1 = x_1' - x_1'' \\ x_2 = x_2' - x_2''$$

The given problem can be written as

$$\text{Max. } z = 2(x_1' - x_1'') + 3(x_2' - x_2'')$$

Subject to

$$\begin{aligned} -(x_1' - x_1'') + 2(x_2' - x_2'') &\leq 4 \\ (x_1' - x_1'') + (x_2' - x_2'') &\leq 6 \\ (x_1' - x_1'') + 3(x_2' - x_2'') &\leq 9 \end{aligned}$$

Introducing slack variables

$$\text{Max. } z = 2x_1' - 2x_1'' + 3x_2' - 3x_2''$$

Subject to

$$\begin{aligned} -x_1' + x_1'' + 2x_2' - 2x_2'' + x_3 &= 4 \\ x_1' - x_1'' + x_2' - x_2'' + x_4 &= 6 \\ x_1' - x_1'' + 3x_2' - 3x_2'' + x_5 &= 9 \end{aligned}$$

where x_3 , x_4 and x_5 are slack variables

Iteration 1:

	c_j	2	-2	3	-3	0	0	0	
c_B	Basic variables B	x_1'	x_1''	x_2'	x_2''	x_3	x_4	x_5	Solution values $b (=X_B)$
0	x_3	-1	1	2	-2	1	0	0	4
0	x_4	1	-1	1	-1	0	1	0	6
0	x_5	1	-1	3	-3	0	0	1	9
$Z_j - c_j$		-2	2	-3	3	0	0	0	

Key column = x_2' column.
 Minimum $(4/2, 6/1, 9/3) = 2$
 Key row = x_3 row.
 Pivot element = 2
 x_3 departs and x_2' enters.

Iteration 2:

	c_j	2	-2	3	-3	0	0	0	
c_B	Basic variables B	x_1'	x_1''	x_2'	x_2''	x_3	x_4	x_5	Solution values $b (=X_B)$
3	x_2'	$\frac{1}{2}$	$\frac{1}{2}$	1	-1	$\frac{1}{2}$	0	0	2
0	x_4	$\frac{3}{2}$	$\frac{3}{2}$	0	0	$\frac{1}{2}$	1	0	4
0	x_5	$\frac{5}{2}$	$\frac{5}{2}$	0	0	$\frac{3}{2}$	0	1	3
$z_j - c_j$		$\frac{7}{2}$	$\frac{7}{2}$	0	0	$\frac{3}{2}$	0	0	

Iteration 3:

	c_j	2	-2	3	-3	0	0	0	
c_B	Basic variables B	x_1'	x_1''	x_2'	x_2''	x_3	x_4	x_5	Solution values $b (=X_B)$
3	x_2'	0	0	1	-1	$\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{13}{5}$
0	x_4	0	0	0	0	$\frac{2}{5}$	1	$-\frac{3}{5}$	$\frac{11}{5}$
2	x_1'	1	-1	0	0	$-\frac{3}{5}$	0	$\frac{2}{5}$	$\frac{6}{5}$

$Z_j - C_j$		0	0	0	0	-3/5	0	7/5	
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Iteration 4:

	C_j	2	-2	3	-3	0	0	0	
C_B	Basic variables B	x_1'	x_1''	x_2'	x_2''	x_3	x_4	x_5	Solution values $b (=X_B)$
3	x_2'	0	0	1	-1	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
0	x_3	0	0	0	0	1	$\frac{5}{2}$	$-\frac{3}{2}$	$\frac{11}{2}$
2	x_1'	1	-1	0	0	0	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{9}{2}$
$Z_j - C_j$		0	0	0	0	0	$\frac{3}{2}$	$\frac{1}{2}$	

Result:

The optimal solution is $x_1' = 9/2$, $x_1'' = 0$, $x_2' = 3/2$, $x_2'' = 0$.

Solution of the original problem is

$$x_1 = x_1' - x_1'' = 9/2 - 0 = 9/2$$

$$x_2 = x_2' - x_2'' = 3/2 - 0 = 3/2$$

$$\text{Max } z = 2 \times 9/2 + 3 \times 3/2 = 27/2.$$

Transportation Problems

Transportation Problems is a special kind of linear programming problem in which goods are transported from a set of sources to a set of destination subjected to supply and demand for source and destination respectively such that the total cost of transportation is minimized.

Generalisation of transportation problem or formulation of transportation problem

Suppose we are supplying the resources from 'm' source to the 'n' destinations such that

a_i : the quantity available at the source S_i

b_j : the quantity required at the destination D_j

c_{ij} : the cost of transportation of one unit resource from S_i to D_j

x_{ij} : units of resources transported from S_i to D_j where $1 \leq i \leq m$ and $1 \leq j \leq n$

Transportation Table

	D_1	D_2	D_3	...	D_n	supply
S_1	c_{11}	c_{12}	c_{13}	...	c_{1n}	a_1
S_2	c_{21}	c_{22}	c_{23}	...	c_{2n}	a_2
S_3	c_{31}	c_{32}	c_{33}	...	c_{3n}	a_3
.	.	.	.		.	.
.	.	.	.		.	.
.	.	.	.		.	.
S_m	c_{m1}	c_{m2}	c_{m3}	...	c_{mn}	a_m
Demand	b_1	b_2	b_3	...	b_n	

Types of Transportation Problem

The transportation problem has 2 types they are

1) Balanced transportation problem

The sum of the supply or available is equal to the sum of requirements then it is called balanced transportation problem.

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$$

2) UnBalanced transportation problem

Suppose the sum of supply is more than sum of requirements or sum of source is less than sum of destination is called an unbalanced transportation problem. To solve this type problem we shall add one dully source (if supply > requirement) and add dummy source (if demand > supply) with zero cost, to make them balance.

Evaluation of Transportation Problem

To evaluate the transportation problem we have the following methods

- (i) North-West corner rule method
- (ii) Least Cost method
- (iii) Vogel's approximation method (VAM) or penalty method.

(i) North-West corner rule method

Step-1 : select the upper left corner cell of the transportation matrix and allocate the minimum of a_1, b_1 .

Step-2 : subtract this value from supply and demand of respective row and column.

Step-3 : If supply is zero then cross (strike) that row and move down to the next cell.

Step-4 : If the demand is zero then cross (strike) that column and move to the right to next cell.

Step-5 : repeat this step until all the supply and demand values are zero.

Step-6 : count the number of cells allocated and if it is equal to $m+n-1$ then it is called the non degenerate solution.

Problems

Find a solution using North-West corner rule method destination

	D_1	D_2	D_3	D_4	supply
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	

The given transportation problem is balanced TP

	D_1	D_2	D_3	D_4
S_1	5 19	2 30	50	10
S_2	70	6 30	3 40	60
S_3	40	8	4 70	14 20

Minimum transportation cost

$$= (5 \times 19) + (2 \times 30) + (6 \times 30) + (3 \times 40) + (4 \times 70) + (14 \times 20)$$

$$= 1015$$

$$m + n - 1 = 3 + 4 - 1 = 6 = \text{number of cell allocated}$$

The solution is non degenerate

Find the transportation cost using North-West corner method to the following

	D_1	D_2	D_3	D_4	supply
S_1	3	1	7	4	300
S_2	2	6	5	9	400
S_3	8	3	3	2	500
Demand	250	350	400	200	

The given transportation problem is balanced TP

	D_1	D_2	D_3	D_4
S_1	250 3	50 1	7	4
S_2	2	300 6	100 5	9
S_3	8	3	300 3	200 2

Minimum transportation cost

$$= (3 \times 250) + (1 \times 50) + (300 \times 6) + (5 \times 100) + (3 \times 300) + (2 \times 200)$$

$$= 4400$$

$$m + n - 1 = 3 + 4 - 1 = 6 = \text{number of cell allocated}$$

The solution is non degenerate

(ii) Least cost method of transportation problem

Step-1 : Determine the smallest cost in the cost matrix of the transportation table. Let it be c_{ij} allocate $x_{ij} = \min(a_i, b_j)$ in the cell (i,j).

Step-2 : If $x_{ij} = a_i$ then cross of i^{th} row of the transportation table and decrease b_j from a_i . Then go to step-3.

If $x_{ij} = b_j$ then cross of j^{th} column of the transportation table and decrease a_i from b_j . Then go to step-3.

Step-3 : If $x_{ij} = a_i = b_j$ cross of either i^{th} row or j^{th} column but not both.

Between step-1 and step-2 for resulting reduced transportation table until all the rim requirement are satisfied. Whenever the minimum cost is not unique make arbitrary choice along the minima.

Problems

Obtain an initial feasible solution for following transportation problems.

	D_1	D_2	D_3	D_4	supply
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	

The given transportation problem is balanced.

Sum of supply = Sum of demand

	D_1	D_2	D_3	D_4
O_1	1	6 2	3	4
O_2	4	3	2 2	6 0
O_3	4 0	2	6 2	1

Minimum transportation cost

$$= (6 \times 2) + (2 \times 2) + (6 \times 0) + (4 \times 0) + (6 \times 2)$$

$$= 28$$

$$m + n - 1 = 3 + 4 - 1 = 6 = \text{number of cell allocated}$$

But number of cell allocated is 5

The solution is degenerate.

Determine an initial basic feasible solution for the following TP using least cost method

	D_1	D_2	D_3	D_4	supply
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Demand	6	10	15	4	

The given transportation problem is balanced.

Sum of supply = Sum of demand

	D_1	D_2	D_3	D_4
O_1	6	4	14 1	5
O_2	6 8	9 9	1 2	7
O_3	4	1 3	6	4 2

Minimum cost Z

$$= (14 \times 1) + (6 \times 8) + (9 \times 9) + (1 \times 2) + (1 \times 3) + (4 \times 2) \\ = 156$$

$$m + n - 1 = 3 + 4 - 1 = 6 = \text{number of cell allocated}$$

The solution is non degenerate.

(iii) Vogel's approximation method

Step-1 : Find the penalty cost namely the difference between the smallest and next smallest causes in each row and column.

Step-2 : Among the penalties as found in step 1 choose the maximum penalty. If this maximum penalty is more than one (that is if there is a tie) choose any one arbitrary.

Step-3 : In the selected row or column as by step 2 find out the cell having the least cost. Allocate to this cell as much as possible depending on the capacity and requirement.

Step-4 : Delete the row or column which is fully exhausted. Again compute the column and row penalties for the reduced transportation table and then go to step 2. Repeat the procedure until all the rim requirements are satisfied.

Note: If the column is exhausted then there is a change in row penalty and viceversa.

Find the initial basic feasible solution for following transportation by VAM

	A	B	C	D	supply
P	5	4	2	6	20
Q	8	3	5	7	30
R	3	9	4	6	60
Demand	10	40	20	30	

The given transportation problem is balanced.

P	5	4	20	2	6	20		
Q	8	30	3	5	7	30		
R	10	3	10	9	4	30	6	60

Minimum cost Z

$$= (20 \times 2) + (30 \times 3) + (10 \times 3) + (10 \times 9) + (30 \times 6) \\ = 510$$

Find the initial basic feasible solution for following transportation by VAM

	D_1	D_2	D_3	D_4	supply
O_1	11	13	17	14	250
O_2	16	18	14	10	300
O_3	21	24	13	10	400
Demand	200	225	275	250	

The given transportation problem is balanced TP.

	D_1	D_2	D_3	D_4
O_1	200 11	50 13	17	14
O_2	16	175 18	14	125 10
O_3	21	24	275 13	125 10

Minimum cost Z

$$\begin{aligned}
 &= (200 \times 11) + (50 \times 13) + (175 \times 18) + (125 \times 10) + (125 \times 10) + (275 \times 13) \\
 &\quad + (125 \times 10) \\
 &= 12075
 \end{aligned}$$

Algorithm for assignment problem by Hungarian Method

The solution of assignment problem can be arrived using Hungarian method. The steps involved in this method are as follows.

Step-1 : Prepare a cost matrix. If the cost matrix is not a square matrix then add a dummy row(column) with zero cost element.

Step-2 : Subtract the minimum element in each row from all the elements of the respective rows.

Step-3 : Further modify the resulting matrix by subtracting the minimum element of each column from all the elements of the respective columns. Thus, obtain the modified matrix.

Step-4 : Then, draw minimum number of horizontal and vertical lines to cover all zeroes in the resulting matrix. Let the minimum number of lines be N. Now there are two possible cases.

Case(i) : If $N = n$ where n is the order of matrix then an optimal assignment can be made. So make the assignment to get the required solution.

Case(ii) : If $N < n$ then proceed to step 5.

Step-5 : Determine the smallest uncovered element in the matrix (element not covered by N lines). Subtract this minimum element from all uncovered elements and add the same element at the intersection of horizontal and vertical lines. Thus, the second modified matrix is obtained.

Step-6 : Repeat step 3 and 4 until we get the case (i) of step 3.

Step-7 : (To make zero assignment) examine the rows successively until a row wise exactly single zero is found. Circle (0) this zero to make the assignment. Then mark a cross (x) over all zeroes if lying in the column of the circled zero, showing that they can't be considered for future assignment. Continue in this matter until all the zeroes have been examined. Repeat the same procedure for column also.

Step-8 : Repeat the step 6 successively until one of the following situation arises.

i) If no unmarked zero is left, then the process ends or

ii) If there lies more than one of the unmarked zero in any column or row then, circle one of the unmarked zeroes arbitrarily and mark a cross in the cells of remaining zeroes in its rows or column. Repeat the process until no unmarked zero is left in the matrix.

Step-9 : Thus exactly one marked circled zero in each row and each column of the matrix is obtained. The assignment corresponding to these marked circled zeroes will give the optimal assignment.

Problems

Solve the assignment problem

	M1	M2	M3	M4
J1	10	12	19	11
J2	5	10	7	8
J3	12	14	13	11
J4	8	12	11	9

The total number of machines = number of jobs

$n=4$

Given assignment problem is balanced.

Step 1: Row operation

	M1	M2	M3	M4
J1	0	2	9	1
J2	0	5	2	3
J3	1	3	2	0
J4	0	4	3	1

Step 2: Column operation

	M1	M2	M3	M4
J1	0	0	7	1
J2	0	3	0	3
J3	1	1	0	0
J4	0	2	1	1

Step 3: Row operation

	M1	M2	M3	M4
J1	0	0	7	1
J2	0	3	0	3
J3	1	1	0	0
J4	0	2	1	1

The number of covered line = order of matrix = 4

The optimum solution is obtained

Job	Machine	Cost
J1	M1	12
J2	M2	7
J3	M3	11
J4	M4	8

Minimum cost $z = 12 + 7 + 11 + 8 = 38$

There are 4 jobs to be assigned to the machines. Only one job could be assigned to 1 machine. The amount of time in hours required for jobs in a machine are given in a following matrix. Find an optimum assignment of jobs to machine to minimum total processing time and also for each machine no job is assigned. What is total processing time to complete all the jobs?

Jobs	Machines				
	A	B	C	D	E
1	4	3	6	2	7
2	10	12	11	14	16
3	4	3	2	1	5
4	8	7	6	9	6

Given assignment problem is not balanced.

So that let's add one dummy job as 5. We get

	A	B	C	D	E
J1	4	3	6	2	7
J2	10	12	11	14	16
J3	4	3	2	1	5
J4	8	7	6	9	6
J5	0	0	0	0	0

Step 1: Row operation

	A	B	C	D	E
J1	2	1	4	0	5
J2	0	2	1	4	6
J3	3	2	1	0	4
J4	2	1	0	3	0
J5	0	0	0	0	0

Step 2 : Column operation

	A	B	C	D	E
J1	2	1	4	0	5
J2	0	2	1	4	6
J3	3	2	1	0	4
J4	2	1	0	3	0
J5	0	0	0	0	0

	A	B	C	D	E
J1	2	1	4	0	5
J2	0	2	1	4	6
J3	3	2	1	0	4
J4	2	1	0	3	0
J5	0	0	0	0	0

Here the number of covered lines is less than the order of matrix that is $4 < 5$.

Identify the least cost among the uncovered table and subtract the same from the uncovered cost and add the same to the intersection points.

	A	B	C	D	E
J1	2	0	3	0	4
J2	0	1	0	4	5
J3	3	1	0	0	3
J4	3	1	0	4	0
J5	1	0	0	1	0

Job	Machine	Cost
J1	A	3
J2	B	10
J3	C	1
J4	D	6
J5	E	0

Minimum cost $z = 3 + 10 + 1 + 6 = 20$