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College of Engineering



Transform Techniques and Optimization Theory

1BMATEC301



Course outcome

- CO1:** Develop Fourier series representations of periodic functions and perform harmonic analysis for functions with different periodicities, discontinuities and symmetry properties.
- CO2:** Apply Fourier transforms, including sine, cosine, discrete Fourier transforms (DFT), and Fast Fourier Transforms (FFT), to analyze signals and solve engineering problems involving differential equations.
- CO3:** Utilize Z-transforms, their properties, inverse transforms, and convolution techniques to analyze discrete-time systems and solve difference equations.
- CO4:** Formulate and solve Linear Programming Problems using graphical, simplex and Big-M methods to obtain optimal solutions in engineering and management applications.
- CO5:** Determine the extremals of functionals using calculus of variations and solve problems arising in dynamics of rigid bodies and vibrational analysis.



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Module-1: Fourier Series

Module-2: Fourier transforms

Module-3: Discrete Transforms - Z transforms

Module-4: Linear programming problems

Module-5: Transportation problems



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Module-1: Fourier Series



TOPICS



Fourier Series: Periodic functions, Dirichlet's condition. Fourier series of periodic functions

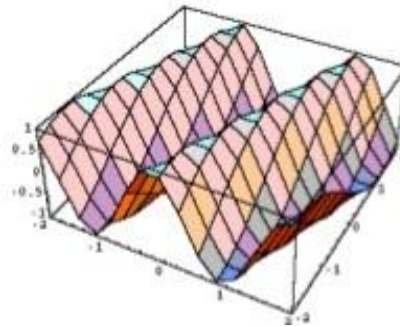
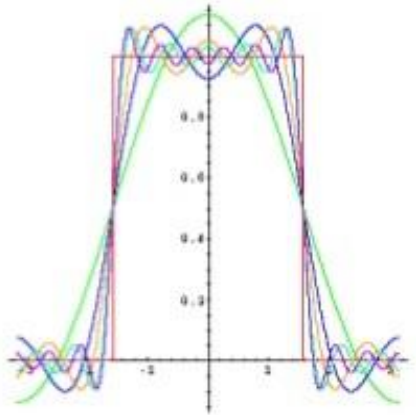


Half range Fourier series.



Practical Harmonic Analysis

Fourier waves



FOURIER SERIES can be generally written as,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx),$$

Where,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \quad \text{..... (1.1)}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \quad \text{..... (1.2)}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \quad \text{..... (1.3)}$$

Fourier series make use of the orthogonality relationships of the sine and cosine functions.

BASIS FORMULAE OF FOURIER SERIES

The Fourier series of a periodic function $f(x)$ with period 2π is defined as the trigonometric series with the coefficient a_0 , a_n and b_n , known as **FOURIER COEFFICIENTS**, determined by formulae (1.1), (1.2) and (1.3).

The individual terms in Fourier Series are known as **HARMONICS**.

Every function $f(x)$ of period 2π satisfying following conditions known as **DIRICHLET'S CONDITIONS**, can be expressed in the form of Fourier series.

FOURIER SERIES FOR EVEN AND ODD FUNCTIONS

EVEN FUNCTIONS



If function $f(x)$ is an even periodic function with the period $2L$ ($-L \leq x \leq L$), then $f(x)\cos(n\pi x/L)$ is even while $f(x)\sin(n\pi x/L)$ is odd.

Thus the Fourier series expansion of an even periodic function $f(x)$ with period $2L$ ($-L \leq x \leq L$) is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

Where, $a_0 = \frac{2}{L} \int_0^L f(x) dx$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx \quad n = 1, 2, \dots$$

$$b_n = 0$$

ODD FUNCTIONS



If function $f(x)$ is an even periodic function with the period $2L$ ($-L \leq x \leq L$), then $f(x)\cos(n\pi x/L)$ is even while $f(x)\sin(n\pi x/L)$ is odd.

Thus the Fourier series expansion of an odd periodic function $f(x)$ with period $2L$ ($-L \leq x \leq L$) is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \left(\frac{n\pi x}{L} \right)$$

Where, $b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \quad n = 1, 2, \dots$



EXAMPLES..

Question.: Find the fourier series of $f(x) = x^2 + x$, $-\pi \leq x \leq \pi$.

Solution.: The fourier series of $f(x)$ is given by,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx),$$

Using above,

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} (x^2 + x) dx \\ &= \frac{1}{\pi} \left(\frac{x^3}{3} + \frac{x^2}{2} \right)_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left(\frac{\pi^3}{3} + \frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} \right) = \frac{2\pi^3}{3} = a_0 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} (x^2 + x) \cos nx dx \\ &= \frac{1}{\pi} \left[(x^2 + x) \left(\frac{\sin nx}{n} \right) - (2x + 1) \left(\frac{-\cos nx}{n^2} \right) + (2) \left(\frac{-\sin nx}{n^3} \right) \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left[(2\pi + 1) \frac{\cos n\pi}{n^2} - (-2\pi + 1) \frac{\cos n\pi}{n^2} \right] \\ &= \frac{1}{\pi} \left[(2\pi + 1) \frac{(-1)^n}{n^2} - (-2\pi + 1) \frac{(-1)^n}{n^2} \right] \\ &= \frac{4(-1)^n}{n^2} \end{aligned}$$

$$\begin{aligned}b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \\&= \frac{1}{\pi} \int_{-\pi}^{\pi} (x^2 + x) \sin nx dx \\&= \frac{1}{\pi} \left[(x^2 + x) \left(-\frac{\cos nx}{n} \right) - (2x + 1) \left(-\frac{\sin nx}{n^2} \right) + (2) \left(\frac{\cos nx}{n^3} \right) \right]_{-\pi}^{\pi} \\&= \frac{1}{\pi} \left[-\frac{(\pi^2 + \pi)}{n} (-1)^n + \frac{(\pi^2 + \pi)}{n} (-1)^n \right] \\&= \frac{(-1)^n}{\pi n} [-\pi^2 - \pi + \pi^2 - \pi] \\&= -\frac{2(-1)^n}{n}\end{aligned}$$

Hence fourier series of, $f(x) = x^2 + x$,

$$x^2 + x = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left[\frac{4(-1)^n}{n^2} \cos nx - \frac{2(-1)^n}{n} \sin nx \right]$$

Half Range Fourier Series

If a function is defined over half the range, say 0 to L , instead of the full range from $-L$ to L , it may be expanded in a series of sine terms only or of cosine terms only. The series produced is then called a **half range Fourier series**.

Conversely, the Fourier Series of an even or odd function can be analyzed using the half range definition

HALF RANGE SINE SERIES

- Suppose that $f(x)$ satisfies Dirichlet's condition in the interval $0 < x < \tau$
- $f(x)$ is an even function of period 2τ , then even periodic $G(x)$ will have a convergent Fourier series representation consisting of cosine terms only and given by

$$G(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\tau}\right)$$

$$b_n = \frac{2}{\tau} \int_0^{\tau} f(x) \sin \frac{n\pi x}{\tau} dx, (n = 0, 1, 2, \dots)$$

HALF RANGE COSINE SERIES

- Suppose that $f(x)$ satisfies Dirichlet's condition in the interval $0 < x < \tau$
- $f(x)$ is an even function of period 2τ , then even periodic $F(x)$ will have a convergent Fourier series representation consisting of cosine terms only and given by

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\tau}\right)$$

$$a_0 = \frac{2}{\tau} \int_0^{\tau} f(x) dx \text{ and } a_n = \frac{2}{\tau} \int_0^{\tau} f(t) \cos \frac{n\pi x}{\tau} dx, \\ \text{for } (n = 0, 1, 2, \dots)$$

Example 1

Find the half-range sine series of the function

$$f(x) = \begin{cases} 4, & \text{if } 0 < x < \pi/2 \\ 0, & \text{if } \pi/2 < x < \pi. \end{cases}$$

Solution: $L = \pi$, so that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \quad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx.$$

where

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx = \\ &= \frac{8}{\pi} \int_0^{\pi/2} \sin(nx) dx = \\ &= \frac{8}{n\pi} \int_0^{n\pi/2} \sin u du = \\ &= -\frac{8}{n\pi} [\cos u]_0^{n\pi/2} \\ &= \frac{8}{n\pi} [1 - \cos(n\pi/2)]. \end{aligned}$$

Example 2

Find the half-range cosine series of the function

$$f(x) = \begin{cases} 4, & \text{if } 0 < x < \pi/2 \\ 0, & \text{if } \pi/2 < x < \pi. \end{cases}$$

NB same function as in the previous example!

Solution: again, $L = \pi$, so that

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx), \quad a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) dx,$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx.$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi/2} 4 dx = 2.$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx = \frac{8}{\pi} \int_0^{\pi/2} \cos(nx) dx \\ &= \frac{8}{n\pi} \int_0^{n\pi/2} \cos u du = \frac{8}{n\pi} [\sin u]_0^{n\pi/2} \\ &= \frac{8}{n\pi} \sin(n\pi/2). \end{aligned}$$

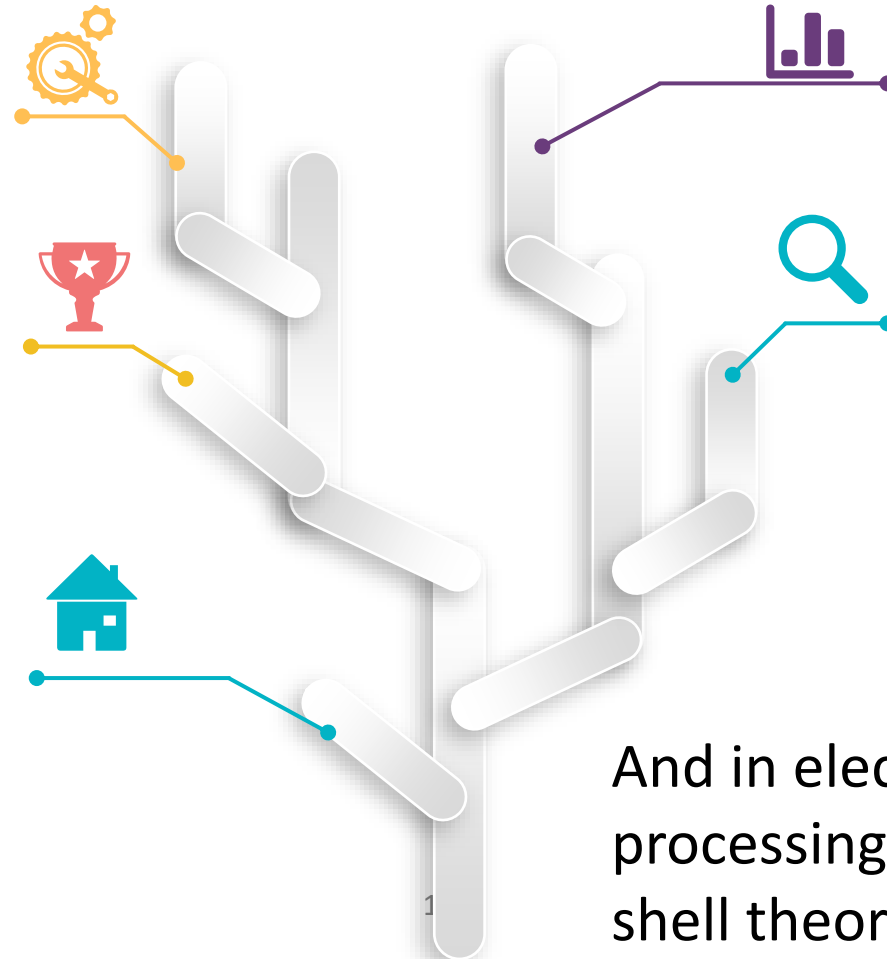


APPLICATIONS OF FOURIER SERIES

Quantum Mechanics

Acoustics

Image Processing



Vibration Analysis

Optics

And in electrical engineering, signal processing, econometrics, thin-walled shell theory, etc.



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Module-2: Fourier transforms

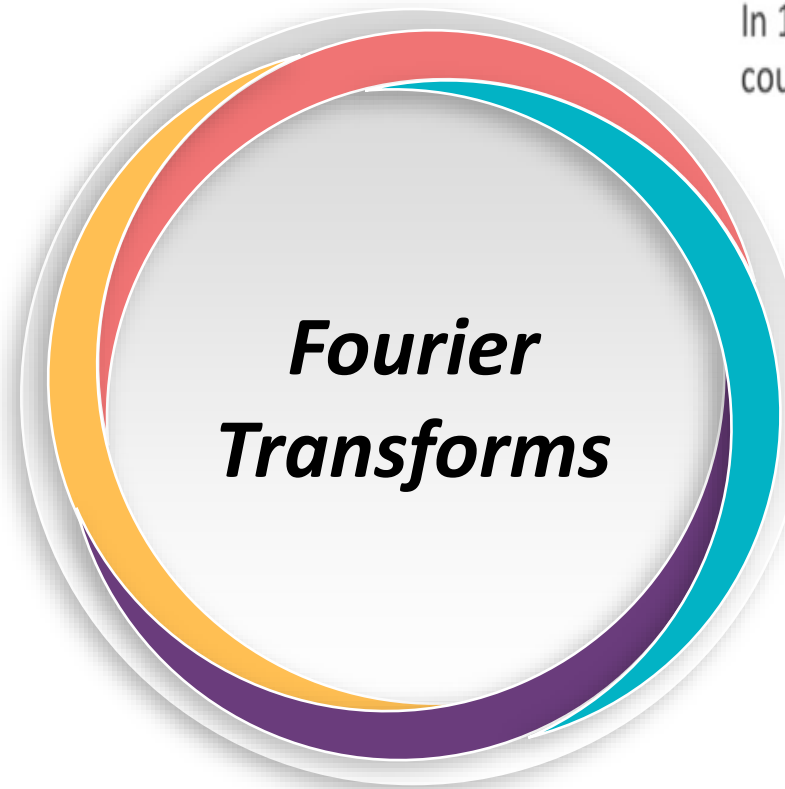
- ***Fourier Transforms:*** Infinite Fourier transforms, Fourier sine and
- cosine transforms.
- Inverse Fourier transforms.
- Problems

Fourier Transform (FT) involves a whole new paradigm of viewing signals in a context different from the frequency domain of their occurrence.
Transform:

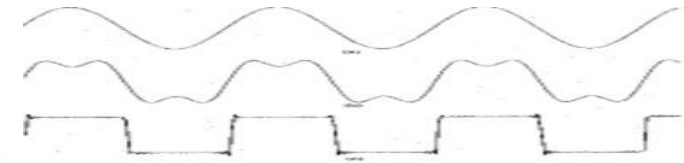
- Four Types of transformation in mathematics
- I. Translation
- II. Rotation
- III. Reflection
- IV. Dilation

What is Fourier Transform?

- The Fourier Transform is a type of mathematical transform.
- The Fourier Transform transforms a non-periodic function $f(t)$ in time domain into a function $F(s)$ in its corresponding frequency domain.
- It is a tool to break a function into a sinusoidal forms characterized by sine and cosine terms.
- To understand the conduction of heat, wave propagation, digital signal processing, image processing, filtering, etc.



In 1807, Joseph Fourier showed that any **periodic** signal could be represented by a series of **sinusoidal** functions



What is Fourier Transform?

- defined on $(-\infty, \infty)$
- piecewise continuous in each finite interval
- absolutely integrable in $(-\infty, \infty)$

$$F(w) = \int_{-\infty}^{\infty} f(t)e^{-iwt} dt$$

The function $F(w)$ is called the Fourier Transformation of $f(t)$

Fourier sine and cosine transform (Infinite)

If $f(t)$ is a function in $0 < t < \infty$ then

➤ $f_s\{w\} = \int_0^{\infty} f(t) \sin wt \, dt$ > Infinite Fourier sine transform

➤ $F_s^{-1}[f_s\{w\}] = \frac{2}{\pi} \int_0^{\infty} f_s\{w\} \sin wt \, dw$ > Inverse form of Infinite Fourier sine transform

➤ $f_c\{w\} = \int_0^{\infty} f(t) \cos wt \, dt$ > Infinite Fourier cosine transform

➤ $F_c^{-1}[f_s\{w\}] = \frac{2}{\pi} \int_0^{\infty} f_s\{w\} \cos wt \, dw$ > Inverse form of Infinite Fourier cosine transform

Fourier sine and cosine transform (finite)

➤ Finite Fourier sine transform of $F(x)$ over $0 < x < l$

$$f_s(n) = \int_0^l F(x) \sin \frac{n\pi x}{l} dx \text{ where } n \text{ is integer.}$$

The function $F(x)$ is called inverse finite Fourier sine transform of $f_s(n)$

$$F(x) = \frac{2}{l} \sum_{n=1}^{\infty} f_s(n) \sin \frac{n\pi x}{l}$$

➤ Finite Fourier cosine transform of $F(x)$ over $0 < x < l$

$$f_c(n) = \int_0^l F(x) \cos \frac{n\pi x}{l} dx \text{ where } n \text{ is integer.}$$

The function $F(x)$ is called inverse finite Fourier cosine transform of $f_s(n)$

$$F(x) = \frac{1}{l} f_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} f_c(n) \cos \frac{n\pi x}{l}$$



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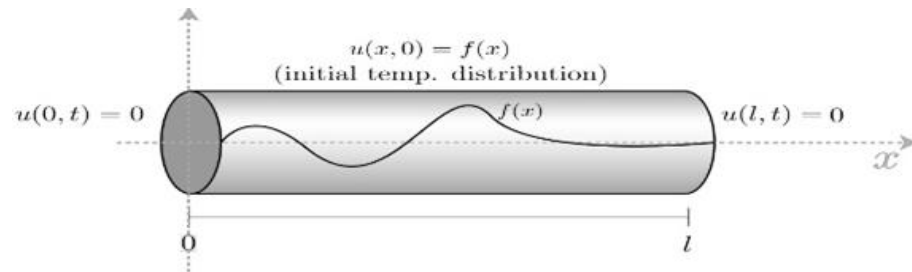
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Using FT to Solve Heat

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}; U(0, t) = 0; U(4, t) = 0;$$
$$U(x, 0) = 2x \text{ where } 0 < x < 4, t > 0.$$

$u(x, t)$ represents temperature at any point x at time t , $l = 0 \rightarrow 4$



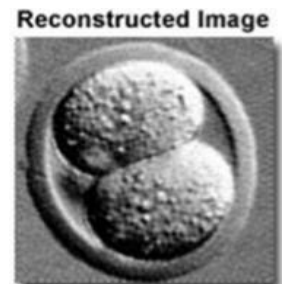
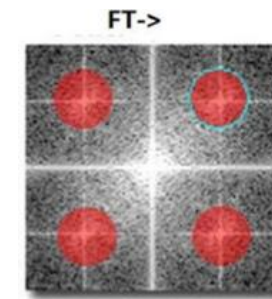
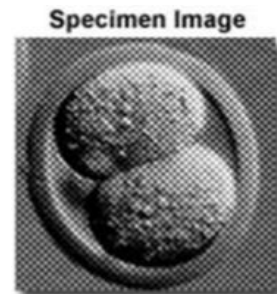
Real life Examples:

1. A doctor examines a patient's heart beat, is done by ECG (Electro- Cardio-Gram)
2. For a musician, all the ragas played are actually in time domain but frequency is more important for him than time
3. For a circuit, the input and output signals are functions of time.

Other Applications of the FT

- Signal analysis
- Sound filtering
- Partial differential equations
- Image processing
- Conduction of heat
- Wave propagation.

Applications: Image Filtering





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Module-3: Z transforms

Difference equations and Z-transforms

Difference Equations

Difference equations arise in problems where the time is assumed to have a discrete set of possible values.

$$y(k + n) + a_{n-1}y(k + n - 1) + \dots + a_1y(k + 1) + a_0y(k) = b_nu(k + n) + b_{n-1}u(k + n - 1) + \dots + b_1u(k + 1) + b_0u(k)$$

Where coefficients $a_{n-1}, a_{n-2}, \dots$ and $b_n, b_{n-1}, \dots$ are constant. $u(k)$ is forcing function

Example 1: For the given difference equation, determine the (a) order of the equation. Is the equation (b) linear, (c) time invariant, or (d) homogeneous? $y(k + 2) + 0.8y(k + 1) + 0.07y(k) = u(k)$

Solution:

- a) The equation is second order.
- b) All terms enter the equation linearly
- c) All the terms if the equation have constant coefficients. Therefore the equation is therefore LTI.
- d) A forcing function appears in the equation, so it is nonhomogeneous.



Z-transform Definition

Given the causal sequence $\{x(kT)\}$, its z-transform is defined as
$$X(z) = \sum_{k=0}^{\infty} x(kT) z^{-k}$$

$$X(z) = x(0) + x(T) z^{-1} + x(2T) z^{-2} + \dots + x(kT) z^{-k}$$

The variable z^{-1} in the above equation can be regarded as a time delay operator.

Example

- Obtain the Z-transform of the sequence

$$x(kT) = \{1, 1, 3, 2, 0, 4, 0, 0, 0, \dots\}$$

- Solution:

$$X(z) = 1 + z^{-1} + 3z^{-2} + 2z^{-3} + 4z^{-5}$$

z-transform of a Unit-Step Function

Let $x(kT) = u(kT)$

$$u(kT) = \begin{cases} 1 & \text{for } k \geq 0 \\ 0 & \text{for } k < 0 \end{cases}$$

Then

$$X(z) = \sum_{k=0}^{\infty} x(kT) z^{-k} = \sum_{k=0}^{\infty} u(kT) z^{-k} = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$



Z-Transform Properties: Linearity

Notation $x_1(kT) \xleftrightarrow{Z} X_1(z)$ & $x_2(kT) \xleftrightarrow{Z} X_2(z)$

Linearity $a x_1(kT) + b x_2(kT) \xleftrightarrow{Z} a X_1(z) + b X_2(z)$

Example:

$$x(kT) = 4u(kT) - 3a^k u(kT)$$

$$X(z) = \frac{4}{1-z^{-1}} - \frac{3}{1-az^{-1}} = \frac{1-(3-4a)z^{-1}}{(1-z^{-1})(1-az^{-1})}$$

Complex Differentiation $k x(kT) \xleftrightarrow{Z} -z \frac{dX(z)}{dz}$
 $k^m x(kT) \xleftrightarrow{Z} \left(-z \frac{d}{dz}\right)^m X(z)$

Multiplication by Exponential

$$a^k x(kT) \xleftrightarrow{Z} X(z/a)$$



Time Right Shifting (Time Delay)

Use the time delay property of the Laplace Transform

$$x(t - T_d) \xrightarrow{Z} e^{-sT} X(s)$$

Then

$$x(kT - nT) \xrightarrow{Z} z^{-n} X(z)$$

Example:

$$x(kT) = (0.2)^{(k-3)T} u((k-3)T)$$

$$X(Z) = z^{-3} \frac{1}{1 - 0.2z^{-1}}$$

Time Left Shifting (Time Advance)

$$x(kT + nT) \xrightarrow{Z} z^n [X(z) - \sum_{k=0}^{n-1} x(kT) z^{-k}]$$

Example:

$$x(kT) = (0.2)^{(k+2)T} u((k+2)T)$$

$$X(Z) = z^2 \left[\frac{1}{1 - 0.2z^{-1}} - \sum_{k=0}^1 x(kT) z^{-k} \right]$$

$$X(Z) = z^2 \left[\frac{1}{1 - 0.2z^{-1}} - (1 + 0.2z^{-1}) \right] = \frac{0.04}{1 - 0.2z^{-1}}$$



Convolution

$$x_1(kT) \xleftrightarrow{Z} X_1(z) \quad \& \quad x_2(kT) \xleftrightarrow{Z} X_2(z)$$

Convolution in time domain is multiplication in z-domain

$$x_1(kT) * x_2(kT) \xleftrightarrow{Z} X_1(z) X_2(z)$$

Example: Let's calculate the convolution of

$$x_1(kT) = a^{kT} u(kT) \quad \text{and} \quad x_2(kT) = u(kT)$$

Multiplications of z-transforms is

$$Y(z) = X_1(z) X_2(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})}$$

The Inverse Z-Transform

Formal inverse z-transform is based on a Cauchy integral

$$x(kT) = Z^{-1}[X(z)] = \frac{1}{2\pi j} \oint_C X(z) z^{k-1} dz$$

Initial and Final Value Theorems

Initial Value Theorem:

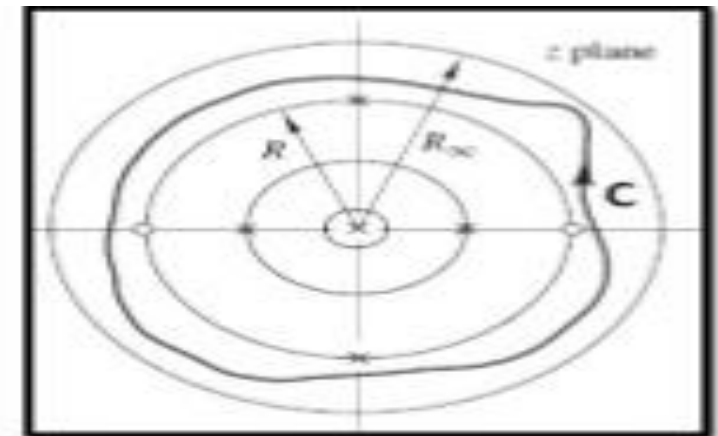
- The value of $x(n)$ as $k \rightarrow 0$ is given by:

$$x(0) = \lim_{k \rightarrow 0} x(kT) = \lim_{z \rightarrow \infty} X(z)$$

Final Value Theorem:

- The value of $x(n)$ as $k \rightarrow \infty$ is given by:

$$x(\infty) = \lim_{k \rightarrow \infty} x(kT) = \lim_{z \rightarrow 1} [(z - 1) X(z)]$$



APPLICATIONS OF Z- TRANSFORM

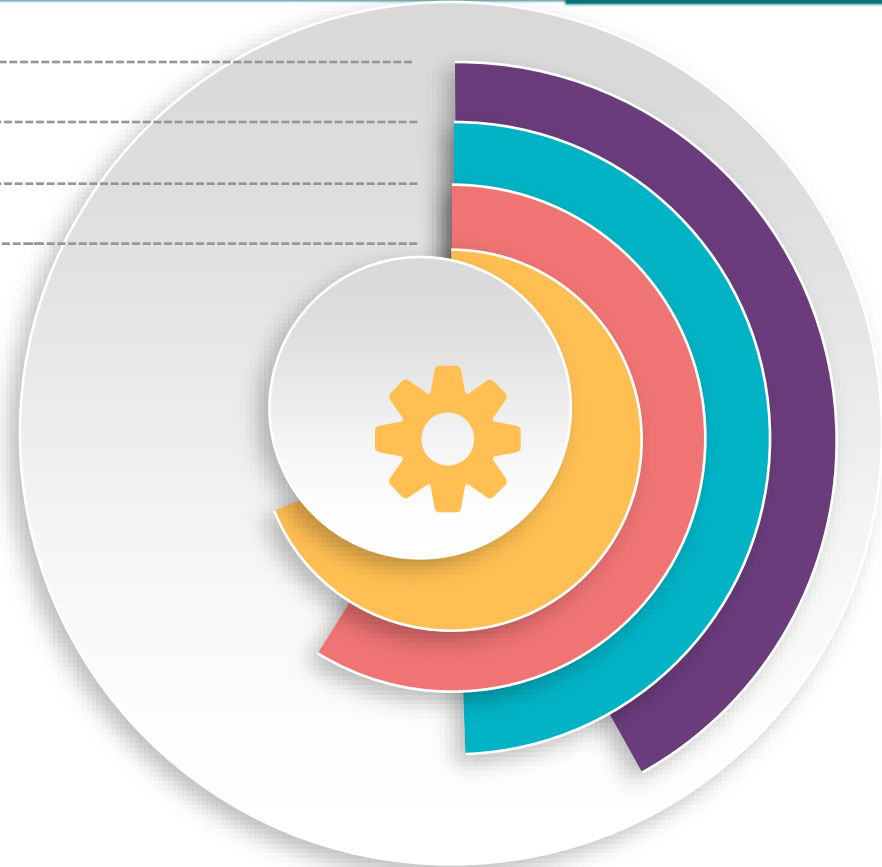
Z transform is used to convert discrete time domain signal into discrete frequency domain signal.

It has wide range of applications in mathematics and digital signal processing.

It is mainly used to analyze and process digital data. For example to analyze JPEG images, MP3 and MP4 songs, ZIP files etc,

Applications of Z transform in digital signal processing are,

1. Used in system designing
- 2.Used to find out stability of a system
- 3.Used to find frequency response of a signal
- 4.Analysis of linear discrete system
- 5.For designing digital filters





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Module-4: Linear programming problems

Linear programming

A technique that allows decision makers to solve **maximization** and **minimization** problems where there are certain **constraints** that limit what can be done, given that all objectives and constraints are **linear**.

Definitions

- The **objective function** is the function to be maximized (or minimized)
- The **constraints** are given in inequalities
- Both the **objective** and **constraints** are **linear** in choice variables

Graphical Method

- Let us consider a typical LP problem TechEdge Company Manufacturers of assembled PC's
Two models - *TE DeskPro, TE Portable*
❖ *Data (for weekly plan)*

	Assembly Time (Hours)	Special Display Units (Units)	Warehouse Space (Sq. Ft.)	Profit Contribn Per unit
DeskPro	3	0	8	50
Portable	5	1	5	40
Max. Availability	150	20	300	



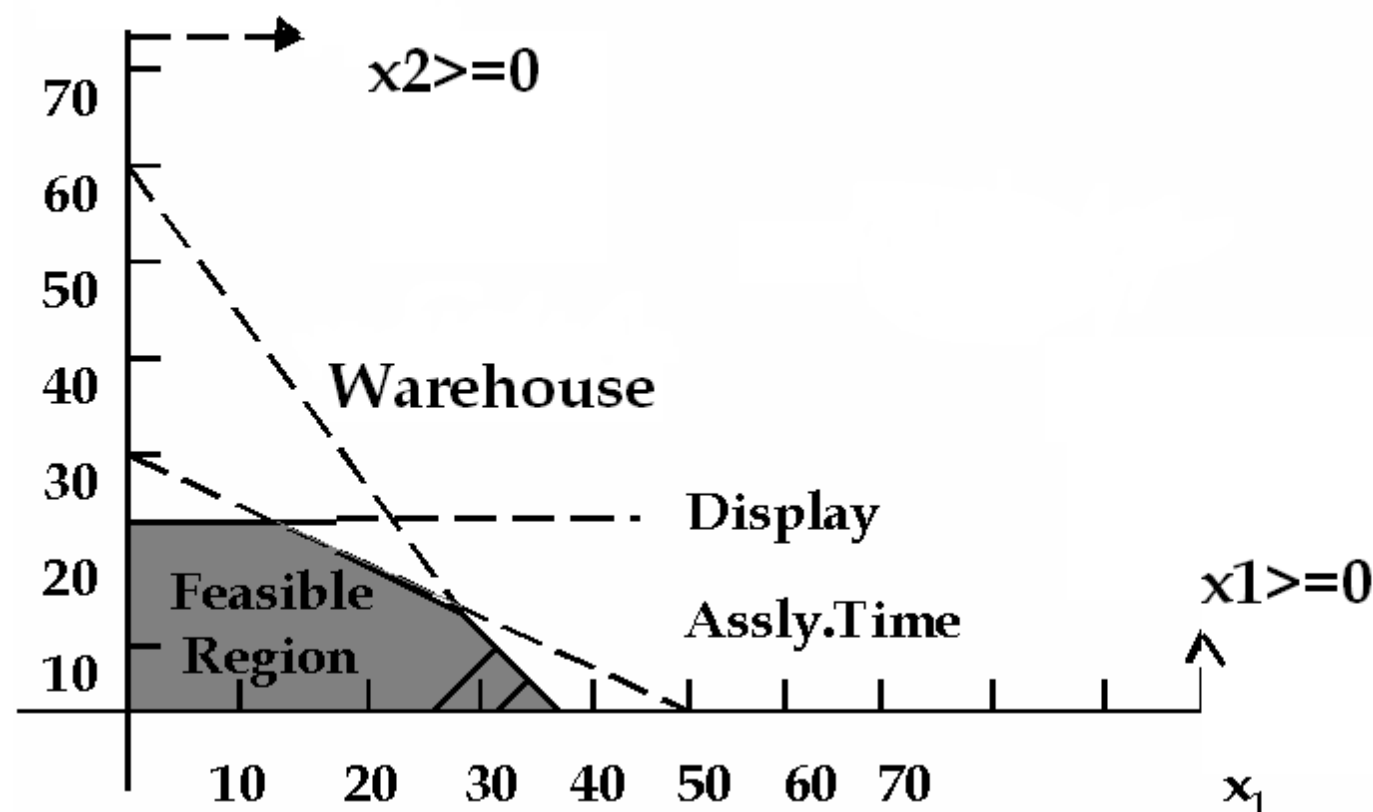
➤ **Objective – To find weekly production schedule that maximizes total profit**

▪ **Decision variables**

x_1 – number of units of DeskPro assembled in a week

x_2 – number of units of Portable assembled in a week

- Maximize $Z = 50x_1 + 40x_2$
- Subject to
 - $3x_1 + 5x_2 \leq 150$ (Assembly time)
 - $x_2 \leq 20$ (Special display unit)
 - $8x_1 + 5x_2 \leq 300$ (Warehouse space)
 - $x_1, x_2 \geq 0$



- Finding optimal solution – Pick that point in the feasible region which will maximize the objective function value
- A trivial solution – Check value of the function at each point in the feasible region
- However, special properties of LP problems allow us to find it more efficiently
- Specifically, optimal solution would be at one of the edges – Corner Points (CP)

Consider corner points of feasible region

<u>C.P.</u>	<u>?</u>	<u>Z</u>
• (0,0)	0	
• (37.5,0)	$50 * 37.5$	$= 1875$
• (30,12)	$50 * 30 + 40 * 12$	$= 1980$
• (50,20)	$50 * (50/3) + 40 * 20$	$= 1633.3$
• (0,20)	$40 * 20$	$= 800$
• (30,12) is optimal (x_1^* , x_2^*)		
• $Z^* = 1980$		

Simplex method

A systematic process for comparing **extreme point** or **corner solutions** to linear programming problems

Convert to Augmented Form

Constraints $\rightarrow$ Equations

Add slack ($\leq$ case) or subtract surplus ($\geq$ case) variables

$$(a) 3x_1 + 5x_2 \leq 150 \rightarrow 3x_1 + 5x_2 + x_3 = 150, x_3 \geq 0$$

$\rightarrow x_3$ is slack variable (amount by which resource availability exceeds its usage)

$$(b) 3x_1 + 5x_2 \geq 150 \rightarrow 3x_1 + 5x_2 - x_4 = 150, x_4 \geq 0$$

$\rightarrow x_4$ is surplus variable (e.g., in diet problem, surplus of a nutrient in diet plan over its minimum requirement)

Thus, TechEdge problem could be converted to its augmented form

Original Form	Augmented Form
Maximize $Z = 50x_1 + 40x_2$ Subject to $3x_1 + 5x_2 \leq 150$ $x_2 \leq 20$ $8x_1 + 5x_2 \leq 300$ and $x_1, x_2 \geq 0$	Maximize $Z = 50x_1 + 40x_2$ Subject to $3x_1 + 5x_2 + x_3 = 150$ $x_2 + x_4 = 20$ $8x_1 + 5x_2 + x_5 = 300$ and $x_1, x_2, x_3, x_4, x_5 \geq 0$

- Solve the following L.P.P using simplex method: $\text{Max } z = 4x_1 + 10x_2$ subject to

$$-2x_1 + x_2 \leq 50, 2x_1 + 5x_2 \leq 100, 2x_1 + 3x_2 \leq 90 \text{ and } x_1, x_2 \geq 0.$$

- Standard form:

$$\text{Max } z = 4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3$$

$$\text{subject to } -2x_1 + x_2 + s_1 = 50$$

$$2x_1 + 5x_2 + s_2 = 100$$

$$2x_1 + 3x_2 + s_3 = 90$$

$$\text{And } x_1, x_2, s_1, s_2, s_3 \geq 0.$$

C_j
 4 10 0 0 0

B	x_1	x_2	s_1	s_2	s_3	x_B
s_1	-2	1	1	0	0	50
s_2	2	5	0	1	0	100
s_3	2	3	0	0	1	90
Z_j	0	0	0	0	0	0
$Z_j - C_j$	-4	-10	0	0	0	

x_B/x_2
 $50/1 = 50$
 $100/5 = 20 \rightarrow$
 $90/3 = 30$



Entering variable = x_2 , leaving variable = s_2 , Pivotal element = 5. Column with pivotal element (Entering column) is called pivotal column. Modify entering column to $(0 \quad 1 \ 0)^T$ using elementary row operation.

$R_2' \rightarrow R_2/5$	$R_1' \rightarrow R_1 - R_2'$	$R_3' \rightarrow R_3 - 3R_2'$
$2/5$	$2 - 2/5 = 8/5$	$2 - 3(2/5) = 4/5$
1	$1 - 1 = 0$	$3 - 3(1) = 0$
0	$1 - 0 = 1$	$0 - 3(0) = 0$
$1/5$	$0 - 1/5 = -1/5$	$0 - 3(1/5) = -3/5$
0	$0 - 0 = 0$	$1 - 3(0) = 1$
$100/5 = 20$	$50 - 20 = 30$	$90 - 3(20) = 30$

Simplex table:2

	C_j	4	10	0	0	0	
C_B	B_v	x_1	x_2	s_1	s_2	s_3	x_B
0	s_1	8/5	0	1	-1/5	0	30
10	x_2	2/5	1	0	1/5	0	20
0	s_3	4/5	0	0	-3/5	1	30
	Z_j	4	10	0	2	0	200
	$Z_j - C_j$	0	0	0	2	0	

Since all $z_j - c_j \geq 0$, current basic feasible solution is optimal.

Therefore, optimal feasible solution is $Max\ z_j = 200, x_1 = 0, x_2 = 20$.

- Solve the following L.P.P using Big M method:

**Max $z = -2x_1 - x_2$ subject to $3x_1 + x_2 = 3$, $4x_1 + 3x_2 \geq 6$,
 $x_1 + 2x_2 \leq 4$, , $x_1, x_2 \geq 0$.**

- **Standard form:**

**Max $z = -2x_1 - x_2 + 0s_1 + 0s_2 - MA_1 - MA_2$ subject to $3x_1 + x_2 + A_1 = 3$
 $4x_1 + 3x_2 - s_1 + A_2 = 6$
 $x_1 + 2x_2 + s_2 = 4$ and $x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$.**

C_j

-2
 -1
 0
 0
 $-M$
 $-M$

B_v	x_1	x_2	s_1	s_2	A_1	A_2	x_B
A_1	3	1	0	0	1	0	3
A_2	4	3	-1	0	0	1	6
s_2	1	2	0	1	0	0	4
z_j	$-7M$	$-4M$	M	0	$-M$	$-M$	$-9M$
$z_j - C_j$	$2 - 7M$	$1 - 4M$	M	0	0	0	-

x_B/x_1

$3/3 \Rightarrow$
 $6/4$
 $4/1$



Entering variable = x_1 , leaving variable = A_1 , Pivotal element = 3.
 Ignore the leaving artificial variable column. Modify Pivotal column as $(1 \ 0 \ 0)^T$

$R_1 \rightarrow R_1/3$	$R_1 \rightarrow R_2 - 4R_1$	$R_3 \rightarrow R_3 - R_1$
1	$4 - 4(1) = 0$	$1 - 1 = 0$
$\frac{1}{3}$	$3 - 4\left(\frac{1}{3}\right) = \frac{5}{3}$	$2 - \frac{1}{3} = \frac{5}{3}$
0	$-1 - 4(0) = -1$	$0 - 0 = 0$
0	$0 - 4(0) = 0$	$1 - 0 = 1$
0	$1 - 4(0) = 1$	$0 - 0 = 0$
1	$6 - 4(1) = 2$	$4 - 1 = 3$

C_j		-1	3	-2	0	0	
C_B	B_v	x_1	x_2	s_1	s_2	A_2	x_B
-1	x_1	1	1/3	0	0	0	1
-M	A_2	0	5/3	-1	0	1	2
0	s_2	0	5/3	0	1	0	3
	z_j^*	-1	$-1 - 5M$	M	0	$-M$	$-1 - 6M$
	$z_j^* - c_j$	0	$\frac{3}{8 - 5M}$	$M + 2$	0	0	-
			3				

Entering variable = x_2 , leaving variable = A_2 , Pivotal element = $5/3$.
 Ignore the leaving artificial variable column. Modify entering column
 as $(0 \ 1 \ 0)^T$

$R_2'' \rightarrow \frac{3}{5} R_2'$	$R_1'' \rightarrow R_1' - \frac{1}{3} R_2''$	$R_3'' \rightarrow R_3' - \frac{5}{3} R_2''$
0	$1 - \frac{1}{3}(0) = 1$	$0 - \frac{5}{3}(0) = 0$
1	$\frac{1}{3} - \frac{1}{3}(1) = 0$	$\frac{5}{3} - \frac{5}{3}(1) = 0$
$-\frac{3}{5}$	$0 - \frac{1}{3}(-\frac{3}{5}) = \frac{1}{5}$	$0 - \frac{5}{3}(-\frac{3}{5}) = 1$
0	$0 - \frac{1}{3}(0) = 0$	$1 - \frac{5}{3}(0) = 1$
$\frac{6}{5}$	$1 - \frac{1}{3}(\frac{6}{5}) = \frac{3}{5}$	$3 - \frac{5}{3}(\frac{6}{5}) = 1$

		C_j				
		-1 3 -2 0				
C_B	B_v	x_1	x_2	s_1	s_2	x_B
	x_1	1	0	1/5	0	3/5
	x_2	0	1	-3/5	0	6/5
	s_2	0	0	1	1	1
	z_j^*	-1	3	-2	0	-12/5
	$z_j^* - c_j$	0	0	0	0	-

Since all $z^* - c_j \geq 0$, current basic feasible solution is optimal.
Therefore, optimal feasible solution is $x_1 = 3/5, x_2 = 6/5, x_3 = 0$ and $\text{Max } z = -12/5$



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Module-5: Transportation problems

Introduction:

The objective of the transportation problem is to transport commodity from different origins to different destinations in such a way that the total transportation cost is minimum.

To achieve this, we must know quantity of supplies and demands. In addition to that, we must know per unit transportation cost to transport the commodity from different origins to different destinations.

Notation:

- a_i – Supply from i^{th} source.
- b_j – Demand of j^{th} destination.
- c_{ij} – Per unit transportation cost to transport from i^{th} source to j^{th} destination.
- x_{ij} – Quantity which can be transported from i^{th} source to j^{th} destination.

Formulation of L.P.P of the transportation problem:

- **Feasible solution:** A set of non-negative values x_{ij} that satisfies the constraints.
- **Basic feasible solution:** A feasible solution with the total number of non-negative allocation is not more than $m + n - 1$. Where, m – number of rows, n – number of columns of the transportation table.
- **Optimal basic feasible solution:** A basic feasible solution which minimizes the total transportation cost.
- **Non-degenerate basic feasible solution:** A basic feasible solution with the total number of non-negative allocation is exactly $m + n - 1$.

- **Degenerate basic feasible solution:** A basic feasible solution with the total number of non-negative allocation is less than $m + n - 1$.
- **Balanced transportation problems:** Total supply from all the sources is equal to the total demand in all the destinations. ($\sum a_i = \sum b_j$).
- **Unbalanced transportation problems:** Total supply from all the sources is not equal to the total demand in all the destinations. ($\sum a_i \neq \sum b_j$).

Working rule to find the solution of the transportation problem:

(i) Make a balanced transportation problem:

- If $\sum a_i < \sum b_j$ then create a dummy source. (Dummy row)
- If $\sum a_i > \sum b_j$ then create a dummy destination. (Dummy column)

(ii) Find initial solution using any one of the following methods:

- North West Corner Method (NWCM)
- Least Cost Method (or) Matrix Minima Method
- Vogel's Approximation Meth

(iii) Test for optimality using MODI's method (Modified distribution method)

Find the feasible solution to the following transportation problem using North West corner method.

	<i>I</i>	<i>II</i>	<i>III</i>	<i>IV</i>	
<i>A</i>					20
<i>B</i>					10
<i>C</i>					15
	15	15	8	7	

$\sum a_i = \sum b_j = 45$. Therefore, this is the balanced transportation problem.

By North West Corner Method,

	D_1	D_2	D_3	D_4		
A	15	5	—	—	~~20~~	5
B	—	10	0	—	~~10~~	
C	—	—	8	7	<u>15</u>	<u>7</u>
	~~15~~	~~15~~	8	7		
		10	1			

Feasible solution is,

	D_1	D_2	D_3	D_4
A	15	5	—	—
B	—	10	0	—
C	—	—	8	7

Find the initial basic feasible solution to the following transportation problem using North West corner method.

Factory	I	II	III	Availability
A	4	3	2	10
B	5	6	1	8
C	6	4	3	5
D	3	5	4	6
Requirement	7	12	4	

$\sum a_i = 29, \sum b_j = 23. \sum a_i > \sum b_j.$
 This is the unbalanced problem. Create a dummy column.
 By North West Corner Method,

	I	II	III	IV		
A	4	3	2	0	10	3
	7	3	—	—		
B	5	6	1	0	8	
	—	8	—	—		
C	6	4	3	0	5	4
	—	1	4	—		
D	3	5	4	0	6	
	—	—	—	6		
	7	12	4	6		
		9				

Initial basic feasible solution is

	<i>I</i>	<i>II</i>	<i>III</i>	
<i>A</i>	7	3	—	10
<i>B</i>	—	8	—	8
<i>C</i>	—	1	4	5
<i>D</i>	—	—	—	6
	7	12	4	

Total transportation cost is

$$7(4) + 3(3) + 6(8) + 4(1) + 3(4) + 0(6) = 101$$

Solve the following transportation problem by least cost method.

5	4	3	6
4	7	6	8
2	5	8	12
8	6	7	4
8	10	12	

$\sum a_i = \sum b_j = 30$. This is the balanced transportation problem.

By Least Cost Method,

	S ₁	S ₂	S ₃		
w ₁	5	4	3	6	
	—	—	6		
w ₂	4	7	6	8	2
	—	2	6		
w ₃	2	5	8	<u>12</u>	<u>4</u>
	8	4	—		
w ₄	8	6	7	4	
	—	4	—		
	8	10	12		
		6	6		

Initial solution is

—	—	6	6
—	2	6	8
8	4	—	12
—	4	—	4
8	10	12	

Total shipping cost is

$$6(3) + 2(7) + 6(6) + 8(2) + 4(5) + 4(6) = 128$$

Solve the following transportation problem by least cost method.

	To			Supply
From	5	1	7	10
	6	4	6	80
	3	2	5	15
Demand	75	20	50	

$\sum a_i = 105, \sum b_j = 145$. Total supply is less than total demand.

This is the unbalanced transportation problem. Create a dummy row.

Balanced transportation problem is

	To			Supply
From	5	1	7	10
	6	4	6	80
	3	2	5	15
	0	0	0	40
Demand	75	20	50	

By Least Cost Method,

	S ₁	S ₂	S ₃	
w ₁	5 —	1 10	7 —	10
w ₂	6 30	4 —	6 50	80
w ₃	3 5	2 10	5 —	15 5
w ₄	0 40	0 —	0 —	40
	75	20	50	
	70	10		
	30			

Initial solution is

—	10	—	10
30	—	50	80
5	10	—	15
40	—	—	40
75	20	50	

Total shipping cost is $10(1) + 30(6) + 50(6) + 5(3) + 10(2) + 40(0) = 525$

Find optimal solution of the given balanced transportation problem using Vogel’s approximation method.

	<i>A</i>	<i>B</i>	<i>C</i>	
<i>F</i> ₁	10	9	8	8
<i>F</i> ₂	10	7	10	7
<i>F</i> ₃	11	9	7	9
<i>F</i> ₅	12	14	10	4
	10	10	8	

Step 1: Find initial solution

$\sum a_i = \sum b_j = 28$. This is the balanced Transportation problem.
By Vogel's approximation method,

	A	B	C					
F_1	10	9	8	8	6	1	1	1
	6	2	—					
F_2	10	7	10	7		3	—	—
	—	7	—					
F_3	11	9	7	9		2	2	2
	—	1	8	1				
F_4	12	14	10	4		2	2	2
	4	—	—					
	10	10	8					
		3						
	0	2	1					
	1	0	1					
	1	0	—					
	2	5	—					

Initial solution is

	<i>A</i>	<i>B</i>	<i>C</i>	
F_1	6	2	—	8
F_2	—	7	—	7
F_3	—	1	8	9
F_5	4	—	—	4
	10	10	8	

Step 2: Test for optimality using MODI's Method

Number of occupied cells = 6 and $m + n - 1 = 6$

Since number of occupied cells = $m + n - 1$

Therefore, initial solution is non-degenerate.

Choose u_1 as 0. Find other u_i 's, v_j 's using $c_{ij} = u_i + v_j$ for occupied cells.
 Find the sign of $\Delta_{ij} = c_{ij} - (u_i + v_j)$ using c_{ij} for unoccupied cells.

				u_i
	10	9	8	9
★		★	(+)	
	10	7	10	7
(+)	★	(+)		
	10	9	7	9
(+)	★	★		
	12	14	10	11
★	(+)	(+)		
v_j	1	0	-2	

Since all Δ_{ij} are positive, initial solution is itself optimal solution.

	A	B	C	
F_1	6	2	—	8
F_2	—	7	—	7
F_3	—	1	8	9
F_5	4	—	—	4
	10	10	8	

Total minimum transportation cost is

$$6(10) + 2(9) + 7(7) + 1(9) + 8(7) + 4(12) = 240$$

Four Gasoline dealers A, B, C and D require 50, 40, 60 and 40 kl of gasoline respectively. It is possible to supply these from locations 1, 2, 3 which have 80, 100, 50 kl respectively. The cost in rupees for shipping every kl as shown in the table below. Determine the most economical supply pattern.

	<u>A</u>	<u>B</u>	<u>C</u>	<u>D</u>
1	7	6	6	6
2	5	7	6	7
3	8	5	8	6

Step 1: Find balanced transportation problem.

By data,

	<u>A</u>	<u>B</u>	<u>C</u>	<u>D</u>	a_i
1	7	6	6	6	80
2	5	7	6	7	100
3	8	5	8	6	50
b_j	50	40	60	40	

$\Sigma a_i = 230, \Sigma b_j = 190$. Therefore, $\Sigma a_i > \Sigma b_j$.

It is an unbalanced transportation problem. Create a dummy column.

Balanced transportation problem is

	<u>A</u>	<u>B</u>	<u>C</u>	<u>D</u>	<u>E</u>	a_i
1	7	6	6	6	0	80
2	5	7	6	7	0	100
3	8	5	8	6	0	50
b_j	50	40	60	40	40	

Step 2: Find initial solution

By Vogel’s approximation method,



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	A	B	C	D	E				Penalty										
1	<table><tr><td>7</td></tr><tr><td>—</td></tr></table>	7	—	<table><tr><td>6</td></tr><tr><td>—</td></tr></table>	6	—	<table><tr><td>6</td></tr><tr><td>10</td></tr></table>	6	10	<table><tr><td>6</td></tr><tr><td>30</td></tr></table>	6	30	<table><tr><td>0</td></tr><tr><td>40</td></tr></table>	0	40	80 40			6 0 0 0
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8																			
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40																			
8																			
—																			
6																			
10																			
0																			
—																			
	50	40	60	40	40														
			20	30															
	2	1	0	0	0														
	2	1	0	0	—														
	—	1	0	0	—														
	—	1	2	0	—														

Initial solution is

	A	B	C	D	E	a_i
1	—	—	10	30	40	80
2	50	—	50	—	—	100
4	—	40	—	10	—	50
b_j	50	40	60	40	40	

Step 3: Test for optimality using MODI's Method

Number of occupied cells= 7,

$$m + n - 1 = 5 + 3 - 1 = 7$$

Therefore, initial solution is non-degenerate.

2nd row has more number of occupied cells. Choose u_2 as 0.

Find other u_i 's, v_j 's using $c_{ij} = u_i + v_j$ for occupied cells denoted by *.

Find the sign of $\Delta_{ij} = c_{ij} - (u_i + v_j)$ using c_{ij} for unoccupied cells.

	A	B	C	D	E	u_i
1	7 (+)	6 (+)	6 *	6 *	0 *	0
2	5 *	7 (+)	6 *	7 (+)	0 (+)	0
3	8 (+)	5 *	8 (+)	6 *	0 (+)	-1
v_j	5	6	6	7	0	

All Δ_{ij} are positive. Optimum solution is reached.

Optimum solution is

	<u>A</u>	<u>B</u>	<u>C</u>	<u>D</u>	<u>E</u>	a_i
1	—	—	10	30	40	80
2	50	—	50	—	—	100
4	—	40	—	10	—	50
b_j	50	40	60	40	40	

This is the most economical supply pattern.

Minimum cost = $10(6) + 30(6) + 50(5) + 50(6) + 40(5) + 10(6) = ₹1050$



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Thank You