



A T M E
College of Engineering



Affiliated to VTU, Belagavi, Approved by AICTE, New Delhi and Recognized by Government of Karnataka.

Course: Complex Analysis, Transform Techniques and Optimization

Course Code 1BMATEE301

Semester – III

Department of Mathematics
(Academic Year 2026-27)

Complex Analysis, Transform Techniques and Optimization			
Course Code	1BMATEE301	Scheme	2025
Type of Course	Applied Science Course (ASC)	Semester	III
Teaching Hours/Week (L:T:P)	70 (3:2:0)	CIE Marks	50
Total Hours of Pedagogy per semester L /T/P/SL&TW:	120 (42:28:0:50)	SEE Marks	50
Credits	4	Total Marks	100
Examination type (SEE)	Theory	Exam Hours	3
Course outcome (Course Skill Set): At the end of the course, the student will be able to: C01: Apply the concepts of analytic functions, Cauchy-Riemann equations, complex integration, Cauchy's theorem, and Cauchy's integral formula to solve engineering and mathematical problems involving complex variables. C02: Develop Fourier series representations of periodic functions and perform harmonic analysis for functions with different periodicities, discontinuities, and symmetry properties. C03: Utilize Fourier transforms and Z-transforms, including their inverse transforms and properties, to analyze signals and solve difference equations arising in engineering applications. C04: Apply probability distributions, sampling theory, and statistical hypothesis testing techniques for analyzing uncertainty and making data-driven decisions. C05: Formulate and solve linear programming problems using graphical, simplex, and Big-M methods to obtain optimal solutions in engineering and management applications.			
Module-1: Complex Analysis			
Review of a function of a complex variable, limits, continuity and differentiability. Analytic functions-Cauchy-Riemann equations in Cartesian and polar forms. Construction of analytic functions by Milne-Thomson method, Problems on electric flux. Complex integration: Line integral of a complex function, Cauchy's theorem and Cauchy's integral formula (statement only) and problems.			
Number of Hours:8+5T			
Module-2: Fourier Series			
Periodic functions, Euler's Formula, Conditions for Fourier series expansion(Dirichlet's condition), Fourier series of functions -having points of discontinuity, with period 2π , with change of interval (arbitrary period) and, Even and odd functions. Half range Fourier series. Practical harmonic analysis.			
Number of Hours:8+5T			
Module-3: Fourier transforms and Z-transforms			
Infinite Fourier transforms: Definition, Fourier transforms, Fourier sine, and cosine transforms. Inverse Fourier Transforms, inverse Fourier sine and cosine transforms. Problems. Z-transforms: Definition, Standard Z-transforms, Damping, and shifting rules, Statements of initial value theorem and Final Value Theorem (statements only). Inverse Z-transform and applications to solve difference equations.			
Number of Hours:9+6T			

Module-4: Probability distributions and Sampling theory
Probability Distributions: Review of basic probability theory, Random variables-discrete and continuous. Probability distribution function, cumulative distribution function, Mathematical Expectation-mean and variance. Repeated Trials- Binomial, Poisson, Normal and exponential distribution (without proofs for mean and SD). Sampling Theory: Introduction to sampling distributions, standard error, Type-I and Type-II errors. Test of hypothesis for single means, single proportions. Student's t-distribution, Chi-square distribution as a test of goodness of fit. Number of Hours:9+6T
Module-5: Linear Programming Problems
Linear Programming Problems: Introduction to optimization techniques, formulation of the problems. Introduction to Linear programming, graphical solution, solution by simplex and Big M method. Number of Hours:8+6T
Suggested Learning Resources: (Text Book/ Reference Book/ Manuals): Textbooks: (i) Higher Engineering Mathematics, B.S. Grewal, 45th Edition, Khanna Publishers. (ii) Operations Research: An Introduction, Hamdy A. Taha, Pearson Education. (iii) Higher Engineering Mathematics, H.K. Dass and Er. Rajnish Verma, S. Chand Publications. Reference books: (i) Complex Variables and Applications by James Ward Brown and Ruel V. Churchill, McGraw-Hill Education. (ii) Advanced Engineering Mathematics, Erwin Kreyszig, 10th Edition, Wiley India. (iii) Fourier Series and Boundary Value Problems by Ruel V. Churchill and James Ward Brown. (iv) Probability and Statistics for Engineers and Scientists by Ronald E. Walpole et al., Pearson Education. (v) Operations Research by Hamdy A. Taha, Pearson Education. (vi) Schaum's Outline of Complex Variables, McGraw-Hill.
Web links and Video Lectures (e-Resources): • NPTEL Courses on Engineering Mathematics • MIT OpenCourseWare Mathematics • SWAYAM Online Courses
Teaching-Learning Process (Innovative Delivery Methods): The following are sample strategies that educators may adopt to enhance the effectiveness of the teaching-learning process and facilitate the achievement of course outcomes. 1. Flipped Classroom Approach ○ Students review video lectures and reading materials before class. ○ Classroom sessions focus on problem-solving, discussions, and applications of transform techniques and optimization models. 2. Problem-Based Learning (PBL) ○ Real-world engineering problems involving Fourier Transforms, Z-Transforms, and Linear Programming are assigned for collaborative solution development.

3. **ICT-Enabled Teaching** (may be conducted in Tutorial hours)
 - Use of MATLAB, Python, and online simulation tools to visualize transforms, optimization problems, and signal-processing applications.
4. **Collaborative Learning** (may be conducted in Tutorial hours)
Group assignments and mini-projects on optimization models and transform applications.

Assessment Structure:

The assessment in each course is divided equally between Continuous Internal Evaluation (CIE) and the Semester End Examination (SEE), with each carrying 50% weightage.

- To qualify and become eligible to appear for SEE, in the **CIE**, a student must score at least **40% of 50 marks**, i.e., **20 marks**.
- To pass the **SEE**, a student must score at least **35% of 50 marks**, i.e., **18 marks**.
- Notwithstanding the above, a student is considered to have **passed the course**, provided the combined total of **CIE and SEE** is at least **40 out of 100 marks**.

Continuous Comprehensive Assessments (CCA)+ Internal Assessment Test (IA) =

Continuous Internal Evaluation CIE [25+25=50 marks]

A minimum of **two Internal Assessment Tests (IA)** shall be conducted, carrying a total of **25 marks**.

In addition, **Continuous Comprehensive Assessment (CCA)** shall be conducted for a total of **25 marks**.

It is recommended to include **a maximum of two learning activities** as part of the CCA to foster the **holistic development of students**. These activities shall be:

Sl. No.	Name of the Learning activity	Maximum Marks
1.	Assignment at RBL3 and RBL4 levels based on Team Work(TW) activities.	10
2.	GATE-based Aptitude Test	15

Rubrics for Learning Activity (Based on the nature of learning activity, design the rubrics for each activity):

Rubrics for Learning Activity–1: Assignment (10 Marks)

Performance Indicator	Superior (10)	Good (8)	Fair (6)	Needs Improvement (5)
Understanding of Concepts	Demonstrates excellent conceptual clarity and application	Good understanding with minor errors	Basic understanding	Poor understanding
Problem Solving Ability	Solves all problems correctly with proper methodology	Solves most problems correctly	Partial solutions with errors	Unable to solve problems effectively
Mathematical Accuracy	Calculations and derivations are error-free	Minor computational errors	Several errors	Significant errors
Presentation & Organization	Well-structured, neat, and professionally presented	Organized with minor deficiencies	Average presentation	Poor presentation

Rubrics for Learning Activity–2: GATE-Based Aptitude Test (15 Marks)

Performance Indicator	Superior (15)	Good (12)	Fair (10)	Needs Improvement (8)
Quantitative Aptitude	$\geq 90\%$ correct responses	75–89% correct	60–74% correct	$< 60\%$ correct
Analytical Thinking	Excellent logical reasoning and interpretation	Good reasoning ability	Moderate reasoning	Weak reasoning
Time Management	Completes all questions within allotted time	Completes most questions	Requires additional time	Unable to complete
Accuracy & Precision	Very high accuracy	Good accuracy	Moderate accuracy	Low accuracy

Term Work (TW) and Self Learning (SL) components in Number of Hours per semester		
Term work (includes assignments, seminars, micro projects, industrial visits, any other student activities etc.)		
Sl. No.	Term Work (TW) Activity	Number of Hours / Semester
1.	Group Assignment on Applications of Complex Variables in Electrical Engineering (AC circuits, electric flux, potential theory)	5
2.	MATLAB/Python Exploration of Fourier and Z-Transforms	5
3.	Mini Project on Linear Programming Models in Engineering Applications	5
4.	Case Study on Signal Processing using Transform Techniques	5
SL: Self Learning, MOOCs, spoken tutorials, online educational resources etc.		
Sl. No.	Self-Learning (SL) Activity	Number of Hours / Semester
1.	Conformal transformations: Discussion of transformations: $w = z^2$, $w = e^z$, $w = z + \frac{1}{z}$, $z \neq 0$, Bilinear transformations	6
2.	Complex form of Fourier Series	6
3.	Convolution theorem for Fourier transforms and Z transforms. Applications of transforms to boundary value problems	6
4.	Joint probability distributions.	6
5.	Dual simplex method.	6

Continuous Internal Evaluation (CIE)

1. The CIE marks shall be decided based on internal tests and other outcome-based activities. The continuous internal evaluation shall be carried out by the teacher handling the course.
2. Out of 50 marks earmarked for CIE, 25 marks shall be assigned for internal tests. The first test shall be conducted after completing two modules of the syllabus and the second one after completing the remaining three modules.
3. The remaining 25 marks shall be assigned for Continuous Course Assessment (CCA), conducted as per the rubrics listed in the assessment section of the respective syllabus.
4. A student shall obtain a minimum of 40% marks or 20 out of 50 marks allotted to CIE to become eligible to appear for the SEE.

Passing requirement in SEE

1. For a pass in the Semester End Examination (SEE), a student shall secure a minimum of 35 marks out of 100.
2. For the declaration of a pass grade, the sum of the Continuous Internal Evaluation (CIE) marks (out of 50) marks and the SEE marks scaled down to 50 shall be greater than or equal to 40 marks.
3. If the total (CIE + SEE) is less than 40 marks, the student shall be awarded the grade 'F' (Fail).

Module-1

Complex Analysis

Recapitulation of Basic concepts

- $Z=x+iy$ [Cartesian form of 'Z']
- $Z=r[\cos\theta + i\sin\theta] = re^{i\theta}$ [Polar form of 'Z']
- $|z| = \sqrt{x^2 + y^2} = r$
Amplitude of $z = \tan^{-1}(y/x) = \theta$
- $\bar{z} = x - iy$ is called the complex conjugate of z .
- $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$
- $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$
 $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$
- $\cos(i\theta) = \cosh\theta$ and $\sin(i\theta) = i\sinh\theta$
- **De-Moivre's theorem:**
 $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, where n is a real number.

Review of function of a complex variable

Functions of a complex variable:

Let C be a set complex numbers. If to each complex number z in C there corresponds a unique complex number w , then w is called a complex function of z defined on C , and we write $w = f(z) = u + iv$.

Hence, w has a real part, say u and an imaginary part, say v .

Then, w has the representation

$$w = f(z) = u(x,y) + i v(x,y) \quad (\text{Cartesian form})$$

$$w = u(r,\theta) + i v(r,\theta) \quad (\text{polar form})$$

Continuity:

A complex valued function $f(z)$ is said to be continuous at a point z_0 if $f(z)$ is defined at z_0 and $\lim_{z \rightarrow z_0} f(z) = f(z_0)$.

Derivative of a complex function:

The derivative of a complex function f at a point $z = z_0$ is defined as

$$f'(z) = \lim_{\delta z \rightarrow 0} \frac{f(z+\delta z) - f(z)}{\delta z} \text{ provided the limit exists.}$$

We should remember that by the definition of limit $f(z)$ is defined in a neighborhood of z_0 and z may approach z_0 from any direction in the complex plane. The derivative of a function at a point is unique if it exists.

Analytic Functions

In complex analysis we are interested in the functions, which are differentiable in some domain, called the analytic functions. A large variety of functions of complex variables which are useful for applications purpose are analytic.

A function $f(z)$ is said to be analytic at a point z_0 , if it is differentiable at z_0 and, in addition, it is differentiable throughout some neighborhood of z_0 .

That is, a complex valued function $w = f(z)$ is said to be analytic at a point $z = z_0$ if

$$\frac{dw}{dz} = f'(z) = \lim_{\delta z \rightarrow 0} \frac{f(z+\delta z) - f(z)}{\delta z} \text{ exists and is unique at } z_0 \text{ and in the neighbourhood of } z_0$$

Analytic function is also called a regular function or holomorphic function.

Cauchy-Riemann Equations in the Cartesian form

The necessary conditions that the function $w = f(z) = u(x,y) + i v(x,y)$ be analytic at any point is that, there exists four continuous first order partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ and

satisfy the equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

These are known as Cauchy-Riemann (C-R) equations.

Equivalently, $u_x = v_y$ and $v_x = -u_y$

Proof:

Let $f(z)$ be analytic at a point $z = x + iy$ and hence by the definition,

$$f'(z) = \lim_{\delta z \rightarrow 0} \frac{f(z+\delta z) - f(z)}{\delta z} \text{ exists and is unique.}$$

In the Cartesian form, $f(z) = u(x,y) + i v(x,y)$ and let δz be the increment in z corresponding to the increments δx , δy in x , y .

$$f'(z) = \lim_{\delta z \rightarrow 0} \frac{u(x + \delta x, y + \delta y) + i v(x + \delta x, y + \delta y) - (u(x, y) + i v(x, y))}{\delta z}$$

$$f'(z) = \lim_{\delta z \rightarrow 0} \frac{u(x + \delta x, y + \delta y) - u(x, y)}{\delta z} + \lim_{\delta z \rightarrow 0} i \frac{v(x + \delta x, y + \delta y) - v(x, y)}{\delta z} \quad \text{--- (1)}$$

Now, $\delta z = (z + \delta z) - z$ where $z = x + iy$

$$\delta z = [(x + \delta x) + i(y + \delta y)] - [x + iy]$$

$$\text{i.e., } \delta z = \delta x + i\delta y$$

Since δz tends to zero, we have the following two possibilities.

Case-1: Let $\delta y = 0$ then $\delta z = \delta x + i\delta y$ becomes $\delta z = \delta x$ and $\delta z \rightarrow 0$ then $\delta x \rightarrow 0$

$$\text{Now } (1) \Rightarrow f'(z) = \lim_{\delta x \rightarrow 0} \frac{u(x+\delta x, y) - u(x, y)}{\delta x} + \lim_{\delta x \rightarrow 0} i \frac{v(x+\delta x, y) - v(x, y)}{\delta x}$$

By the definitions of the partial derivatives of u and v w.r.t x . It can be written as

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \text{---} \quad (2)$$

Case-2: Let $\delta x = 0$ then $\delta z = \delta x + i\delta y$ becomes $\delta z = i\delta y$ as $\delta z \rightarrow 0$ then $\delta y \rightarrow 0$

$$(1) \Rightarrow f'(z) = \lim_{\delta y \rightarrow 0} \frac{u(x, y+\delta y) - u(x, y)}{i\delta y} + \lim_{\delta y \rightarrow 0} i \frac{v(x, y+\delta y) - v(x, y)}{i\delta y}$$

$$\text{It can be written as } f'(z) = \frac{\partial u}{i \partial y} + \frac{\partial v}{\partial y}$$

$$\text{But } \frac{1}{i} = \frac{i}{i^2} = -i$$

$$f'(z) = \frac{\partial u}{i \partial y} + \frac{\partial v}{\partial y} = \left(\frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \right) \quad \text{---} \quad (3)$$

Equating RHS of equation (2) and (3) we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

These are known as the Cauchy-Riemann equations.

Cauchy –Riemann equations in the polar form

If $f(z)=f(re^{i\theta})=u(r, \theta) + iv(r, \theta)$ is analytic at a point z , then there exists four continuous partial derivatives $\frac{\partial u}{\partial r}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}, \frac{\partial u}{\partial \theta}$ and satisfies the equation

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

These are known as Cauchy –Riemann (C-R) equations in the polar form

Proof:

Let $f(z)$ be analytic at a point $z=re^{i\theta}$ and hence by definition

$$f'(z) = \lim_{\delta z \rightarrow 0} \frac{f(z+\delta z) - f(z)}{\delta z} \quad \text{---} \quad (1) \quad \text{exists and unique}$$

In the polar form $f(z) = u(r, \theta) + iv(r, \theta)$ and let δz be the increment in z corresponding to the increments are δr and $\delta \theta$ in r and θ respectively.

$$(1) \Rightarrow f'(z) = \lim_{\delta z \rightarrow 0} \frac{u(r+\delta r, \theta+\delta \theta) + iv(r+\delta r, \theta+\delta \theta) - (u(r, \theta) + iv(r, \theta))}{\delta z}$$

$$f'(z) = \lim_{\delta z \rightarrow 0} \frac{u(r + \delta r, \theta + \delta \theta) - u(r, \theta)}{\delta z} + \lim_{\delta z \rightarrow 0} i \frac{v(r + \delta r, \theta + \delta \theta) - v(r, \theta)}{\delta z} \quad \text{---} \quad (2)$$

Consider $z = re^{i\theta}$, since z is a function of two variables r and θ we have,

$$\begin{aligned}\delta z &= \frac{\partial z}{\partial r} \delta r + \frac{\partial z}{\partial \theta} \delta \theta \\ &= \frac{\partial}{\partial r}(re^{i\theta}) \delta r + \frac{\partial}{\partial \theta}(re^{i\theta}) \delta \theta\end{aligned}$$

$$\delta z = e^{i\theta} \delta r + ire^{i\theta} \delta \theta$$

Since δz tends to zero, we have the following two possibilities.

Case-1: Let $\delta \theta = 0$ then $\delta z = e^{i\theta} \delta r + ire^{i\theta} \delta \theta$ becomes $\delta z = e^{i\theta} \delta r$ and as $\delta z \rightarrow 0$ then $\delta r \rightarrow 0$

$$\begin{aligned}(2) \Rightarrow f'(z) &= \lim_{\delta r \rightarrow 0} \frac{u(r+\delta r, \theta) - u(r, \theta)}{e^{i\theta} \delta r} + \lim_{\delta r \rightarrow 0} i \frac{v(r+\delta r, \theta) - v(r, \theta)}{e^{i\theta} \delta r} \\ f'(z) &= \frac{\partial u}{e^{i\theta} \partial r} + i \frac{\partial v}{e^{i\theta} \partial r} = \frac{1}{e^{i\theta}} \left(\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right) \quad \text{--- (3)}\end{aligned}$$

Case-2: Let $\delta r = 0$ then $\delta z = e^{i\theta} \delta r + ire^{i\theta} \delta \theta$ becomes $\delta z = ire^{i\theta} \delta \theta$ as $\delta z \rightarrow 0$ then $\delta \theta \rightarrow 0$

$$f'(z) = \lim_{\delta \theta \rightarrow 0} \frac{u(r, \theta + \delta \theta) - u(r, \theta)}{ire^{i\theta} \delta \theta} + \lim_{\delta \theta \rightarrow 0} i \frac{v(r, \theta + \delta \theta) - v(r, \theta)}{ire^{i\theta} \delta \theta}$$

It can be written as

$$f'(z) = \frac{\partial u}{ire^{i\theta} \partial \theta} - \frac{\partial v}{re^{i\theta} \partial \theta} = \frac{1}{re^{i\theta}} \left(-i \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta} \right) \quad \text{--- (4)}$$

Equating RHS of equation (3) and (4) we get

$$e^{-i\theta} \left(\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right) = e^{-i\theta} \left(-i \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{1}{r} \frac{\partial v}{\partial \theta} \right)$$

Cancelling $e^{-i\theta}$ on both sides and equating real and imaginary parts, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

OR

$$ru_r = v_\theta \text{ and } rv_r = -u_\theta$$

Thus we have established Cauchy-Riemann equations in polar form.

Consequences of C-R Equations:

Harmonic function – Definition

A function ϕ is said to be **harmonic** if it satisfies Laplace equation $\nabla^2 \phi = 0$.

In the Cartesian form $\phi(x,y)$ is harmonic if $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$.

In the Polar form $\phi(r,\theta)$ is harmonic if $\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0$.

Harmonic Property: The real and imaginary parts of an analytic function are harmonic

- If 'u' is harmonic then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
- If 'v' is harmonic then $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$
- If 'u' is harmonic then $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$

If 'v' is harmonic then $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} = 0$

Note:

1. The real and imaginary parts of an analytic function when expressed in Cartesian and polar form satisfy Laplace's equation in the Cartesian and polar form.
2. The converse of this property is not true. That is we can say that if a function is harmonic it may not be analytic.

Ex. $u = x^2 - y^2$; $v = x^3 - 3xy^2$. In this example u and v are harmonic functions but its not analytic [Since C-R equations are not satisfied].

Orthogonal Property

If $f(z) = u + iv$ is an analytic function, then the family of curves $u(x, y) = c_1$ and $v(x, y) = c_2$, c_1 and c_2 being constants, intersect each other orthogonally.

Proof:

We know that two curves intersect each other orthogonally if the tangents at the point of intersection are at right angles. Further we know that $\frac{dy}{dx}$ represents slope of the tangent and the condition for the perpendicularity of two lines is that the product of their slopes must be equal to -1.

Let $u(x,y) = c_1$ -- (i)

$v(x, y) = c_2$ -- (ii)

Differentiating eqn (i) partially w.r.t x treating y as a function of x, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx} = 0 \text{ so that } \frac{dy}{dx} = - \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = m_1$$

Similarly, the slope of the member of the family $v(x, y) = c_2$

Passing through the point (x,y) is $\frac{dy}{dx} = - \frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = m_2$

$$m_1 m_2 = \frac{\frac{\partial u}{\partial x} \frac{\partial v}{\partial y}}{\frac{\partial u}{\partial y} \frac{\partial v}{\partial x}}$$

But $f(z)$ is analytic, then CR equation hold we get $m_1 m_2 = -1$.

Hence the curves intersect orthogonally at every point of intersection.

Note: Converse of this theorem is not true

Problems on finding the derivative of an analytic function

1. Given $w = f(z)$, we substitute $z = x + iy$ or $z = re^{i\theta}$ to find the real and imaginary parts u and v as functions of x, y or r, θ .
2. We find first order partial derivatives and verify Cauchy-Riemann equations in the Cartesian or polar form to conclude that $f(z)$ is analytic.
3. To find the derivative of $f(z)$ we make use of the fundamental results derived while establishing Cauchy-Riemann equations. They are as follows

$$f'(z) = u_x + iv_x \quad [\text{Cartesian form}]$$

$$f'(z) = e^{-i\theta}[u_r + iv_r] \quad [\text{Polar form}]$$
4. We substitute for the partial derivatives and rearrange as a function of $(x + iy)$ or $re^{i\theta}$ which is z , with the result $f'(z)$ is obtained as a function of z .

1. Show that $w = z + e^z$ is analytic and hence find $\frac{dw}{dz}$.

Sol: By data $w = z + e^z$, Put $z = x + iy$

$$\begin{aligned} u + iv &= (x + iy) + e^{(x+iy)} \\ &= (x + iy) + e^x \cdot e^{iy} \\ &= (x + iy) + e^x(\cos y + i \sin y) \end{aligned}$$

$$u + iv = (x + e^x \cos y) + i(y + e^x \sin y)$$

$$u = x + e^x \cos y \quad v = y + e^x \sin y$$

$$u_x = 1 + e^x \cos y \quad v_x = e^x \sin y$$

$$u_y = -e^x \sin y \quad v_y = 1 + e^x \cos y$$

We observe C-R equations $u_x = v_y$ and $v_x = -u_y$ are satisfied.

Thus, $w = z + e^z$ is analytic.

$$\text{Now, } \frac{dw}{dz} = f'(z) = u_x + iv_x$$

$$\begin{aligned} \frac{dw}{dz} &= (1 + e^x \cos y) + i(e^x \sin y) \\ &= 1 + e^x (\cos y + i \sin y) = 1 + e^x \cdot e^{iy} = 1 + e^{x+iy} \end{aligned}$$

$$\frac{dw}{dz} = 1 + e^z$$

2. Show that $f(z) = \sin z$ is analytic and hence find $f'(z)$.

Sol: Given: $f(z) = \sin z$, Put $z = x + iy$

$$f(z) = \sin(x + iy)$$

$$u + iv = \sin x \cos(iy) + \cos x \sin(iy)$$

$$[\text{Since } \sin(i\theta) = i \sinh \theta \text{ and } \cos(i\theta) = \cosh \theta]$$

$$u + iv = \sin x \cosh y + i \cos x \sinh y$$

$$u = \sin x \cosh y \quad v = \cos x \sinh y$$

$$u_x = \cos x \cosh y \quad v_x = -\sin x \sinh y$$

$$u_y = \sin x \sinh y \quad v_y = \cos x \cosh y$$

We observe C-R equations $u_x = v_y$ and $v_x = -u_y$ are satisfied

Thus, $f(z) = \sin z$ is analytic.

$$\text{Now, } f'(z) = u_x + iv_x$$

$$f'(z) = (\cos x \cosh y) + i(-\sin x \sinh y)$$

$$f'(z) = \cos x (\cosh y) - \sin x (i \sinh y)$$

$$f'(z) = \cos x \cos(iy) - \sin x \sin(iy)$$

$$f'(z) = \cos(x + iy)$$

$$f'(z) = \cos z$$

3. Show that $w = \log z$, $z \neq 0$ is analytic and hence find $\frac{dw}{dz}$.

Sol:

By data $w = \log z$, $z \neq 0$, Put $z = re^{i\theta}$

$$u + iv = \log(re^{i\theta})$$

$$u + iv = \log r + \log(e^{i\theta})$$

$$u + iv = \log r + i\theta \log_e e$$

$$u + iv = \log r + i\theta \quad [\log_e e = 1]$$

$$u = \log r \quad v = \theta$$

$$u_r = \frac{1}{r} \quad v_r = 0$$

$$ru_r = 1 \quad rv_r = 0$$

$$u_\theta = 0 \quad v_\theta = 1$$

C-R equations $ru_r = v_\theta$ and $rv_r = -u_\theta$ are satisfied.

Thus, $w = \log z$ is analytic.

Now, $f'(z) = e^{-i\theta}[u_r + iv_r]$

$$= e^{-i\theta}\left[\frac{1}{r} + i \cdot 0\right]$$

$$= \frac{1}{re^{i\theta}}$$

$$f'(z) = \frac{1}{z}$$

4. Show that $f(z) = z^n$, where n is a constant is analytic and hence find its derivative.

Sol:

By data, $f(z) = z^n$, Put $z = re^{i\theta}$

$$u + iv = (re^{i\theta})^n$$

$$u + iv = r^n \cdot e^{in\theta}$$

$$u + iv = r^n(\cos n\theta + i \sin n\theta)$$

$$u + iv = r^n \cos n\theta + i r^n \sin n\theta$$

$$u = r^n \cos n\theta \quad v = r^n \sin n\theta$$

$$u_r = nr^{n-1} \cos n\theta \quad v_r = nr^{n-1} \sin n\theta$$

$$ru_r = nr^n \cos n\theta \quad rv_r = nr^n \sin n\theta$$

$$u_\theta = -r^n \sin n\theta \cdot n \quad v_\theta = r^n \cos n\theta \cdot n$$

C-R equations $ru_r = v_\theta$ and $rv_r = -u_\theta$ are satisfied.

Thus, $f(z) = z^n$ is analytic.

Now, $f'(z) = e^{-i\theta}[u_r + iv_r]$

$$f'(z) = e^{-i\theta}[nr^{n-1} \cos n\theta + i \cdot nr^{n-1} \sin n\theta]$$

$$f'(z) = ne^{-i\theta}r^{n-1}[\cos n\theta + i \sin n\theta]$$

$$f'(z) = ne^{-i\theta}r^{n-1}e^{in\theta}$$

$$f'(z) = nr^{n-1}e^{in\theta - i\theta}$$

$$f'(z) = nr^{n-1}e^{i\theta(n-1)}$$

$$f'(z) = n(re^{i\theta})^{n-1}$$

$$f'(z) = nz^{n-1}$$

Construction of analytic function by Milne-Thomson Method

- Given u or v as functions of x and y we find u_x , u_y or v_x , v_y and consider $f'(z) = u_x + iv_x$.
- Given u , we use C-R equations $v_x = -u_y$ or given u , we use $u_x = v_y$ so that $f'(z) = u_x - iu_y$ or $f'(z) = v_y + iv_x$.
- We substitute the expression for the partial derivatives in the RHS and then put $x = z$, $y = 0$ to obtain $f'(z)$ as a function of z .
- Integrating w.r.t z we get $f(z)$.
- Also in the case of polar co-ordinates r, θ we consider $f'(z) = e^{-i\theta}[u_r + iv_r]$ and use C-R equations in the RHS, $v_r = -\frac{1}{r}u_\theta$ given u or $u_r = \frac{1}{r}v_\theta$ given v . We use the substitution $r = z$, $\theta = 0$ to obtain $f'(z)$ as a function of z . Integrating w.r.t z we get $f(z)$.

1. Construct the analytic function whose real part is $u = \log\sqrt{x^2 + y^2}$.

Sol:

$$\text{Given } u = \log\sqrt{x^2 + y^2}$$

$$u = \log(x^2 + y^2)^{1/2}$$

$$u = \frac{1}{2} \log(x^2 + y^2)$$

$$u_x = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot 2x \quad u_y = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot 2y$$

$$u_x = \frac{x}{x^2 + y^2} \quad u_y = \frac{y}{x^2 + y^2}$$

$$\text{Consider, } f'(z) = u_x + iv_x$$

$$\text{By C-R equations } v_x = -u_y$$

$$f'(z) = u_x - iu_y$$

$$f'(z) = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}$$

Putting $x = z$ and $y = 0$ we have,

$$f'(z) = \frac{z}{z^2 + 0} - i \cdot 0$$

$$f'(z) = \frac{1}{z}$$

$$f(z) = \int \frac{1}{z} dz + c$$

$$\text{Thus, } f(z) = \log z + c$$

2. Find the analytic function $f(z)$ whose imaginary part is $e^x(x\sin y + y\cos y)$.

Sol:

$$\text{Given } v = e^x(x\sin y + y\cos y)$$

$$v_x = e^x(\sin y \cdot 1 + 0) + (x\sin y + y\cos y)e^x$$

$$v_x = e^x(\sin y + x\sin y + y\cos y)$$

$$\text{Also, } v_y = e^x(x\cos y - y\sin y + \cos y \cdot 1)$$

$$\text{Consider, } f'(z) = u_x + iv_x$$

$$\text{By C-R equations } u_x = v_y$$

$$f'(z) = v_y + iv_x$$

$$f'(z) = e^x(x\cos y - y\sin y + \cos y) + ie^x(\sin y + x\sin y + y\cos y)$$

Putting $x = z$ and $y = 0$ we have,

$$f'(z) = e^z(z - 0 + 1) + ie^z(0 + 0 + 0)$$

$$f'(z) = e^z(z + 1)$$

$$f(z) = \int e^z(z + 1) dz + c$$

$$f(z) = e^z(z + 1) - \int e^z \cdot 1 dz + c$$

$$f(z) = e^z(z + 1) - e^z + c$$

$$f(z) = e^z z + e^z - e^z + c$$

$$\text{Thus, } f(z) = ze^z + c$$

3. Find the analytic function whose real part is $\frac{x^4 - y^4 - 2x}{x^2 + y^2}$. Hence determine v .

Sol:

$$\text{Given } u = \frac{x^4 - y^4 - 2x}{x^2 + y^2}$$

$$u_x = \frac{(x^2 + y^2)(4x^3 - 2) - (x^4 - y^4 - 2x)(2x)}{(x^2 + y^2)^2}$$

$$u_y = \frac{(x^2 + y^2)(-4y^3) - (x^4 - y^4 - 2x)(2y)}{(x^2 + y^2)^2}$$

$$\text{Consider, } f'(z) = u_x + iv_x$$

$$\text{By C-R equations } v_x = -u_y$$

$$f'(z) = u_x - iu_y$$

Putting $x = z$ and $y = 0$ we have,

$$f'(z) = [u_x]_{(z,0)} - i[u_y]_{(z,0)}$$

$$f'(z) = \frac{(z^2)(4z^3 - 2) - (z^4 - 2z)(2z)}{(z^2)^2} - i \cdot 0$$

$$f'(z) = \frac{4z^5 - 2z^2 - 2z^5 + 4z^2}{z^4}$$

$$f'(z) = \frac{2z^5 + 2z^2}{z^4}$$

$$f'(z) = \frac{2z^5}{z^4} + \frac{2z^2}{z^4}$$

$$f'(z) = 2z + \frac{2}{z^2}$$

$$f(z) = \int (2z + \frac{2}{z^2}) dz + c$$

$$f(z) = 2 \cdot \frac{z^2}{2} + 2 \cdot \left(-\frac{1}{z}\right) + c$$

$$f(z) = z^2 - \frac{2}{z} + c$$

$$f(z) = (x + iy)^2 - \frac{2}{(x + iy)} + c$$

$$f(z) = (x^2 + i^2y^2 + 2xiy) - \frac{2(x - iy)}{(x + iy)(x - iy)} + c$$

$$f(z) = (x^2 - y^2 + 2xiy) - \frac{2(x - iy)}{(x^2 + y^2)} + c$$

$$f(z) = \left[(x^2 - y^2) - \frac{2x}{(x^2 + y^2)} \right] + i \left[2xy + \frac{2y}{(x^2 + y^2)} \right] + c$$

$$u + iv = \left[\frac{x^4 - y^4 - 2x}{(x^2 + y^2)} \right] + i \left[\frac{2x^3y + 2xy^3 + 2y}{(x^2 + y^2)} \right] + c$$

Equating imaginary part we get

$$v = \left[\frac{2x^3y + 2xy^3 + 2y}{(x^2 + y^2)} \right]$$

4. Determine the analytic function $f(z) = u + iv$ given that the real part $u = e^{2x}(x \cos 2y - y \sin 2y)$.

Sol: Given

$$u = e^{2x}(x\cos 2y - y\sin 2y)$$

$$u_x = e^{2x}(\cos 2y \cdot 1 - 0) + (x\cos 2y - y\sin 2y) \cdot 2e^{2x}$$

$$u_x = e^{2x}(\cos 2y + 2x\cos 2y - 2y\sin 2y)$$

$$u_y = e^{2x}(-2x\sin 2y - 2y\cos 2y - \sin 2y)$$

$$u_y = -e^{2x}(2x\sin 2y + 2y\cos 2y + \sin 2y)$$

$$\text{Consider, } f'(z) = u_x + iv_x$$

$$\text{By C-R equations } v_x = -u_y$$

$$f'(z) = u_x - iu_y$$

Putting $x = z$ and $y = 0$ we have,

$$f'(z) = [u_x]_{(z,0)} - i[u_y]_{(z,0)}$$

$$f'(z) = e^{2z}(1 + 2z)$$

$$f(z) = \int e^{2z}(1 + 2z)dz + c$$

$$f(z) = (1 + 2z)\frac{e^{2z}}{2} - 2 \cdot \frac{e^{2z}}{4} + c$$

$$f(z) = \frac{e^{2z}}{2} + ze^{2z} - \frac{e^{2z}}{2} + c$$

$$f(z) = ze^{2z} + c$$

5. Find the analytic function $f(z)$ whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$.

Sol: Given $u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

$$u_x = \frac{(\cosh 2y - \cos 2x) \cdot 2\cos 2x - \sin 2x(0 - (-2\sin 2x))}{(\cosh 2y - \cos 2x)^2}$$

$$u_x = \frac{(\cosh 2y - \cos 2x) \cdot 2\cos 2x - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$u_y = \frac{0 - \sin 2x(2\sinh 2y - 0)}{(\cosh 2y - \cos 2x)^2}$$

$$u_y = \frac{-\sin 2x(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$\text{Consider, } f'(z) = u_x + iv_x$$

$$\text{By C-R equations } v_x = -u_y$$

$$f'(z) = u_x - iu_y$$

Putting $x = z$ and $y = 0$ we have,

$$f'(z) = [u_x]_{(z,0)} - i[u_y]_{(z,0)}$$

$$f'(z) = \frac{(1 - \cos 2z) \cdot 2\cos 2z - 2\sin^2 2z}{(1 - \cos 2z)^2} - i \cdot 0$$

$$f'(z) = \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$f'(z) = \frac{2\cos 2z - 2(\cos^2 2z + \sin^2 2z)}{(1 - \cos 2z)^2}$$

$$f'(z) = \frac{2\cos 2z - 2(1)}{(1 - \cos 2z)^2}$$

$$f'(z) = \frac{-2(1 - \cos 2z)}{(1 - \cos 2z)^2}$$

$$f'(z) = \frac{-2}{(1 - \cos 2z)}$$

$$f'(z) = \frac{-2}{2\sin^2 z}$$

$$f'(z) = -\operatorname{cosec}^2 z$$

$$f(z) = -\int \operatorname{cosec}^2 z \, dz + c$$

$$f(z) = -(-\cot z) + c$$

$$\mathbf{f(z) = \cot z + c}$$

6. If $\phi + i\psi$ represents the complex potential of an electrostatic field where $\psi = (x^2 - y^2) + \frac{x}{(x^2 + y^2)}$ find the complex potential as a function of the complex variable z and hence determine ϕ .

Sol: Given

$$\psi = (x^2 - y^2) + \frac{x}{(x^2 + y^2)}$$

$$\psi_x = 2x + \frac{(x^2 + y^2) \cdot 1 - x \cdot 2x}{(x^2 + y^2)^2}$$

$$\psi_x = 2x + \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\psi_y = -2y + \frac{(x^2 + y^2) \cdot 0 - x \cdot 2y}{(x^2 + y^2)^2}$$

$$\psi_y = -2y - \frac{2xy}{(x^2 + y^2)^2}$$

Consider, $f'(z) = \phi_x + i\psi_x$

By C-R equations $\phi_x = \psi_y$

$$f'(z) = \psi_y + i\psi_x$$

$$f'(z) = [\psi_y]_{(z,0)} + i[\psi_x]_{(z,0)}$$

$$f'(z) = 0 + i \left[2z + \frac{(-z^2)}{(z^2)^2} \right]$$

$$f'(z) = i \left[2z - \frac{1}{z^2} \right]$$

$$f(z) = i \int \left(2z - \frac{1}{z^2} \right) dz + c$$

$$f(z) = i \left(z^2 + \frac{1}{z} \right) + c$$

To find ϕ

$$\phi + i\psi = i \left\{ (x + iy)^2 + \frac{1}{(x + iy)} \right\} + c$$

$$\phi + i\psi = i \left\{ (x^2 + i^2y^2 + 2xiy) + \frac{(x - iy)}{(x + iy)(x - iy)} \right\} + c$$

$$\phi + i\psi = i \{ (x^2 - y^2) + 2xiy \} + i \frac{(x - iy)}{(x^2 + y^2)} + c$$

$$\phi + i\psi = i(x^2 - y^2) - 2xy + \frac{ix}{(x^2 + y^2)} + \frac{y}{(x^2 + y^2)} + c$$

$$\phi + i\psi = \left[-2xy + \frac{y}{(x^2 + y^2)} \right] + i \left[(x^2 - y^2) + \frac{x}{(x^2 + y^2)} \right] + c$$

Equating real and imaginary parts, we observe that the imaginary part is same as that of the given problem and the required real part is

$$\phi = \left[-2xy + \frac{y}{(x^2 + y^2)} \right]$$

7. Construct the analytic function whose real part is $r^2 \cos 2\theta$.

Sol: Given $u = r^2 \cos 2\theta$

$$u_r = 2r \cos 2\theta \quad u_\theta = -2r^2 \sin 2\theta$$

$$f'(z) = e^{-i\theta} [u_r + iv_r]$$

By C-R equations in the RHS,

$$v_r = -\frac{1}{r}u_\theta$$

$$f'(z) = e^{-i\theta}[u_r - i\frac{1}{r}u_\theta]$$

Putting $r = z$, $\theta = 0$

$$f'(z) = 1 \cdot [2z - i\frac{1}{z} \cdot 0]$$

$$f'(z) = 2z$$

$$f(z) = 2 \int z \, dz + c$$

$$f(z) = 2 \cdot \frac{z^2}{2} + c$$

$$f(z) = z^2 + c$$

Problems on finding the conjugate harmonic function and the analytic function

1. Show that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic and find its harmonic conjugate. Also find the corresponding analytic function $f(z)$.

Sol: Given

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

$$u_x = 3x^2 - 3y^2 + 6x \quad u_{xx} = 6x + 6$$

$$u_y = -6xy - 6y \quad u_{yy} = -6y - 6$$

$$u_{xx} + u_{yy} = 6x + 6 - 6x - 6$$

$$u_{xx} + u_{yy} = 0.$$

Thus, **u is harmonic.**

Consider C-R equation

$$v_y = u_x \quad \text{and} \quad v_x = -u_y$$

Substituting u_x and u_y , we have

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 6x \quad \& \quad \frac{\partial v}{\partial x} = -(-6xy - 6y)$$

$$v = \int (3x^2 - 3y^2 + 6x) dy + f(x)$$

$$v = 3x^2 \int 1 \cdot dy - 3 \int y^2 dy + 6x \int 1 \, dy + f(x)$$

$$v = 3x^2 y - 3 \frac{y^3}{3} + 6xy + f(x)$$

$$v = 3x^2 y - y^3 + 6xy + f(x)$$

Now, $\frac{\partial v}{\partial x} = 6xy + 6y$

$$v = 6y \int x \, dx + 6y \int dx + g(y) \quad v = 6y \frac{x^2}{2} + 6y \cdot x + g(y)$$

$$v = 3x^2y + 6xy + g(y)$$

We have to choose $f(x)$ and $g(y)$ to obtain a unique expression for v . By comparison $f(x) = 0$ and $g(y) = -y^3$.

Thus, $v = 3x^2y - y^3 + 6xy$ is the harmonic conjugate.

The analytic function is, $f(z) = u + iv$

$$f(z) = (x^3 - 3xy^2 + 3x^2 - 3y^2 + 1) + i(3x^2y - y^3 + 6xy)$$

Putting $x = z$ and $y = 0$ we have,

$$f(z) = z^3 + 3z^2 + 1$$

2. Show that $v = \cos x \sinh y$ is harmonic and find its harmonic conjugate. Also find the corresponding analytic function $f(z)$.

Sol: Given $v = \cos x \sinh y$

$$v_x = -\sin x \cosh y \quad v_{xx} = -\cos x \cosh y$$

$$v_y = \cos x \cosh y \quad v_{yy} = \cos x \sinh y$$

$$u_{xx} + u_{yy} = -\cos x \cosh y + \cos x \cosh y$$

$$u_{xx} + u_{yy} = 0.$$

Thus, u is harmonic.

Consider C-R equation

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

Substituting v_y and v_x , we have

$$\frac{\partial u}{\partial x} = \cos x \cosh y \quad \& \quad \frac{\partial u}{\partial y} = -(-\sin x \cosh y)$$

$$u = \int \cos x \cosh y \, dx + f(y)$$

$$u = \cosh y \int \cos x \, dx + f(y)$$

$$u = \cosh y \cdot \sin x + f(y)$$

$$\text{Also, } u = \int \sin x \cosh y \, dy + g(x)$$

$$u = \sin x \int \cosh y \, dy + g(x)$$

$$u = \sin x \sinh y + g(x)$$

We have to choose $f(y)$ and $g(x)$ to obtain a unique expression for u . By comparison $f(y) = 0$ and $g(x) = 0$.

Thus, $u = \cosh y \sin x$ is the harmonic conjugate.

The analytic function is, $f(z) = u + iv$

$$f(z) = (\cosh y \sin x) + i (\cos x \sinh y)$$

Putting $x = z$ and $y = 0$ we have,

$$f(z) = (1 \cdot \sin z) + i \cdot 0 \quad (\text{since } \sinh 0 = 0, \cosh 0 = 1)$$

$$f(z) = \sin z$$

3. Given that $u = x^2 + 4x - y^2 + 2y$ as the real part of an analytic function, find v and hence find $f(z)$ in terms of z .

Sol: Given $u = x^2 + 4x - y^2 + 2y$

$$u_x = 2x + 4 \quad u_y = -2y + 2$$

Consider C-R equation

$$v_y = u_x \quad \text{and}$$

$$v_x = -u_y$$

$$\frac{\partial v}{\partial y} = 2x + 4 \quad \text{and}$$

$$\frac{\partial v}{\partial x} = 2y - 2$$

$$v = \int (2x + 4) dy + f(x)$$

$$v = \int (2y - 2) dx + g(y)$$

$$v = 2x \int 1 dy + 4 \int dy + f(x)$$

$$v = 2y \int 1 dx - 2 \int dx + g(y)$$

$$v = 2xy + 4y + f(x)$$

$$v = 2yx - 2x + g(y)$$

We have to choose $f(x)$ and $g(y)$ to obtain a unique expression for v . By comparison $f(x) = -2x$ and $g(y) = 4y$

$$\text{Thus, } v = 2yx - 2x + 4y$$

The analytic function is, $f(z) = u + iv$

$$f(z) = (x^2 + 4x - y^2 + 2y) + i (2yx - 2x + 4y)$$

Putting $x = z$ and $y = 0$ we have,

$$f(z) = z^2 + 4z - i \cdot 2z$$

4. Show that $u = \left(r + \frac{1}{r}\right) \cos \theta$ is harmonic. Find its harmonic conjugate and also

the corresponding analytic function.

Sol: Given $u = \left(r + \frac{1}{r}\right) \cos\theta$

$$u_r = \left(1 - \frac{1}{r^2}\right) \cos\theta$$

$$u_{rr} = \left(0 - \frac{(-2)}{r^3}\right) \cos\theta = \frac{2}{r^3} \cos\theta$$

$$u_\theta = \left(r + \frac{1}{r}\right) (-\sin\theta)$$

$$u_{\theta\theta} = \left(r + \frac{1}{r}\right) (-\cos\theta) = -\left(r + \frac{1}{r}\right) \cos\theta$$

$$\begin{aligned} \text{Consider } u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} &= \frac{2}{r^3} \cos\theta + \frac{1}{r} \left(1 - \frac{1}{r^2}\right) \cos\theta + \frac{1}{r^2} \left\{ -\left(r + \frac{1}{r}\right) \cos\theta \right\} \\ &= \frac{2}{r^3} \cos\theta + \frac{1}{r} \cos\theta - \frac{1}{r^3} \cos\theta - \frac{1}{r} \cos\theta - \frac{1}{r^3} \cos\theta \end{aligned}$$

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

Thus, **u is harmonic.**

To find v

Consider the C-R equation, $v_\theta = ru_r$ $v_r = -\frac{1}{r} u_\theta$

$$v_\theta = r \left(1 - \frac{1}{r^2}\right) \cos\theta$$

$$v_r = -\frac{1}{r} \left(r + \frac{1}{r}\right) (-\sin\theta)$$

$$v_\theta = \left(r - \frac{1}{r}\right) \cos\theta$$

$$v_r = \left(1 + \frac{1}{r^2}\right) \sin\theta$$

$$v = \int \left(r - \frac{1}{r}\right) \cos\theta \, d\theta + f(r)$$

$$v = \int \left(1 + \frac{1}{r^2}\right) \sin\theta \, dr + g(\theta)$$

$$v = \left(r - \frac{1}{r}\right) \sin\theta + f(r)$$

$$v = \sin\theta \int \left(1 + \frac{1}{r^2}\right) \, dr + g(\theta)$$

$$v = \left(r - \frac{1}{r}\right) \sin\theta + f(r)$$

$$v = \sin\theta \left(r - \frac{1}{r}\right) + g(\theta)$$

Comparing, we must have $f(r) = 0$ and $g(\theta) = 0$

$$\text{Thus, } \mathbf{v} = \left(r - \frac{1}{r}\right) \sin\theta$$

Also, wkt $f(z) = u + iv$

$$f(z) = \left(r + \frac{1}{r}\right) \cos\theta + i \left(r - \frac{1}{r}\right) \sin\theta$$

Put $r = z$ and $\theta = 0$ ($\cos 0 = 1$, $\sin 0 = 0$)

$$\mathbf{f(z)} = \mathbf{z + \frac{1}{z}}$$

Miscellaneous Problems

1. Find the analytic function $f(z) = u + iv$ given $u - v = e^x(\cos y - \sin y)$

Sol: Given $u - v = e^x(\cos y - \sin y)$

Differentiate w.r.t x and y partially

$$u_x - v_x = (\cos y - \sin y)e^x \quad \text{---- (1)}$$

$$\text{And } u_y - v_y = e^x(-\sin y - \cos y) \quad \text{---- (2)}$$

Using C-R equations $u_y = -v_x$ and $v_y = u_x$ for the LHS of (2), we get

$$-v_x - u_x = -e^x(\sin y + \cos y)$$

$$u_x + v_x = e^x(\sin y + \cos y) \quad \text{----(3)}$$

$$(1) + (3): u_x - v_x + u_x + v_x = e^x \cos y - e^x \sin y + e^x \sin y + e^x \cos y$$

$$2u_x = 2e^x \cos y \quad ; \quad u_x = e^x \cos y$$

$$(1) - (3): u_x - v_x - u_x - v_x = e^x \cos y - e^x \sin y - e^x \sin y - e^x \cos y$$

$$-2v_x = -2e^x \sin y \quad ; \quad v_x = e^x \sin y$$

$$\text{Wkt, } f'(z) = u_x + iv_x$$

$$f'(z) = e^x \cos y + ie^x \sin y$$

$$f'(z) = e^x(\cos y + i \sin y)$$

$$f'(z) = e^x \cdot e^{iy}$$

$$f'(z) = e^{x+iy}$$

$$f'(z) = e^z$$

$$\text{Thus, } f(z) = e^z + c$$

2. If $f(z)$ is analytic, show that $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] |f(z)|^2 = 4|f'(z)|^2$

Sol: Let $f(z) = u + iv$ be analytic.

$$|f(z)| = \sqrt{u^2 + v^2} \quad \text{or} \quad |f(z)|^2 = u^2 + v^2 = \phi(\text{say})$$

$$\text{To Prove that } \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] \phi = 4|f'(z)|^2$$

$$\text{i. e., to prove that } \phi_{xx} + \phi_{yy} = 4|f'(z)|^2$$

Consider, $\phi = u^2 + v^2$ and differentiate w.r.t 'x' partially

$$\phi_x = 2uu_x + 2vv_x = 2[uu_x + vv_x]$$

$$\phi_{xx} = 2[uu_{xx} + u_x u_x + vv_{xx} + v_x v_x]$$

$$\phi_{xx} = 2[uu_{xx} + u_x^2 + vv_{xx} + v_x^2]$$

$$\text{Similarly, } \phi_{yy} = 2[uu_{yy} + u_y^2 + vv_{yy} + v_y^2]$$

$$\text{Now, } \phi_{xx} + \phi_{yy} = 2[uu_{xx} + u_x^2 + vv_{xx} + v_x^2 + uu_{yy} + u_y^2 + vv_{yy} + v_y^2]$$

$$\phi_{xx} + \phi_{yy} = 2[u(u_{xx} + u_{yy}) + v(v_{xx} + v_{yy}) + u_x^2 + v_x^2 + u_y^2 + v_y^2] \quad \text{--- (1)}$$

Since $f(z) = u + iv$ is analytic, u and v are harmonic.

Hence, $u_{xx} + u_{yy} = 0$, $v_{xx} + v_{yy} = 0$.

By C-R equations: $v_y = u_x$ and $u_y = -v_x$

Using above results in RHS of (1), we get

$$\phi_{xx} + \phi_{yy} = 2[u(0) + v(0) + u_x^2 + v_x^2 + (-v_x)^2 + (u_x)^2]$$

$$\phi_{xx} + \phi_{yy} = 2[2(u_x^2 + v_x^2)]$$

$$\phi_{xx} + \phi_{yy} = 4[(u_x^2 + v_x^2)] \quad \text{---- (2)}$$

But, $f'(z) = u_x + iv_x$

$$|f'(z)| = \sqrt{u_x^2 + v_x^2}$$

$$|f'(z)|^2 = u_x^2 + v_x^2$$

Using this in RHS of (2), we get $\phi_{xx} + \phi_{yy} = 4|f'(z)|^2$

3. If $f(z)$ is a regular function of z , show that $\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$

Sol: Let $f(z) = u + iv$ be the regular(analytic) function.

$$|f(z)| = \sqrt{u^2 + v^2} = \phi \quad (\text{Say})$$

We have to prove that, $\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 = |f'(z)|^2$

That is prove that, $\phi_x^2 + \phi_y^2 = |f'(z)|^2$ where $\phi = \sqrt{u^2 + v^2}$

Consider $\phi^2 = u^2 + v^2$

Differentiating w.r.t x partially, we get

$$2\phi\phi_x = 2uu_x + 2vv_x, \text{ dividing by 2}$$

$$\phi\phi_x = uu_x + vv_x \quad \text{--- (1)}$$

$$\text{Similarly} \quad \phi\phi_y = uu_y + vv_y \quad \text{--- (2)}$$

Squaring and adding (1) and (2) we have,

$$\phi^2(\phi_x^2 + \phi_y^2) = (uu_x + vv_x)^2 + (uu_y + vv_y)^2$$

$$\varphi^2(\varphi_x^2 + \varphi_y^2) = (u^2u_x^2 + v^2v_x^2 + 2uvu_xv_x) + (u^2u_y^2 + v^2v_y^2 + 2uvu_yv_y)$$

Since $f(z) = u + iv$ is analytic, by C-R equation $u_y = -v_x$ and $v_y = u_x$

By using these in the second bracket of the RHS , we have

$$\begin{aligned}\varphi^2(\varphi_x^2 + \varphi_y^2) &= (u^2u_x^2 + v^2v_x^2 + 2uvu_xv_x) + (u^2u_y^2 + v^2v_y^2 - 2uvv_xu_x) \\ &= u^2(u_x^2 + v_x^2) + v^2(u_x^2 + v_x^2)\end{aligned}$$

$$= (u_x^2 + v_x^2)(u^2 + v^2)$$

But $\varphi^2 = u^2 + v^2$ and using this in RHS , we have

$$\varphi^2(\varphi_x^2 + \varphi_y^2) = \varphi^2(u_x^2 + v_x^2)$$

$$(\varphi_x^2 + \varphi_y^2) = (u_x^2 + v_x^2) \text{ ----- (3)}$$

But $f'(z) = u_x + iv_x$

$$|f'(z)| = \sqrt{u_x^2 + v_x^2}$$

$$\text{Or } |f'(z)|^2 = u_x^2 + v_x^2$$

Using this in RHS of (3), we get

$$\varphi_x^2 + \varphi_y^2 = |f'(z)|^2$$

$$\varphi_x^2 + \varphi_y^2 = |f'(z)|^2$$

Complex Integration

Let $f(z)$ be a continuous function of the complex variable $z = x + iy$ and C be a curve in the xy - plane. The integral of $f(z)$ along the path C is called the **Complex line integral** usually denoted by $\int_C f(z)dz$. If C is any simple closed curve, the notation $\oint_C f(z)dz$ is also used.

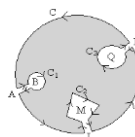
Properties of Complex Integral

1. If $-c$ denotes the curve traversed in the anticlockwise direction, then

$$\int_{-c} f(z)dz = - \int_c f(z)dz$$

2. If c is split into a number of parts $c_1, c_2, c_3, \dots$ then

$$\int_c f(z)dz = \int_{c_1} f(z)dz + \int_{c_2} f(z)dz + \int_{c_3} f(z)dz + \dots$$



3. If λ_1 and λ_2 are constant, then

$$\int_c [\lambda_1 f_1(z) \pm \lambda_2 f_2(z)]dz = \lambda_1 \int_c f_1(z)dz \pm \lambda_2 \int_c f_2(z)dz$$

Line integral of a complex valued function

Let $f(z) = u(x, y) + iv(x, y)$ be a complex valued function defined over region R and C be a curve in the region. Then

$$\int_C f(z)dz = \int_C (u + iv)(dx + idy)$$

i.e.,
$$\int_C f(z)dz = \int_C (u dx - v dy) + i \int_C (v dx + u dy)$$

CAUCHY'S THEOREM

Statement: If $f(z)$ is analytic at all points inside and on a simple closed curve C then ,

$$\int_C f(z)dz = 0.$$

Proof: Let $f(z) = u + iv$

$$dz = dx + idy$$

$$f(z)dz = (u + iv)(dx + idy) = udx + ivdx + iudy + i^2vdy$$

Then ,
$$\int_C f(z)dz = \int_C (u + iv)(dx + idy)$$

i.e.,
$$\int_C f(z)dz = \int_C (u dx - v dy) + i \int_C (v dx + u dy) \quad \text{--- (1)}$$

We know the Green's theorem in a plane

$$\int_C (M dx + N dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Using Green's theorem on RHS of (1), we get

$$\int_C f(z) dz = \iint_R \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy$$

Since $f(z)$ is analytic, we have C-R equations ; $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$.

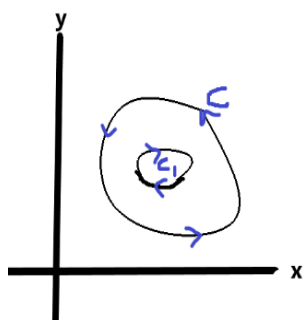
$$\text{Hence, } \int_C f(z) dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} \right) dx dy + i \iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y} \right) dx dy$$

Thus, we get $\int_C f(z) dz = 0$.

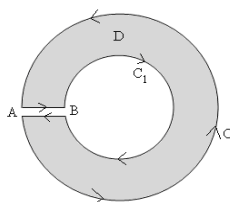
This proves **Cauchy's theorem**.

Consequences of Cauchy's theorem

If C and C_1 are two simple closed curves such that C_1 lies entirely within C and if $f(z)$ is analytic on C, C_1 then



or

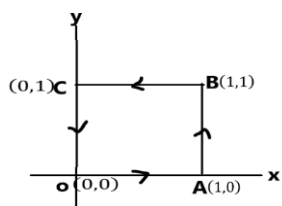


$$\int_C f(z) dz = \int_{C_1} f(z) dz$$

Problems

1. Verify Cauchy's theorem for the function $f(z) = z^2$ where C is the square having vertices $(0,0), (1,0), (1,1), (0,1)$.

Sol: C is the square OABC. By Cauchy's theorem, we have to show that $\int_C f(z) dz = 0$.



Now, $f(z) = z^2$, $dz = dx + idy$

$$\therefore f(z)dz = (x + iy)^2(dx + idy) \quad \text{--- (1)}$$

Along $OA(C_1)$: $y = 0 \Rightarrow dy = 0$

$$(1) \Rightarrow f(z)dz = x^2 dx, \quad 0 \leq x \leq 1$$

Along $AB(C_2)$: $x = 1 \Rightarrow dx = 0$

$$(1) \Rightarrow f(z)dz = (1 + iy)^2 idy, \quad 0 \leq y \leq 1$$

Along $BC(C_3)$: $y = 1 \Rightarrow dy = 0$

$$(1) \Rightarrow f(z)dz = (x + i)^2 dx, \quad 1 \leq x \leq 0$$

Along $CO(C_4)$: $x = 0 \Rightarrow dx = 0$

$$(1) \Rightarrow f(z)dz = (iy)^2 idy = -iy^2 dy, \quad 1 \leq y \leq 0$$

$$\text{Wkt, } \int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \int_{C_4} f(z)dz$$

$$\int_C f(z)dz = \int_{x=0}^1 x^2 dx + \int_{y=0}^1 (1 + iy)^2 idy + \int_{x=1}^0 (x + i)^2 dx + \int_{y=1}^0 -iy^2 dy$$

$$= \left[\frac{x^3}{3} \right]_0^1 + i \left[\frac{(1+iy)^3}{3i} \right]_0^1 + \left[\frac{(x+i)^3}{3} \right]_1^0 - i \left[\frac{y^3}{3} \right]_1^0$$

$$\int_C f(z)dz = \left[\frac{1}{3} - 0 \right] + \left[\frac{(1+i)^3}{3} - \frac{1}{3} \right] + \left[\frac{i^3}{3} - \frac{(1+i)^3}{3} \right] - i \left[0 - \frac{1}{3} \right]$$

$$\int_C f(z)dz = \frac{1}{3} + \frac{(1+i)^3}{3} - \frac{1}{3} - \frac{i}{3} - \frac{(1+i)^3}{3} - \frac{i}{3} \quad (i^3 = i \cdot i^2 = -i)$$

$$\int_C f(z)dz = 0$$

2. Verify Cauchy's theorem for the function $f(z) = 3z^2 + iz - 4$ where C is the square having vertices as $1 \pm i, -1 \pm i$.

Sol: Given $f(z) = 3z^2 + iz - 4$

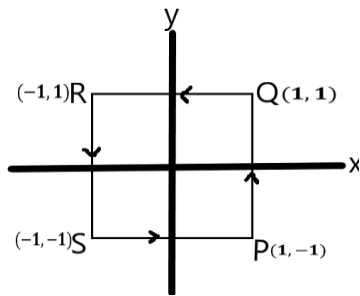
$$f(z) = 3(x + iy)^2 + i(x + iy) - 4$$

$$f(z) = 3(x^2 - y^2 + 2xiy) + i(x + iy) - 4$$

$$f(z) = 3x^2 - 3y^2 + 6xiy + ix - y - 4$$

$$f(z) = (3x^2 - 3y^2 - y - 4) + i(6xy + x)$$

$$f(z)dz = [(3x^2 - 3y^2 - y - 4) + i(6xy + x)][dx + idy] \quad \text{---(1)}$$



Along SP(C₁): $y = -1 \Rightarrow dy = 0$

$$(1) \Rightarrow f(z)dz = [(3x^2 - 3 + 1 - 4) + i(-6x + x)]dx$$

$$f(z)dz = [(3x^2 - 6) - i5x]dx, \quad -1 \leq x \leq 1$$

Along PQ(C₂): $x = 1 \Rightarrow dx = 0$

$$(1) \Rightarrow f(z)dz = [(3 - 3y^2 - y - 4) + i(6y + 1)][idy]$$

$$f(z)dz = i[(-3y^2 - y - 1) + i(6y + 1)]dy, \quad -1 \leq y \leq 1$$

Along QR(C₃): $y = 1 \Rightarrow dy = 0$

$$(1) \Rightarrow f(z)dz = [(3x^2 - 3 - 1 - 4) + i(6x + x)]dx$$

$$f(z)dz = [(3x^2 - 8) + i7x]dx, \quad 1 \leq x \leq -1$$

Along RS(C₄): $x = -1 \Rightarrow dx = 0$

$$(1) \Rightarrow f(z)dz = [(3 - 3y^2 - y - 4) + i(-6y - 1)][idy]$$

$$f(z)dz = i[(-3y^2 - y - 1) - i(6y + 1)]dy, \quad 1 \leq y \leq -1$$

$$\text{Wkt, } \int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \int_{C_4} f(z)dz$$

$$\begin{aligned} \int_C f(z)dz &= \int_{x=-1}^1 [(3x^2 - 6) - i5x]dx \\ &+ \int_{y=-1}^1 i[(-3y^2 - y - 1) + i(6y + 1)]dy \\ &+ \int_{x=1}^{-1} [(3x^2 - 8) + i7x]dx + \int_{y=1}^{-1} i[(-3y^2 - y - 1) - i(6y + 1)]dy \end{aligned}$$

$$\begin{aligned} \int_C f(z)dz &= 3 \left[\frac{x^3}{3} \right]_{-1}^1 - 6[x]_{-1}^1 - 5i \left[\frac{x^2}{2} \right]_{-1}^1 - 3i \left[\frac{y^3}{3} \right]_{-1}^1 - i \left[\frac{y^2}{2} \right]_{-1}^1 - i[y]_{-1}^1 - 6 \left[\frac{y^2}{2} \right]_{-1}^1 - [y]_{-1}^1 \\ &+ 3 \left[\frac{x^3}{3} \right]_1^{-1} - 8[x]_1^{-1} + 7i \left[\frac{x^2}{2} \right]_1^{-1} - 3i \left[\frac{y^3}{3} \right]_1^{-1} - i \left[\frac{y^2}{2} \right]_1^{-1} - i[y]_1^{-1} + 6 \left[\frac{y^2}{2} \right]_1^{-1} + [y]_1^{-1} \end{aligned}$$

$$\int_c f(z) dz = 2 - 12 - \frac{5i}{2}(0) - 2i - i(0) - 2i - 3(0) - 2 - 2 + 16 + \frac{7i}{2}(0) + 2i - \frac{i}{2}(0) + 2i + 3(0) - 2$$

$$\int_c f(z) dz = 2 - 12 - 2i - 2i - 2 - 2 + 16 + 2i + 2i - 2$$

$$\int_c f(z) dz = 0$$

3. Verify Cauchy's theorem for the function $f(z) = ze^{-z}$ over the unit circle with origin as the centre.

Sol: Given $f(z) = ze^{-z}$, C is the circle $|z| = 1$

$$z = e^{i\theta}, \quad 0 \leq \theta \leq 2\pi$$

$$dz = ie^{i\theta} d\theta$$

$$f(z) dz = e^{i\theta} e^{-e^{i\theta}} ie^{i\theta} d\theta$$

$$\text{Now, } \int_c f(z) dz = \int_{\theta=0}^{2\pi} e^{i\theta} e^{-e^{i\theta}} \cdot i e^{i\theta} d\theta$$

$$\text{Put, } e^{i\theta} = t$$

$$\therefore ie^{i\theta} d\theta = dt$$

$$\text{When } \theta = 0, t = 1$$

$$\theta = 2\pi, t = e^{i.2\pi} = \cos(2\pi) + i\sin(2\pi) = 1 + i.0 = 1$$

$$\int_c f(z) dz = \int_{t=1}^1 t e^{-t} \cdot dt$$

$$\int_c f(z) dz = 0 \quad (\text{since both the limits are same, the value of the integral is zero})$$

Cauchy's Integral formula

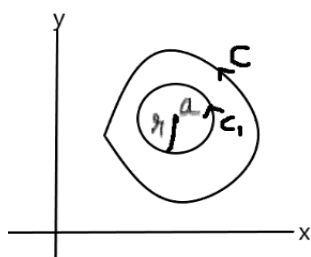
Statement: If $f(z)$ is analytic inside and on a simple closed curve C and if ' a ' is

$$\text{any point within } C \text{ then } f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z-a} dz$$

Proof: Consider the function $\frac{f(z)}{(z-a)}$, which is analytic at all points within C except at

$$z = a.$$

Draw a circle C_1 with a as centre and radius r such that C_1 lies entirely inside C .



Since $\frac{f(z)}{(z-a)}$ analytic in the region between C and C_1 , by Cauchy's theorem we have

$$\int_C \frac{f(z)}{(z-a)} dz = \int_{C_1} \frac{f(z)}{(z-a)} dz \quad \text{--- (1)}$$

The equation of circle C_1 is

$$|z - a| = r$$

$$z - a = re^{i\theta}$$

$$z = a + re^{i\theta}, \quad 0 \leq \theta \leq 2\pi$$

$$dz = i \cdot re^{i\theta} d\theta$$

$$(1) \Rightarrow \int_C \frac{f(z)}{(z-a)} dz = \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}} i \cdot re^{i\theta} d\theta$$

$$\int_C \frac{f(z)}{(z-a)} dz = i \int_{\theta=0}^{2\pi} f(a + re^{i\theta}) d\theta$$

In the limiting form, as the circle shrinks to the point 'a' i.e., $r \rightarrow 0$

$$\begin{aligned} \int_C \frac{f(z)}{(z-a)} dz &= i \int_{\theta=0}^{2\pi} f(a) d\theta \\ &= i f(a) \int_{\theta=0}^{2\pi} 1 \cdot d\theta \\ &= i f(a) [\theta]_0^{2\pi} \end{aligned}$$

$$\int_C \frac{f(z)}{z-a} dz = i f(a) \cdot 2\pi$$

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

Problems on Cauchy's integral formula

Working rule

1. Firstly we have to find out whether the point $z = a$ lies inside or outside the given curve C.
2. If $z = a$ is inside C then we use Cauchy's integral formula

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a) \quad ; \quad \int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

3. If the point $z = a$ is outside C, the value of the integral is zero.

1. Evaluate $\int_C \frac{e^z}{z+i\pi} dz$ over each of the following contours C:

a) $|z| = 2\pi$ b) $|z| = \frac{\pi}{2}$ c) $|z - 1| = 1$

Sol: Here $f(z) = e^z$

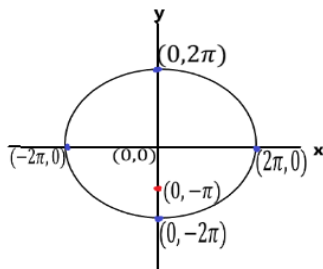
$$z - a = z + i\pi = z - (-i\pi)$$

$$\therefore a = -i\pi \Rightarrow a = (0, -\pi)$$

a) C: $|z| = 2\pi$ is a circle

$$\Rightarrow \text{Centre} = (0,0) \quad ; \quad r = 2\pi$$

The point $a = (0, -\pi)$ lies within the circle $|z| = 2\pi$.



From Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\int_C \frac{e^z}{z+i\pi} dz = 2\pi i f(-i\pi)$$

$$= 2\pi i \cdot e^{-i\pi} \quad (\text{since } f(z) = e^z, z = -i\pi)$$

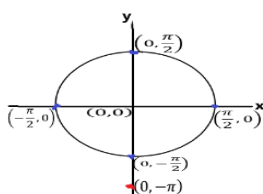
$$= 2\pi i \cdot (\cos\pi - i\sin\pi) \quad (\cos\pi = -1, \sin\pi = 0)$$

$$\int_C \frac{e^z}{z+i\pi} dz = -2\pi i$$

b) C: $|z| = \frac{\pi}{2}$ is a circle

$$\Rightarrow \text{Centre} = (0,0) \quad ; \quad r = \frac{\pi}{2}$$

The point $a = (0, -\pi)$ lies outside the circle $|z| = \frac{\pi}{2}$.



From Cauchy's integral formula,

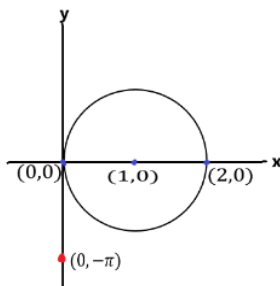
$$\int_C \frac{f(z)}{z-a} dz = 0$$

$$\int_C \frac{e^z}{z+i\pi} dz = 0$$

c) C: $|z - 1| = 1$ is a circle

$\Rightarrow$ Centre = (1,0) ; $r = 1$

The point $a = (0, -\pi)$ lies outside the circle $|z - 1| = 1$



From Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a} dz = 0$$

$$\int_C \frac{e^z}{z+i\pi} dz = 0$$

2. Evaluate $\int_C \frac{dz}{(z^2-4)}$ over each of the following contours C:

a) $|z| = 1$ b) $|z| = 3$ c) $|z + 2| = 1$

Sol: Here $f(z) = 1$

$$\text{Consider } \int_C \frac{1}{(z^2-4)} dz = \int_C \frac{1}{(z+2)(z-2)} dz \quad \text{--- (1)}$$

$$\text{Now } \frac{1}{(z+2)(z-2)} = \frac{A}{(z+2)} + \frac{B}{(z-2)} \quad \text{--- (2)}$$

$$1 = A(z-2) + B(z+2)$$

$$\text{Put } z = 2 ; \quad 1 = A(0) + B(4) \quad \Rightarrow 1 = 4B \quad ; \quad \Rightarrow B = \frac{1}{4}$$

$$\text{Put } z = -2 ; \quad 1 = A(-4) + B(0) \quad \Rightarrow 1 = -4A \quad ; \quad \Rightarrow A = -\frac{1}{4}$$

$$(2) \Rightarrow \frac{1}{(z+2)(z-2)} = \frac{-\frac{1}{4}}{(z+2)} + \frac{\frac{1}{4}}{(z-2)} = \frac{1}{4(z-2)} - \frac{1}{4(z+2)}$$

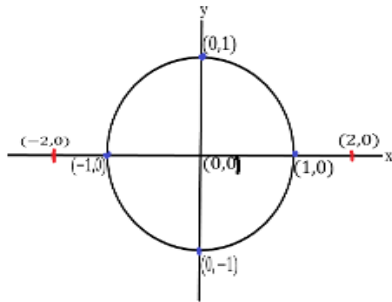
$$(1) \Rightarrow \int_c \frac{1}{(z^2-4)} dz = \frac{1}{4} \int_c \frac{1}{(z-2)} dz - \frac{1}{4} \int_c \frac{1}{(z+2)} dz \quad \text{--- (*)}$$

a) $C: |z| = 1$ is a circle

$\Rightarrow$ Centre = $(0,0)$; $r = 1$

The point $a = (2, 0)$

$a = (-2,0)$ lies outside the circle $|z| = 1$.



From Cauchy's integral formula,

$$\int_c \frac{dz}{z-2} = 0 = \int_c \frac{dz}{z+2}$$

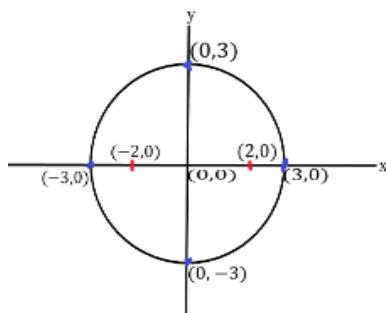
$$(*) \Rightarrow \int_c \frac{1}{(z^2-4)} dz = 0$$

b) $C: |z| = 3$ is a circle

$\Rightarrow$ Centre = $(0,0)$; $r = 3$

The point $a = (2, 0)$

$a = (-2,0)$ lies inside the circle $|z| = 3$.



$$\int_c \frac{dz}{z-2} = 2\pi i f(a)$$

$$\int_c \frac{1}{z-2} dz = 2\pi i f(2) \quad (\text{Here } a = 2, f(z) = 1)$$

$$\int_c \frac{1}{z-2} dz = 2\pi i (1)$$

$$\int_c \frac{1}{z-2} dz = 2\pi i$$

$$\text{Also, } \int_c \frac{dz}{z+2} = 2\pi i f(a)$$

$$\int_c \frac{1}{z+2} dz = 2\pi i f(-2) \quad (\text{Here } a = -2, f(z) = 1)$$

$$\int_c \frac{1}{z+2} dz = 2\pi i (1)$$

$$\int_c \frac{1}{z+2} dz = 2\pi i$$

$$\text{Thus, } (*) \Rightarrow \int_c \frac{1}{(z^2-4)} dz = \frac{1}{4}(2\pi i) - \frac{1}{4}(2\pi i)$$

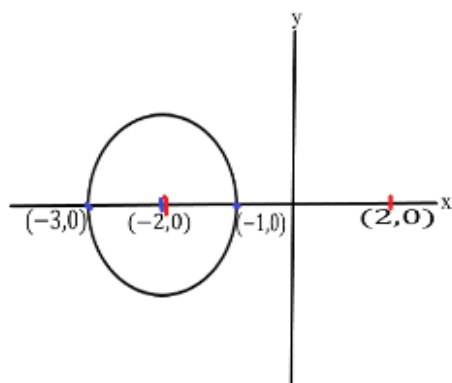
$$\int_c \frac{1}{(z^2-4)} dz = 0$$

c) $C: |z+2| = 1$ is a circle

$$\Rightarrow \text{Centre} = (-2, 0) \quad ; \quad r = 1$$

The point $a = (2, 0)$ lies outside the circle $|z+2| = 1$

$a = (-2, 0)$ lies inside the circle $|z+2| = 1$.



$$\int_c \frac{dz}{z-2} = 0 \quad (\text{Since } a = (2, 0) \text{ lies outside } C)$$

$$\text{Also, } \int_c \frac{1}{z+2} dz = 2\pi i f(a)$$

$$\int_c \frac{1}{z+2} dz = 2\pi i f(-2) \quad (\text{Here } a = -2, f(z) = 1)$$

$$\int_c \frac{1}{z+2} dz = 2\pi i(1)$$

$$\int_c \frac{1}{z+2} dz = 2\pi i$$

$$\text{Thus, } (*) \Rightarrow \int_c \frac{1}{(z^2-4)} dz = \frac{1}{4}(0) - \frac{1}{4}(2\pi i)$$

$$\int_c \frac{1}{(z^2-4)} dz = -\frac{\pi i}{2}$$

3. Evaluate $\int_c \frac{e^{2z}}{(z+1)(z-2)} dz$ where C is the circle $|z| = 3$.

Sol: Here $f(z) = e^{2z}$

$$\text{Consider } \frac{1}{(z+1)(z-2)} = \frac{A}{(z+1)} + \frac{B}{(z-2)} \quad \text{--- (1)}$$

$$1 = A(z-2) + B(z+1)$$

$$\text{Put } z = 2; \quad 1 = A(0) + B(3) \quad \Rightarrow 1 = 3B \quad ; \quad \Rightarrow B = \frac{1}{3}$$

$$\text{Put } z = -1; \quad 1 = A(-3) + B(0) \quad \Rightarrow 1 = -3A \quad ; \quad \Rightarrow A = -\frac{1}{3}$$

$$(1) \Rightarrow \frac{1}{(z+1)(z-2)} = \frac{-\frac{1}{3}}{(z+1)} + \frac{\frac{1}{3}}{(z-2)} = \frac{1}{3(z-2)} - \frac{1}{3(z+1)}$$

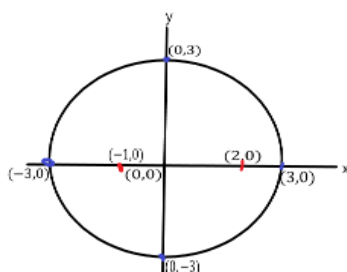
$$\int_c \frac{e^{2z}}{(z+1)(z-2)} dz = \frac{1}{3} \int_c \frac{e^{2z}}{(z-2)} dz - \frac{1}{3} \int_c \frac{e^{2z}}{(z+1)} dz \quad \text{--- (*)}$$

Now, C: $|z| = 3$ is a circle

$$\Rightarrow \text{Centre} = (0,0) \quad ; \quad r = 3$$

The point $a = (2, 0)$

$a = (-1,0)$ lies inside the circle $|z| = 3$.



Using Cauchy's integral formula

$$\int_c \frac{e^{2z}}{z-2} dz = 2\pi i f(a)$$

$$\int_c \frac{e^{2z}}{z-2} dz = 2\pi i f(2) \quad (\text{Here } a = 2, f(z) = e^{2z})$$

$$\int_c \frac{e^{2z}}{z-2} dz = 2\pi i (e^{2 \cdot 2})$$

$$\int_c \frac{e^{2z}}{z-2} dz = 2\pi i e^4$$

$$\text{Also, } \int_c \frac{e^{2z}}{z+1} dz = 2\pi i f(a)$$

$$\int_c \frac{e^{2z}}{z+1} dz = 2\pi i f(-1) \quad (\text{Here } a = -1, f(z) = e^{2z})$$

$$\int_c \frac{e^{2z}}{z+1} dz = 2\pi i (e^{2 \cdot (-1)})$$

$$\int_c \frac{e^{2z}}{z+1} dz = 2\pi i e^{-2}$$

$$(*) \Rightarrow \int_c \frac{e^{2z}}{(z+1)(z-2)} dz = \frac{1}{3} (2\pi i e^4) - \frac{1}{3} (2\pi i e^{-2})$$

$$\int_c \frac{e^{2z}}{(z+1)(z-2)} dz = \frac{2\pi i}{3} [e^4 - e^{-2}]$$

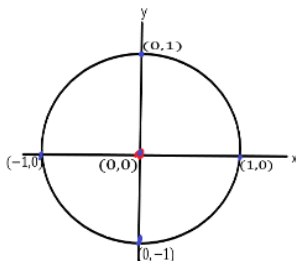
4. Evaluate $\int_C \frac{e^{3z}}{z^2} dz$ where C is the circle $|z| = 1$.

Sol: Here $f(z) = e^{3z}$

Now, C: $|z| = 1$ is a circle

$$\Rightarrow \text{Centre} = (0,0) \quad ; \quad r = 1$$

The point $a = (0, 0)$ lies inside the circle $|z| = 1$.



By Cauchy's integral formula

$$\int_C \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(a)$$

$$\int_C \frac{e^{3z}}{z^2} dz = \frac{2\pi i}{(2-1)!} f^{(2-1)}(0) \quad (\text{Since } f(z) = e^{3z}, a = 0, n = 2)$$

$$\int_C \frac{e^{3z}}{z^2} dz = \frac{2\pi i}{1} f^{(1)}(0) \quad (f^{(1)}(z) = f'(z) = 3 \cdot e^{3z})$$

$$\int_C \frac{e^{3z}}{z^2} dz = 2\pi i [3 \cdot e^{3 \cdot 0}] \quad (e^{3 \cdot 0} = e^0 = 1)$$

$$\int_C \frac{e^{3z}}{z^2} dz = 6\pi i$$

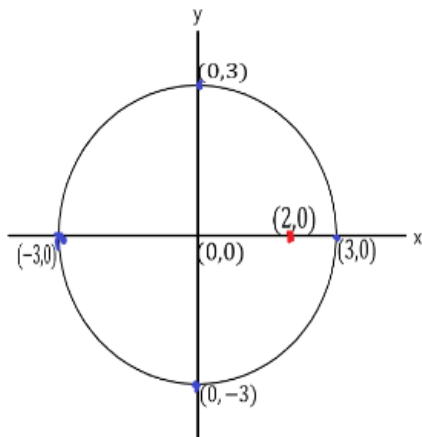
5. Evaluate $\int_C \frac{z^2+z+1}{(z-2)^3} dz$ where C is the circle $|z| = 3$.

Sol: Here $f(z) = z^2 + z + 1$

Now, C: $|z| = 3$ is a circle

$$\Rightarrow \text{Centre} = (0,0) \quad ; \quad r = 3$$

The point $a = (2, 0)$ lies inside the circle $|z| = 3$



By Cauchy's integral formula

$$\int_C \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(a)$$

$$\int_C \frac{z^2+z+1}{(z-2)^3} dz = \frac{2\pi i}{(3-1)!} f^{(3-1)}(2) \quad (\text{Since } f(z) = z^2 + z + 1, a = 2, n = 3)$$

$$\int_C \frac{z^2+z+1}{(z-2)^3} dz = \frac{2\pi i}{2} f^{(2)}(2) \quad (f^{(2)}(z) = f''(z) = 2)$$

$$\int_C \frac{z^2+z+1}{(z-2)^3} dz = \pi i [2] \int_C \frac{z^2+z+1}{(z-2)^3} dz = 2\pi i$$

6. Evaluate $\int_C \frac{e^{\pi z}}{(2z-i)^3} dz$ where C is the circle $|z| = 1$.

Sol: Given $\int_C \frac{e^{\pi z}}{(2z-i)^3} dz = \int_C \frac{e^{\pi z}}{2^3(z-\frac{i}{2})^3} dz$

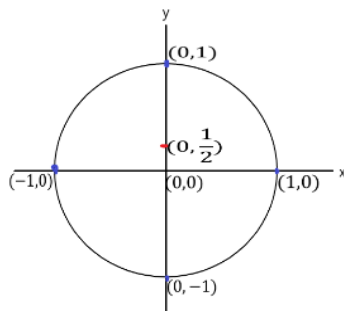
$$\int_C \frac{e^{\pi z}}{(2z-i)^3} dz = \frac{1}{8} \int_C \frac{e^{\pi z}}{(z-\frac{i}{2})^3} dz$$

Here $f(z) = e^{\pi z}$

Now, C: $|z| = 1$ is a circle

$\Rightarrow$ Centre = (0,0) ; $r = 1$

The point $a = \frac{i}{2} = (0, \frac{1}{2})$ lies inside the circle $|z| = 1$.



By Cauchy's integral formula

$$\int_C \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(a)$$

$$\int_C \frac{e^{\pi z}}{(2z-i)^3} dz = \frac{1}{8} \cdot \frac{2\pi i}{(3-1)!} f^{(3-1)}\left(\frac{i}{2}\right) \quad (\text{Since } f(z) = e^{\pi z}, a = \frac{i}{2}, n = 3)$$

$$\int_C \frac{e^{\pi z}}{(2z-i)^3} dz = \frac{1}{8} \cdot \frac{2\pi i}{2} f^{(2)}\left(\frac{i}{2}\right) \quad (f^{(2)}(z) = f''(z) = \pi^2 \cdot e^{\pi z})$$

$$\int_C \frac{e^{\pi z}}{(2z-i)^3} dz = \frac{\pi i}{8} \left[\pi^2 e^{\frac{i\pi}{2}} \right] \int_C \frac{e^{\pi z}}{(2z-i)^3} dz = \frac{\pi^3 i}{8} \left[\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right]$$

$$\int_C \frac{e^{\pi z}}{(2z-i)^3} dz = \frac{\pi^3 i}{8} [0 + i.1]$$

$$\int_C \frac{e^{\pi z}}{(2z-i)^3} dz = \frac{\pi^3 i^2}{8}$$

$$\int_C \frac{e^{\pi z}}{(2z-i)^3} dz = -\frac{\pi^3}{8}$$

7. Evaluate $\int_C \frac{e^{2z}}{(z+1)^2(z-2)} dz$ where C is the circle $|z| = 3$.

Sol: Here $f(z) = e^{2z}$

$$\text{Consider } \frac{1}{(z+1)^2(z-2)} = \frac{A}{(z+1)} + \frac{B}{(z+1)^2} + \frac{C}{(z-2)} \quad \text{--- (1)}$$

$$1 = A(z-2)(z+1) + B(z-2) + C(z+1)^2$$

$$\text{Put } z = 2; \quad 1 = A(0) + B(0) + C(2+1)^2 \quad \Rightarrow 1 = 9C \quad ; \quad \Rightarrow C = \frac{1}{9}$$

$$\text{Put } z = -1; \quad 1 = A(0) + B(-1-2) + C(0) \quad \Rightarrow 1 = -3B \quad ; \quad \Rightarrow B = -\frac{1}{3}$$

$$\text{Put } z = 0; \quad 1 = A(-2)(1) + B(0-2) + C(0+1)^2 \Rightarrow 1 = -2A - 2B + C \quad ;$$

$$\Rightarrow -2A = 1 - \frac{2}{3} - \frac{1}{9} \quad \Rightarrow -2A = \frac{2}{9} \quad \Rightarrow A = -\frac{1}{9}$$

$$(1) \Rightarrow \frac{1}{(z+1)^2(z-2)} = \frac{A}{(z+1)} + \frac{B}{(z+1)^2} + \frac{C}{(z-2)} = -\frac{1}{9(z+1)} - \frac{1}{3(z+1)^2} + \frac{1}{9(z-2)}$$

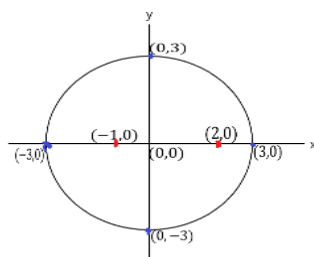
$$\int_C \frac{e^{2z}}{(z+1)^2(z-2)} dz = \frac{1}{9} \int_C \frac{e^{2z}}{(z-2)} dz - \frac{1}{3} \int_C \frac{e^{2z}}{(z+1)^2} dz - \frac{1}{9} \int_C \frac{e^{2z}}{(z+1)} dz \quad \text{--- (*)}$$

Now, C: $|z| = 3$ is a circle

$$\Rightarrow \text{Centre} = (0,0) \quad ; \quad r = 3$$

The point $a = (2, 0)$

$a = (-1,0)$ lies inside the circle $|z| = 3$.



Using Cauchy's integral formula

$$(*) \Rightarrow \int_C \frac{e^{2z}}{(z+1)^2(z-2)} dz = \frac{1}{9} (2\pi i f(2)) - \frac{1}{3} \frac{2\pi i}{(2-1)!} f^{(2-1)}(-1) - \frac{1}{9} (2\pi i f(-1))$$

$$(*) \Rightarrow \int_C \frac{e^{2z}}{(z+1)^2(z-2)} dz = \frac{1}{9} (2\pi i \cdot e^{2 \cdot 2}) - \frac{1}{3} \frac{2\pi i}{1} f'(-1) - \frac{1}{9} (2\pi i \cdot e^{2(-1)})$$

$$\int_C \frac{e^{2z}}{(z+1)^2(z-2)} dz = \frac{2\pi i e^4}{9} - \frac{4\pi i e^{-2}}{3} - \frac{2\pi i e^{-2}}{9}$$

8. Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where C is the circle $|z| = 3$.

Sol: Here $f(z) = \sin \pi z^2 + \cos \pi z^2$

$$\text{Consider } \frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)} \quad \text{--- (1)}$$

$$1 = A(z-2) + B(z-1)$$

$$\text{Put } z = 2; \quad 1 = A(0) + B(1) \quad \Rightarrow 1 = B \quad ; \quad \Rightarrow \mathbf{B = 1}$$

$$\text{Put } z = 1; \quad 1 = A(-1) + B(0) \quad \Rightarrow 1 = -A \quad ; \quad \Rightarrow \mathbf{A = -1}$$

$$(1) \Rightarrow \frac{1}{(z-1)(z-2)} = \frac{-1}{(z-1)} + \frac{1}{(z-2)} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

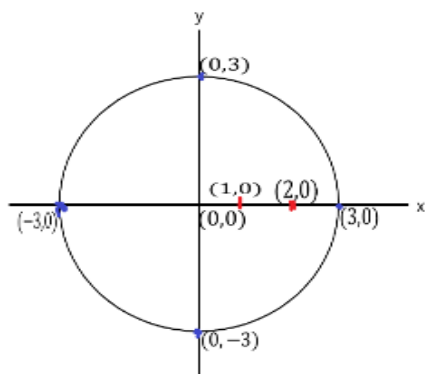
$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)} dz - \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)} dz \quad \text{--- (*)}$$

Now, C: $|z| = 3$ is a circle

$$\Rightarrow \text{Centre} = (0,0) \quad ; \quad r = 3$$

The point $a = (2, 0)$

$a = (1,0)$ lies inside the circle $|z| = 3$.



Using Cauchy's integral formula

$$(*) \Rightarrow \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i f(2) - 2\pi i f(1)$$

$$\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i [\sin 4\pi + \cos 4\pi] - 2\pi i [\sin \pi + \cos \pi]$$

$$\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i [0 + 1] - 2\pi i [0 - 1]$$

$$\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = 4\pi i$$

Module-2

FOURIER SERIES

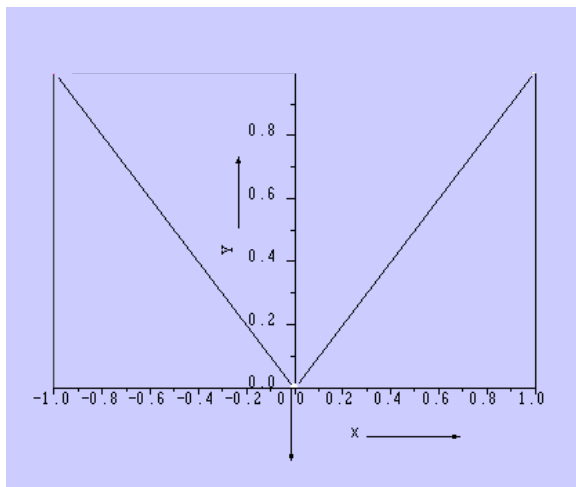
Introduction: In many engineering problems, especially in the study of periodic phenomena in conduction of heat, electro-dynamics and acoustics, it is necessary to express a function in a series of sines and cosines. Such a series is known as the **Fourier series**.

DEFINITIONS:

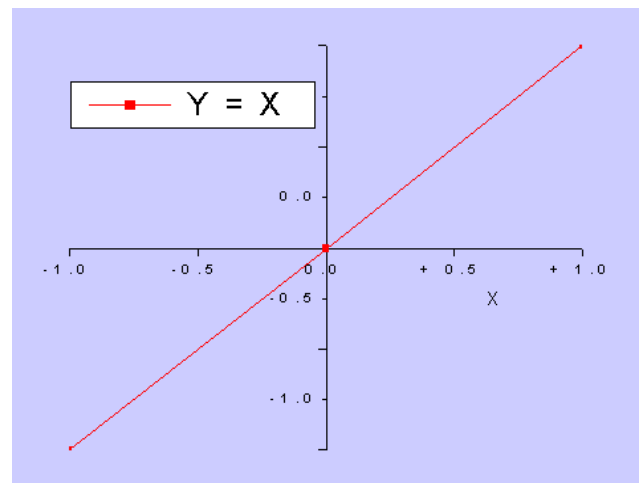
A function $y = f(x)$ is said to be even, if $f(-x) = f(x)$. The graph of the even function is always symmetrical about the y-axis.

A function $y = f(x)$ is said to be odd, if $f(-x) = -f(x)$. The graph of the odd function is always symmetrical about the origin.

For example, the function $f(x) = |x|$ in $[-1, 1]$ is even as $f(-x) = |-x| = |x| = f(x)$ and the function $f(x) = x$ in $[-1, 1]$ is odd as $f(-x) = -x = -f(x)$. The graphs of these functions are shown below:



Graph of $f(x) = |x|$



Graph of $f(x) = x$

Note that the graph of $f(x) = |x|$ is symmetrical about the y-axis and the graph of $f(x) = x$ is symmetrical about the origin.

1. If $f(x)$ is even and $g(x)$ is odd, then

- $h(x) = f(x) \times g(x)$ is odd
- $h(x) = f(x) \times f(x)$ is even
- $h(x) = g(x) \times g(x)$ is even

For example,

1. $h(x) = x^2 \cos x$ is even, since both x^2 and $\cos x$ are even functions
2. $h(x) = x \sin x$ is even, since x and $\sin x$ are odd functions
3. $h(x) = x^2 \sin x$ is odd, since x^2 is even and $\sin x$ is odd.

2. If $f(x)$ is even, then
$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

3. If $f(x)$ is odd, then
$$\int_{-a}^a f(x) dx = 0$$

For example,

$$\int_{-a}^a \cos x dx = 2 \int_0^a \cos x dx, \text{ as } \cos x \text{ is even}$$

$$\text{and } \int_{-a}^a \sin x dx = 0, \text{ as } \sin x \text{ is odd}$$

PERIODIC FUNCTIONS:-

A function $f(x)$ is said to be periodic function with period T if $f(x+T) = f(x)$

Here $f(x)$ is a real-valued function and T is a positive real number.

As a consequence, it follows that

$$f(x) = f(x+T) = f(x+2T) = f(x+3T) = \dots = f(x+nT)$$

Thus, $f(x) = f(x+nT)$, $n = 1, 2, 3, 4, \dots$

The function $f(x) = \sin x$ is periodic of period 2π since

$$\sin(x+2n\pi) = \sin x, \quad n=1,2,3,\dots$$

Note that the graph of the function between 0 and 2π is the same as that between 2π and 4π and so on. It may be verified that a linear combination of periodic functions having period T is also periodic of period T .

EULER'S FORMULAE:

The Fourier series for the function $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\text{Where } a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx, \quad b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx dx,$$

These values of a_0, a_n, b_n are known as Euler's formulae.

Proof: Let $f(x)$ be represented in the interval $(\alpha, \alpha + 2\pi)$ by the Fourier series-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{-----} (1)$$

To find a_0, a_n, b_n , assume that the series (1) can be integrated term by term from $x = \alpha$ to $x = \alpha + 2\pi$.

$$\begin{aligned} \text{To find } a_0: \int_{\alpha}^{\alpha+2\pi} f(x) dx &= \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} a_n \cos nx \right) dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} b_n \sin nx \right) dx \\ &= \frac{1}{2} a_0 (\alpha + 2\pi - \alpha) + 0 + 0 = a_0 \pi \end{aligned}$$

$$\text{Since } \int_{\alpha}^{\alpha+2\pi} \cos nx dx = \frac{\sin nx}{n} \Big|_{\alpha}^{\alpha+2\pi} = 0 \text{ \& } \int_{\alpha}^{\alpha+2\pi} \sin nx dx = -\frac{\cos nx}{n} \Big|_{\alpha}^{\alpha+2\pi} = 0$$

$$\therefore a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) dx$$

To find a_n , multiply each side of (1) by $\cos nx$ and integrate from $x = \alpha$ to $x = \alpha + 2\pi$, we get

$$\begin{aligned} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx &= \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} \cos nx dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} a_n \cos nx \right) \cos nx dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} b_n \sin nx \right) \cos nx dx \\ &= 0 + \pi a_n + 0 = \pi a_n \end{aligned}$$

Since

$$\begin{aligned} \int_{\alpha}^{\alpha+2\pi} \cos^2 nx dx &= \int_{\alpha}^{\alpha+2\pi} \frac{1 + \cos 2nx}{2} dx = \frac{1}{2} \left(x + \frac{\sin 2nx}{2n} \right) \Big|_{\alpha}^{\alpha+2\pi} = \frac{1}{2} (\alpha + 2\pi - \alpha + 0 - 0) = \pi \\ \& \int_{\alpha}^{\alpha+2\pi} \sin nx \cos nx dx &= \frac{\sin^2 nx}{2n} \Big|_{\alpha}^{\alpha+2\pi} = 0 \end{aligned}$$

$$\text{Hence } a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx$$

Similarly To find b_n , multiply each side of (1) by $\sin nx$ and integrate from $x = \alpha$ to $x = \alpha + 2\pi$, we get

$$\begin{aligned} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx dx &= \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} \sin nx dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} a_n \cos nx \right) \sin nx dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} b_n \sin nx \right) \sin nx dx \\ &= 0 + 0 + \pi b_n = \pi b_n \end{aligned}$$

Since

$$\begin{aligned} \int_{\alpha}^{\alpha+2\pi} \sin^2 nx dx &= \int_{\alpha}^{\alpha+2\pi} \frac{1 - \cos 2nx}{2} dx = \frac{1}{2} \left(x - \frac{\sin 2nx}{2n} \right) \Big|_{\alpha}^{\alpha+2\pi} = \frac{1}{2} (\alpha + 2\pi - \alpha + 0 - 0) = \pi \\ \& \int_{\alpha}^{\alpha+2\pi} \sin nx \cos nx dx &= \frac{\sin^2 nx}{2n} \Big|_{\alpha}^{\alpha+2\pi} = 0 \end{aligned}$$

$$\text{Hence } b_n = \frac{1}{\pi} \int_a^{a+2\pi} f(x) \sin nx dx$$

FOURIER SERIES:

A Fourier series of a periodic function consists of a sum of sine and cosine terms. Sines and cosines are the most fundamental periodic functions.

The Fourier series is named after the French Mathematician and Physicist Jacques Fourier (1768 – 1830).

FORMULA FOR FOURIER SERIES

Consider a real-valued function $f(x)$ which obeys the following conditions called Dirichlet's conditions:

1. $f(x)$ is defined in an interval $(a, a+2l)$, and $f(x+2l) = f(x)$ so that $f(x)$ is a periodic function of period $2l$.
2. $f(x)$ is continuous or has only a finite number of discontinuities in the interval $(a, a+2l)$.
3. $f(x)$ has no or only a finite number of maxima & minima in the interval $(a, a+2l)$.

$$\text{Also, let } a_0 = \frac{1}{l} \int_a^{a+2l} f(x) dx \text{ ----- (1)}$$

$$a_n = \frac{1}{l} \int_a^{a+2l} f(x) \cos\left(\frac{n\pi}{l}\right) x dx, n = 1, 2, 3, \dots \text{ ----- (2)}$$

$$b_n = \frac{1}{l} \int_a^{a+2l} f(x) \sin\left(\frac{n\pi}{l}\right) x dx, n = 1, 2, 3, \dots \text{ ----- (3)}$$

Then, the infinite series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right) x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right) x \text{ ----- (4)}$$

is called the Fourier series of $f(x)$ in the interval $(a, a+2l)$. Also, the real numbers $a_0, a_1, a_2, \dots, a_n$, and $b_1, b_2, \dots, b_n$ are called the Fourier coefficients of $f(x)$. The formulae (1), (2) and (3) are called Euler's formulae.

It can be proved that the sum of the series (4) is $f(x)$ if $f(x)$ is continuous at x . Thus we have $f(x)$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right) x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right) x \dots \dots \dots (5)$$

Suppose $f(x)$ is discontinuous at x , then the sum of the series (4) would be

$$\frac{1}{2} [f(x^+) + f(x^-)]$$

where $f(x^+)$ & $f(x^-)$ are respectively right hand and left hand limits of $f(x)$ given by.

$$f(x^+) = \lim_{h \rightarrow 0} f(x+h), \quad f(x^-) = \lim_{h \rightarrow 0} f(x-h), \quad h > 0$$

Particular Cases:

Case (i)

Suppose $a = 0$. Then $f(x)$ is defined over the interval $(0, 2l)$. Formulae (1), (2), (3) reduce to

$$\begin{aligned} a_0 &= \frac{1}{l} \int_0^{2l} f(x) dx \\ a_n &= \frac{1}{l} \int_0^{2l} f(x) \cos\left(\frac{n\pi}{l}x\right) dx, \quad n = 1, 2, \dots, \infty \text{ ----- (6)} \\ b_n &= \frac{1}{l} \int_0^{2l} f(x) \sin\left(\frac{n\pi}{l}x\right) dx, \end{aligned}$$

Then the right-hand side of (5) is the Fourier expansion of $f(x)$ over the interval $(0, 2l)$.

If we set $l = \pi$, then $f(x)$ is defined over the interval $(0, 2\pi)$. Formulae (6) reduce to

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x) dx \\ a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx, \quad n = 1, 2, \dots, \infty \text{ ----- (7)} \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx \quad n = 1, 2, \dots, \infty \end{aligned}$$

$$\text{Also, in this case, (5) becomes: } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{ ----- (8)}$$

Case (ii)

Suppose $a = -l$. Then $f(x)$ is defined over the interval $(-l, l)$. Formulae (1), (2) (3) reduce to

$$\begin{aligned} a_0 &= \frac{1}{l} \int_{-l}^l f(x) dx \\ a_n &= \frac{1}{l} \int_{-l}^l f(x) \cos\left(\frac{n\pi}{l}x\right) dx, \quad n = 1, 2, \dots, \infty \text{ ----- (9)} \\ b_n &= \frac{1}{l} \int_{-l}^l f(x) \sin\left(\frac{n\pi}{l}x\right) dx \quad n = 1, 2, \dots, \infty \end{aligned}$$

Then the right-hand side of (5) is the Fourier expansion of $f(x)$ over the interval $(-l, l)$.

If we set $l = \pi$, then $f(x)$ is defined over the interval $(-\pi, \pi)$. Formulae (9) reduce to

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad n=1,2,\dots,\infty \quad \text{-----} \quad (10) \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \quad n=1,2,\dots,\infty \end{aligned}$$

$$\text{Putting } l = \pi \text{ in (5), we get } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Some useful results:

1. The following rule called Bernoulli's generalized rule of integration by parts is useful in evaluating the Fourier coefficients.

$$\int u v dx = u v_1 - u' v_2 + u'' v_3 + \dots$$

Here $u', u'', \dots$ are the successive derivatives of u and $v_1 = \int v dx, v_2 = \int v_1 dx, \dots$

We illustrate the rule, through the following examples:

$$\begin{aligned} \int x^2 \sin nx dx &= x^2 \left(\frac{-\cos nx}{n} \right) - 2x \left(\frac{-\sin nx}{n^2} \right) + 2 \left(\frac{\cos nx}{n^3} \right) \\ \int x^3 e^{2x} dx &= x^3 \left(\frac{e^{2x}}{2} \right) - 3x^2 \left(\frac{e^{2x}}{4} \right) + 6x \left(\frac{e^{2x}}{8} \right) - 6 \left(\frac{e^{2x}}{16} \right) \end{aligned}$$

2. The following integrals are also useful:

$$\begin{aligned} \int e^{ax} \cos bxdx &= \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx] \\ \int e^{ax} \sin bxdx &= \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx] \end{aligned}$$

3. If 'n' is integer, then

$$\begin{aligned} \sin n\pi &= 0, \quad \cos n\pi = (-1)^n, \quad \sin 2n\pi = 0, \quad \cos 2n\pi = 1 \\ \sin(n+1/2)\pi &= (-1)^n, \quad \cos(n+1/2)\pi = 0 \end{aligned}$$

Examples

1. Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$

$$\text{We have } a_0 = \frac{1}{\pi} \int_0^{2\pi} e^{-x} dx = \frac{e^{-x}}{-1} \Big|_0^{2\pi} = -\frac{1}{\pi} (e^{-2\pi} - 1) = \frac{1 - e^{-2\pi}}{\pi}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_0^{2\pi} e^{-x} \cos nx dx = \frac{1}{\pi} \left[\frac{e^{-x}}{1+n^2} (-\cos nx + n \sin nx) \right]_0^{2\pi} \\
&= \frac{1}{\pi(1+n^2)} \left[e^{-2\pi} (-\cos 2n\pi + n \sin 2n\pi) - e^0 (-\cos 0 + n \sin 0) \right] \\
&= \frac{1}{\pi(1+n^2)} \left[-e^{-2\pi} + 1 \right] = \left(\frac{1-e^{-2\pi}}{\pi} \right) \cdot \frac{1}{1+n^2} \\
b_n &= \frac{1}{\pi} \int_0^{2\pi} e^{-x} \sin nx dx = \frac{1}{\pi} \left[\frac{e^{-x}}{1+n^2} (-\sin nx - n \cos nx) \right]_0^{2\pi} \\
&= \frac{1}{\pi(1+n^2)} \left[e^{-2\pi} (-\sin 2n\pi - n \cos 2n\pi) - e^0 (-\sin 0 - n \cos 0) \right] \\
&= \frac{1}{\pi(1+n^2)} \left[-ne^{-2\pi} + n \right] = \left(\frac{1-e^{-2\pi}}{\pi} \right) \cdot \frac{n}{1+n^2} \\
\therefore e^{-x} &= \frac{1-e^{-2\pi}}{2\pi} + \sum_{n=1}^{\infty} \left(\frac{1-e^{-2\pi}}{\pi} \cdot \frac{1}{1+n^2} \right) \cos nx + \sum_{n=1}^{\infty} \left(\frac{1-e^{-2\pi}}{\pi} \cdot \frac{n}{1+n^2} \right) \sin nx \\
&= \frac{1-e^{-2\pi}}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{1}{1+n^2} \right) \cos nx + \sum_{n=1}^{\infty} \left(\frac{n}{1+n^2} \right) \sin nx \right]
\end{aligned}$$

2. Obtain the Fourier expansion of $f(x) = \frac{1}{2}(\pi - x)$ in $-\pi < x < \pi$

$$\text{We have, } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2}(\pi - x) dx = \frac{1}{2\pi} \left[\pi x - \frac{x^2}{2} \right]_{-\pi}^{\pi} = \pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2}(\pi - x) \cos nx dx$$

Here we use integration by parts, so that

$$a_n = \frac{1}{2\pi} \left[(\pi - x) \frac{\sin nx}{n} - (-1) \left(\frac{-\cos nx}{n^2} \right) \right]_{-\pi}^{\pi} = \frac{1}{2\pi} [0] = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2}(\pi - x) \sin nx dx = \frac{1}{2\pi} \left[(\pi - x) \frac{-\cos nx}{n} - (-1) \left(\frac{-\sin nx}{n^2} \right) \right]_{-\pi}^{\pi} = \frac{(-1)^n}{n}$$

Using the values of a_0 , a_n and b_n in the Fourier expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\text{we get, } f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$$

This is the required Fourier expansion of the given function.

3. Obtain the Fourier expansion of $f(x) = e^{-ax}$ in the interval $(-\pi, \pi)$. Deduce that

$$\frac{\pi}{\sinh \pi} = 2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 + 1}$$

$$\text{Here, } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-ax} dx = \frac{1}{\pi} \left[\frac{e^{-ax}}{-a} \right]_{-\pi}^{\pi} = \frac{e^{a\pi} - e^{-a\pi}}{a\pi} = \frac{2 \sinh a\pi}{a\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-ax} \cos nx dx = \frac{1}{\pi} \left[\frac{e^{-ax}}{a^2 + n^2} \{-a \cos nx + n \sin nx\} \right]_{-\pi}^{\pi} = \frac{2a}{\pi} \left[\frac{(-1)^n \sinh a\pi}{a^2 + n^2} \right]$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-ax} \sin nx dx = \frac{1}{\pi} \left[\frac{e^{-ax}}{a^2 + n^2} \{-a \sin nx - n \cos nx\} \right]_{-\pi}^{\pi} = \frac{2n}{\pi} \left[\frac{(-1)^n \sinh a\pi}{a^2 + n^2} \right]$$

$$\text{Thus, } f(x) = \frac{\sinh a\pi}{a\pi} + \frac{2a \sinh a\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2 + n^2} \cos nx + \frac{2}{\pi} \sinh a\pi \sum_{n=1}^{\infty} \frac{n(-1)^n}{a^2 + n^2} \sin nx$$

For $x = 0, a = 1$, the series reduces to

$$f(0) = 1 = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 1}$$

$$\text{or } 1 = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \left[-\frac{1}{2} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 + 1} \right]$$

$$\text{or } 1 = \frac{2 \sinh \pi}{\pi} \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 + 1}$$

$$\text{Thus, } \frac{\pi}{\sinh \pi} = 2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 + 1}. \text{ This is the desired deduction.}$$

4. Obtain the Fourier expansion of $f(x) = x^2$ over the interval $(-\pi, \pi)$. Deduce that

$$\text{a) } \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots + \infty \quad \text{b) } \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \infty$$

$$\text{c) } \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots + \infty$$

The function $f(x)$ is even. Hence

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx, \text{ since } f(x) \cos nx \text{ is even} \\ &= \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx, \text{ Integrating by parts, we get} \end{aligned}$$

$$= \frac{2}{\pi} \left[x^2 \left(\frac{\sin nx}{n} \right) - 2x \left(\frac{-\cos nx}{n^2} \right) + 2 \left(\frac{-\sin nx}{n^3} \right) \right]_0^\pi = \frac{4(-1)^n}{n^2}$$

Also, $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = 0$, since $f(x) \sin nx$ is odd.

Thus

$$f(x) = x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2} \text{----- (i)}$$

a) By putting $x = 0$, in (i) we get,

$$0 = f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos 0}{n^2} = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\text{i.e. } -\frac{\pi^2}{3} = 4 \left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots \right)$$

$$\text{i.e. } \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

b) by putting $x = \pi$, we get,

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\text{Hence, } \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

c) by adding (a) and (b) we get,

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots + 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

$$\text{i.e. } \frac{3\pi^2}{12} = 2 + \frac{2}{3^2} + \frac{2}{5^2} + \dots \Rightarrow \frac{\pi^2}{4} = 2 \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right)$$

$$\text{Or } \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

Functions having points of discontinuity:

5. Obtain the Fourier expansion of

$$f(x) = \begin{cases} x, 0 \leq x \leq \pi \\ 2\pi - x, \pi \leq x \leq 2\pi \end{cases}$$

Deduce that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

$$\text{Here, } a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \left\{ \int_0^{\pi} f(x) dx + \int_{\pi}^{2\pi} f(x) dx \right\} = \frac{1}{\pi} \left\{ \int_0^{\pi} x dx + \int_{\pi}^{2\pi} (2\pi - x) dx \right\}$$

$$\begin{aligned}
&= \frac{1}{\pi} \left\{ \frac{x^2}{2} \Big|_0^\pi + \left(2\pi x - \frac{x^2}{2} \right) \frac{2\pi}{\pi} \right\} = \frac{1}{\pi} \left\{ \frac{\pi^2}{2} + 4\pi^2 - \frac{4\pi^2}{2} - (2\pi^2 - \frac{\pi^2}{2}) \right\} \\
&= \frac{1}{2\pi} \left\{ \pi^2 + 8\pi^2 - 4\pi^2 - 4\pi^2 + \pi^2 \right\} = \pi \Rightarrow \frac{a_0}{2} = \frac{\pi}{2}.
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx = \frac{1}{\pi} \left\{ \int_0^\pi f(x) \cos nx dx + \int_\pi^{2\pi} f(x) \cos nx dx \right\} \\
&= \frac{1}{\pi} \left\{ \int_0^\pi x \cos nx dx + \int_\pi^{2\pi} (2\pi - x) \cos nx dx \right\}, \text{ integrating using Bernoulli's generalized rule} \\
&= \frac{1}{\pi} \left\{ \left[x \left(\frac{\sin nx}{n} \right) - (1) \left(\frac{-\cos nx}{n^2} \right) \right]_0^\pi + \left[(2\pi - x) \left(\frac{\sin nx}{n} \right) - (-1) \left(\frac{-\cos nx}{n^2} \right) \right]_\pi^{2\pi} \right\} \\
&= \frac{1}{\pi} \left\{ \left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^\pi + \left[(2\pi - x) \left(\frac{\sin nx}{n} \right) - \frac{\cos nx}{n^2} \right]_\pi^{2\pi} \right\} \\
&= \frac{1}{\pi} \left\{ \left(\frac{(-1)^n}{n^2} - \frac{1}{n^2} \right) + \left(0 - \frac{1}{n^2} - \left(0 - \frac{(-1)^n}{n^2} \right) \right) \right\} \\
&= \frac{1}{\pi} \left\{ \frac{2(-1)^n - 2}{n^2} \right\} = \frac{2}{\pi n^2} \{ (-1)^n - 1 \}
\end{aligned}$$

$$\begin{aligned}
\text{Also, } b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx = \frac{1}{\pi} \left\{ \int_0^\pi f(x) \sin nx dx + \int_\pi^{2\pi} f(x) \sin nx dx \right\}, \\
&= \frac{1}{\pi} \left\{ \int_0^\pi x \sin nx dx + \int_\pi^{2\pi} (2\pi - x) \sin nx dx \right\}, \text{ integrating using Bernoulli's generalized rule} \\
&= \frac{1}{\pi} \left\{ \left[x \left(\frac{-\cos nx}{n} \right) - (1) \left(\frac{-\sin nx}{n^2} \right) \right]_0^\pi + \left[(2\pi - x) \left(\frac{-\cos nx}{n} \right) - (-1) \left(\frac{-\sin nx}{n^2} \right) \right]_\pi^{2\pi} \right\} \\
&= \frac{1}{\pi} \left\{ \left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^\pi + \left[-(2\pi - x) \frac{\cos nx}{n} - \frac{\sin nx}{n^2} \right]_\pi^{2\pi} \right\} \\
&= \frac{1}{\pi} \left\{ \frac{-\pi(-1)^n}{n} + 0 - \left(\frac{-\pi(-1)^n}{n} - 0 \right) \right\} = \frac{1}{\pi} \left\{ \frac{-\pi(-1)^n}{n} + \frac{\pi(-1)^n}{n} \right\} = 0
\end{aligned}$$

Thus the Fourier series of $f(x)$ is $f(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} [(-1)^n - 1] \cos nx$

For $x = \pi$, we get

$$f(\pi) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} [(-1)^n - 1] \cos n\pi$$

But $(-1)^n - 1 = 0$, if n is even & $(-1)^n - 1 = -2$, if n is odd.

(since $f(x)$ is discontinuous at $x = \pi$, $f(\pi) = \frac{1}{2}[f(\pi-0) + f(\pi+0)] = \frac{1}{2}(x + 2\pi - x) = \pi$)

$$\therefore \pi = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{-2 \cos(2n-1)\pi}{(2n-1)^2}$$

$$\text{Thus, } \frac{\pi}{2} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{-2(-1)}{(2n-1)^2} \Rightarrow \frac{\pi^2}{4} = \sum_{n=1}^{\infty} \frac{2}{(2n-1)^2} \Rightarrow \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

This is the series as required.

6. Obtain the Fourier expansion of

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases} \quad \text{hence deduce that } \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

$$\begin{aligned} \text{Here, } a_0 &= \frac{1}{\pi} \left[\int_{-\pi}^0 -\pi dx + \int_0^{\pi} x dx \right] = \frac{1}{\pi} \left\{ [-\pi x]_{-\pi}^0 + \left[\frac{x^2}{2} \right]_0^{\pi} \right\} = \frac{1}{\pi} \left[-\pi(0 - (-\pi)) + \frac{1}{2}(\pi^2 - 0) \right] \\ &= \frac{1}{\pi} \left(-\pi^2 + \frac{\pi^2}{2} \right) = -\frac{1}{\pi} \times \frac{\pi^2}{2} = -\frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \left[\int_{-\pi}^0 -\pi \cos nx dx + \int_0^{\pi} x \cos nx dx \right] \\ &= \frac{1}{\pi} \left[\left(-\pi \frac{\sin nx}{n} \right)_{-\pi}^0 + \left(x \frac{\sin nx}{n} - (1) \frac{-\cos nx}{n^2} \right)_{\pi}^0 \right] \\ &= \frac{1}{\pi} \left[0 + 0 + \frac{(-1)^n}{n^2} - \frac{1}{n^2} \right] = \frac{1}{\pi n^2} [(-1)^n - 1] \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \left[\int_{-\pi}^0 -\pi \sin nx dx + \int_0^{\pi} x \sin nx dx \right] = \\ &= \frac{1}{\pi} \left[\left(-\pi \frac{-\cos nx}{n} \right)_{-\pi}^0 + \left\{ x \left(\frac{-\cos nx}{n} \right) - (1) \left(\frac{-\sin nx}{n^2} \right) \right\}_{\pi}^0 \right] \\ &= \frac{1}{\pi} \left[\frac{\pi}{n} (\cos 0 - (-1)^n) - \frac{\pi}{n} (-1)^n + 0 \right] = \frac{1}{\pi} \left[\frac{\pi}{n} (1 - 2(-1)^n) \right] = \frac{1}{n} [1 - 2(-1)^n] \end{aligned}$$

$\therefore$ Fourier series is

$$f(x) = \frac{-\pi}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} [(-1)^n - 1] \cos nx + \sum_{n=1}^{\infty} \left[\frac{1 - 2(-1)^n}{n} \right] \sin nx$$

Note that the point $x = 0$ is a point of discontinuity of $f(x)$. Here $f(x^+) = 0$, $f(x^-) = -\pi$ at $x = 0$. Hence $\frac{1}{2}[f(x^+) + f(x^-)] = \frac{1}{2}(0 - \pi) = \frac{-\pi}{2}$

The Fourier expansion of $f(x)$ at $x = 0$ becomes

$$\frac{-\pi}{2} = \frac{-\pi}{4} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} [(-1)^n - 1]$$

$$\text{or } \frac{\pi^2}{4} = \sum_{n=1}^{\infty} \frac{1}{n^2} [(-1)^n - 1]$$

But $(-1)^n - 1 = 0$, if n is even & $(-1)^n - 1 = -2$, if n is odd.

$$\therefore \frac{\pi^2}{4} = \sum_{n=1}^{\infty} \frac{2}{(2n-1)^2} \Rightarrow \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

$$\text{Hence } \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

7. Obtain the Fourier series of $f(x) = 1 - x^2$ over the interval $(-1, 1)$.

The given function is even as $f(-x) = f(x)$. Also period of $f(x)$ is $1 - (-1) = 2 \Rightarrow 2l = 2 \Rightarrow l = 1$

$$\text{Here } a_0 = \frac{1}{1} \int_{-1}^1 f(x) dx = 2 \int_0^1 f(x) dx = 2 \int_0^1 (1 - x^2) dx = 2 \left[x - \frac{x^3}{3} \right]_0^1 = \frac{4}{3} \text{ as } f(x) \text{ is even.}$$

$$a_n = \frac{1}{1} \int_{-1}^1 f(x) \cos(n\pi x) dx = 2 \int_0^1 f(x) \cos(n\pi x) dx$$

$$= 2 \int_0^1 (1 - x^2) \cos(n\pi x) dx \text{ [as } f(x) \cos n\pi x \text{ is even. Integrating by parts, we get]}$$

$$= 2 \left[\left(1 - x^2 \right) \left(\frac{\sin n\pi x}{n\pi} \right) - (-2x) \left(\frac{-\cos n\pi x}{(n\pi)^2} \right) + (-2) \left(\frac{-\sin n\pi x}{(n\pi)^3} \right) \right]_0^1$$

$$= 2 \left[[0 - 0 - \frac{2}{(n\pi)^2} [(-1)^n - 0] + \frac{2}{(n\pi)^3} (0)] \right] = \frac{4(-1)^{n+1}}{n^2 \pi^2}$$

$$b_n = \frac{1}{1} \int_{-1}^1 f(x) \sin(n\pi x) dx = 0, \text{ since } f(x) \sin(n\pi x) \text{ is odd.}$$

$$\text{The Fourier series of } f(x) \text{ is } f(x) = \frac{2}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos(n\pi x)$$

8. Obtain the Fourier expansion of

$$f(x) = \begin{cases} 1 + \frac{4x}{3} & -\frac{3}{2} < x \leq 0 \\ 1 - \frac{4x}{3} & 0 \leq x < \frac{3}{2} \end{cases} \quad \text{Hence deduce that } \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

$$\text{The period of } f(x) \text{ is } \frac{3}{2} - \left(-\frac{3}{2} \right) = 3 \Rightarrow 2l = 3 \Rightarrow l = 3/2$$

Also $f(-x) = f(x)$. Hence $f(x)$ is even

$$a_0 = \frac{1}{3/2} \int_{-3/2}^{3/2} f(x) dx = \frac{2}{3/2} \int_0^{3/2} f(x) dx = \frac{4}{3} \int_0^{3/2} \left(1 - \frac{4x}{3}\right) dx = 0$$

$$= \frac{4}{9} \int_0^{3/2} (3 - 4x) dx = \frac{4}{9} \left[3x - \frac{4x^2}{2} \right]_0^{3/2} = \frac{4}{9} \left[3x - 2x^2 \right]_0^{3/2} = \frac{4}{9} \left[\frac{9}{2} - 2 \times \frac{9}{4} \right] = 0$$

$$a_n = \frac{1}{3/2} \int_{-3/2}^{3/2} f(x) \cos\left(\frac{n\pi x}{3/2}\right) dx = \frac{2}{3/2} \int_0^{3/2} f(x) \cos\left(\frac{2n\pi x}{3}\right) dx$$

$$= \frac{4}{3} \int_0^{3/2} \left(1 - \frac{4x}{3}\right) \cos\left(\frac{2n\pi x}{3}\right) dx$$

$$= \frac{4}{3} \left[\left(1 - \frac{4x}{3}\right) \left(\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)} \right) - \left(-\frac{4}{3}\right) \left(\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2} \right) \right]_0^{3/2}$$

$$= \frac{4}{3} \left[0 - \frac{4}{3} \times \frac{9}{4n^2\pi^2} ((-1)^n - 1) \right] = \frac{4}{n^2\pi^2} [1 - (-1)^n] = \frac{8}{n^2\pi^2}, n = 1, 3, 5, \dots$$

Also, $b_n = \frac{1}{3/2} \int_{-3/2}^{3/2} f(x) \sin\left(\frac{n\pi x}{3/2}\right) dx = 0$, as $f(x) \sin\left(\frac{n\pi x}{3/2}\right)$ is an odd function.

Thus $f(x) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{2n\pi x}{3}\right), n = 1, 3, 5, \dots$

putting $x=0$, we get $f(0) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$

or $1 = \frac{8}{\pi^2} \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$

Thus, $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

Exercises: 1) Obtain the Fourier series of $f(x) = \frac{\pi-x}{2}$ in $0 < x < 2\pi$. Hence deduce

that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$

2) If $f(x) = x(2\pi - x)$ in $0 \leq x \leq 2\pi$ show that

$$f(x) = \frac{2\pi^2}{3} - 4 \left(\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \dots \right) \text{ hence deduce that}$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

3) Find a Fourier series in $(-\pi, \pi)$ to represent a) $f(x) = x - x^2$ b) $f(x) = x + x^2$

Hence deduce that

$$a) \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots + \infty$$

$$b) \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \infty$$

$$c) \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots + \infty$$

4) Obtain the Fourier series for the function $f(x) = \begin{cases} 0 & \text{in } -\pi \leq x \leq 0 \\ \sin x & \text{in } 0 \leq x \leq \pi \end{cases}$

Hence deduce that

$$1. \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots = \frac{1}{2} \quad 2. \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} + \dots = \frac{\pi - 2}{4}$$

5) If $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the range 0 to 2π , show that $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$

6) An alternating current after passing through a rectifier has the form

$$i = \begin{cases} I_0 \sin x & \text{for } 0 \leq x \leq \pi \\ 0 & \text{for } \pi \leq x \leq 2\pi \end{cases}$$

Where I_0 is the maximum current and the period is 2π . Express i as a Fourier series.

7) Obtain the Fourier series for the function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 < x < \pi \end{cases} \quad \text{Where } f(x + 2\pi) = f(x)$$

8) Obtain the Fourier series for the function

$$f(x) = \begin{cases} x^2, & 0 \leq x \leq \pi \\ -x^2, & -\pi \leq x \leq 0 \end{cases}$$

9) Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-l, l)$

10) If $f(x) = \begin{cases} \pi x, & 0 \leq x \leq 1 \\ \pi(2 - x), & 1 \leq x \leq 2 \end{cases}$; show that in the interval $(0, 2)$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

Answers: 1) $f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$

$$3) a) x - x^2 = \frac{-\pi^2}{3} + 4 \sum_1^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx + 2 \sum_1^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

$$b) x - x^2 = \frac{\pi^2}{3} + 4 \sum_1^{\infty} \frac{(-1)^n}{n^2} \cos nx + 2 \sum_1^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

$$4) f(x) = \frac{1}{\pi} + \sum_2^{\infty} \frac{-1}{\pi(n^2 - 1)} \{1 + (-1)^n\} \cos nx + \frac{1}{2} \sin x.$$

$$6) i = \frac{I_0}{\pi} + \frac{1}{2} I_0 \sin x - \frac{2I_0}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}$$

$$7) f(x) = \frac{\pi^2}{6} - 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos nx}{n^2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{2}{n^3} - \frac{\pi^2}{n} \right) \sin nx$$

$$8) f(x) = 2 \left(\pi - \frac{4}{\pi} \right) \sin x - \pi \sin 2x + \frac{2}{3} \left(\pi - \frac{4}{9\pi} \right) \sin 3x - \frac{\pi}{2} \sin 4x + \dots$$

$$9) e^{-x} = \sinh l \left\{ \frac{1}{l} - 2l \sum_{n=1}^{\infty} \frac{1}{l^2 + n^2 \pi^2} \cos \frac{n\pi x}{l} - 2\pi \sum_{n=1}^{\infty} \frac{n}{l^2 + n^2 \pi^2} \sin \frac{n\pi x}{l} \right\}$$

HALF-RANGE FOURIER SERIES

The Fourier expansion of the periodic function $f(x)$ of period $2l$ may contain both sine and cosine terms. Many a time it is required to obtain the Fourier expansion of $f(x)$ in the interval $(0, l)$ which is regarded as half interval. The definition can be extended to the other half in such a manner that the function becomes even or odd. This will result in cosine series or sine series only.

Sine series :

Suppose $f(x) = \phi(x)$ is given in the interval $(0, l)$. Then we define $f(x) = -\phi(-x)$ in $(-l, 0)$. Hence $f(x)$ becomes an odd function in $(-l, l)$. The Fourier series then is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \left(\frac{n\pi x}{l} \right) \quad (11)$$

where

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \left(\frac{n\pi x}{l} \right) dx$$

The series (11) is called half-range sine series over $(0, l)$.

Putting $l=\pi$ in (11), we obtain the half-range sine series of $f(x)$ over $(0, \pi)$ given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$$

Cosine series :

Let us define

$$f(x) = \begin{cases} \phi(x) & \text{in } (0, l) \dots \dots \text{given} \\ \phi(-x) & \text{in } (-l, 0) \dots \dots \text{in order to make the function even} \end{cases}$$

Then the Fourier series of f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad (12)$$

where,

$$a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

The series (12) is called half-range cosine series over (0,l)

Putting $l = \pi$ in (12), we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx \quad n = 1, 2, 3, \dots$$

Examples :

1. Expand $f(x) = x(\pi - x)$ as half-range sine series over the interval $(0, \pi)$.

We have,

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \sin nx dx$$

Integrating by parts, we get

$$b_n = \frac{2}{\pi} \left[\left(\pi x - x^2 \right) \left(\frac{-\cos nx}{n} \right) - (\pi - 2x) \left(\frac{-\sin nx}{n^2} \right) + (-2) \left(\frac{\cos nx}{n^3} \right) \right]_0^\pi$$

$$= \frac{4}{n^3 \pi} [1 - (-1)^n]$$

The sine series of f(x) is

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^3} [1 - (-1)^n] \sin nx$$

2. Obtain the cosine series of

$$f(x) = \begin{cases} x, 0 < x < \frac{\pi}{2} \\ \pi - x, \frac{\pi}{2} < x < \pi \end{cases} \quad \text{over}(0, \pi)$$

Here

$$a_0 = \frac{2}{\pi} \left[\int_0^{\pi/2} x dx + \int_{\pi/2}^{\pi} (\pi - x) dx \right] = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi} \left[\int_0^{\pi/2} x \cos nx dx + \int_{\pi/2}^{\pi} (\pi - x) \cos nx dx \right]$$

Performing integration by parts and simplifying, we get

$$a_n = -\frac{2}{n^2 \pi} \left[1 + (-1)^n - 2 \cos \left(\frac{n\pi}{2} \right) \right]$$

$$= -\frac{8}{n^2 \pi}, n = 2, 6, 10, \dots$$

Thus, the Fourier cosine series is

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi} \left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \dots \infty \right]$$

3. Obtain the half-range cosine series of f(x) = c-x in 0 < x < c

$$\text{Here } a_0 = \frac{2}{c} \int_0^c (c-x) dx = c, \quad a_n = \frac{2}{c} \int_0^c (c-x) \cos \left(\frac{n\pi x}{c} \right) dx$$

$$\text{Integrating by parts and simplifying we get, } a_n = \frac{2c}{n^2 \pi^2} [1 - (-1)^n]$$

The cosine series is given by

$$f(x) = \frac{c}{2} + \frac{2c}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} [1 - (-1)^n] \cos \left(\frac{n\pi x}{c} \right)$$

Exercises:

I. Obtain the half-range sine series of the following functions over the specified intervals:

1. $f(x) = \cos x$ over $(0, \pi)$ 2. $f(x) = \sin^3 x$ over $(0, \pi)$ 3. $f(x) = lx - x^2$ over $(0, l)$

II. Obtain the half-range cosine series of the following functions over the specified intervals:

1. $f(x) = x^2$ over $(0, \pi)$ 2. $f(x) = x \sin x$ over $(0, \pi)$ 3. $f(x) = (x-1)^2$ over $(0, 1)$

$$4. f(x) = \begin{cases} kx, 0 \leq x \leq \frac{l}{2} \\ k(l-x), \frac{l}{2} \leq x \leq l \end{cases}$$

HARMONIC ANALYSIS

The Fourier series of a **known** function $f(x)$ in a given interval may be found by finding the Fourier coefficients. The method described cannot be employed when $f(x)$ is not known explicitly, but defined through the values of the function at some equidistant points. In such a case, the integrals in Euler's formulae cannot be evaluated. Harmonic analysis is the process of finding the Fourier coefficients numerically.

To derive the relevant formulae for Fourier coefficients in Harmonic analysis, we employ the following result:

The mean value of a continuous function $f(x)$ over the interval (a, b) denoted by $[f(x)]$ is defined

as
$$[f(x)] = \frac{1}{b-a} \int_a^b f(x) dx.$$

The Fourier coefficients defined through Euler's formulae, (1), (2), (3) may be redefined as

$$a_0 = 2 \left[\frac{1}{2l} \int_a^{a+2l} f(x) dx \right] = 2[f(x)]$$

$$a_n = 2 \left[\frac{1}{2l} \int_a^{a+2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx \right] = 2 \left[f(x) \cos\left(\frac{n\pi x}{l}\right) \right]$$

$$b_n = 2 \left[\frac{1}{2l} \int_a^{a+2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx \right] = 2 \left[f(x) \sin\left(\frac{n\pi x}{l}\right) \right]$$

Using these in (5), we obtain the Fourier series of $f(x)$. The term $a_1 \cos x + b_1 \sin x$ is called the first harmonic or fundamental harmonic, the term $a_2 \cos 2x + b_2 \sin 2x$ is called the second harmonic and so on. The amplitude of the first harmonic is $\sqrt{a_1^2 + b_1^2}$ and that of second harmonic is $\sqrt{a_2^2 + b_2^2}$ and so on.

Examples

1. Find the first two harmonics of the Fourier series of $f(x)$ given the following table:

X	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
f(x)	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Note that the values of $y = f(x)$ are spread over the interval $0 \leq x \leq 2\pi$ and $f(0) = f(2\pi) = 1.0$.

Hence the function is periodic and so we omit the last value $f(2\pi) = 0$. We prepare the following table to compute the first two harmonics.

x°	$y = f(x)$	$\cos x$	$\cos 2x$	$\sin x$	$\sin 2x$	$y \cos x$	$y \cos 2x$	$y \sin x$	$y \sin 2x$
0°	1.0	1	1	0	0	1	1	0	0
60°	1.4	0.5	-0.5	0.866	0.866	0.7	-0.7	1.2124	1.2124
120°	1.9	-0.5	-0.5	0.866	-0.866	-0.95	-0.95	1.6454	-1.6454
180°	1.7	-1	1	0	0	-1.7	1.7	0	0
240°	1.5	-0.5	-0.5	-0.866	0.866	-0.75	-0.75	-1.299	1.299
300°	1.2	0.5	-0.5	-0.866	-0.866	0.6	-0.6	-1.0392	-1.0392
Total						-1.1	-0.3	0.519	-0.1732

$$\text{We have } a_n = 2 \left[f(x) \cos \left(\frac{n\pi x}{l} \right) \right] = 2[y \cos nx] \quad b_n = 2 \left[f(x) \sin \left(\frac{n\pi x}{l} \right) \right] = 2[y \sin nx]$$

as the length of interval $= 2l = 2\pi$ or $l = \pi$

Putting, $n = 1, 2$, we get

$$a_1 = 2[y \cos x] = \frac{2 \sum y \cos x}{6} = \frac{2(-1.1)}{6} = -0.367$$

$$a_2 = 2[y \cos 2x] = \frac{2 \sum y \cos 2x}{6} = \frac{2(-0.3)}{6} = -0.1$$

$$b_1 = 2[y \sin x] = \frac{2 \sum y \sin x}{6} = 0.173$$

$$b_2 = 2[y \sin 2x] = \frac{2 \sum y \sin 2x}{6} = -0.0577$$

The first two harmonics are $a_1 \cos x + b_1 \sin x$ and $a_2 \cos 2x + b_2 \sin 2x$. That is

$(-0.367 \cos x + 1.0392 \sin x)$ and $(-0.1 \cos 2x - 0.0577 \sin 2x)$

2. Express y as a Fourier series up to the third harmonic given the following values:

X	0	1	2	3	4	5
Y	4	8	15	7	6	2

The values of y at $x = 0, 1, 2, 3, 4, 5$ are given and hence the interval of x should be

$0 \leq x < 6$. The length of the interval $= 6 - 0 = 6$, so that $2l = 6$ or $l = 3$.

The Fourier series up to the third harmonic is

$$y = \frac{a_0}{2} + \left(a_1 \cos \frac{\pi x}{l} + b_1 \sin \frac{\pi x}{l} \right) + \left(a_2 \cos \frac{2\pi x}{l} + b_2 \sin \frac{2\pi x}{l} \right) + \left(a_3 \cos \frac{3\pi x}{l} + b_3 \sin \frac{3\pi x}{l} \right) \text{ or}$$

$$y = \frac{a_0}{2} + \left(a_1 \cos \frac{\pi x}{3} + b_1 \sin \frac{\pi x}{3} \right) + \left(a_2 \cos \frac{2\pi x}{3} + b_2 \sin \frac{2\pi x}{3} \right) + \left(a_3 \cos \frac{3\pi x}{3} + b_3 \sin \frac{3\pi x}{3} \right)$$

Put $\theta = \frac{\pi x}{3}$, then

$$y = \frac{a_0}{2} + (a_1 \cos \theta + b_1 \sin \theta) + (a_2 \cos 2\theta + b_2 \sin 2\theta) + (a_3 \cos 3\theta + b_3 \sin 3\theta) \quad (1)$$

We prepare the following table using the given values:

x	$\theta = \frac{\pi x}{3}$	y	y cos θ	y cos 2 θ	y cos 3 θ	y sin θ	y sin 2 θ	y sin 3 θ
0	0	04	4	4	4	0	0	0
1	60°	08	4	-4	-8	6.928	6.928	0
2	120°	15	-7.5	-7.5	15	12.99	-12.99	0
3	180°	07	-7	7	-7	0	0	0
4	240°	06	-3	-3	6	-5.196	5.196	0
5	300°	02	1	-1	-2	-1.732	-1.732	0
Total		42	-8.5	-4.5	8	12.99	-2.598	0

$$a_0 = 2[f(x)] = 2[y] = \frac{2\sum y}{6} = \frac{1}{3}(42) = 14, \quad a_1 = 2[y \cos \theta] = \frac{2}{6}(-8.5) = -2.833,$$

$$b_1 = 2[y \sin \theta] = \frac{2}{6}(12.99) = 4.33, \quad a_2 = 2[y \cos 2\theta] = \frac{2}{6}(-4.5) = -1.5,$$

$$b_2 = 2[y \sin 2\theta] = \frac{2}{6}(-2.598) = -0.866, \quad a_3 = 2[y \cos 3\theta] = \frac{2}{6}(8) = 2.667, \quad b_3 = 2[y \sin 3\theta] = 0$$

Using these in (1), we get

$$y = 7 - 2.833 \cos \left(\frac{\pi x}{3} \right) + (4.33) \sin \left(\frac{\pi x}{3} \right) - 1.5 \cos \left(\frac{2\pi x}{3} \right) - 0.866 \sin \left(\frac{2\pi x}{3} \right) + 2.667 \cos \pi x$$

This is the required Fourier series up to the third harmonic.

3. The following table gives the variations of a periodic current A over a period T:

T (secs)	0	T/6	T/3	T/2	2T/3	5T/6	T
A (amp)	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Show that there is a constant part of 0.75amp. in the current A and obtain the amplitude of the first harmonic.

Note that the values of A at $t=0$ and $t=T$ are the same. Hence A (t) is a periodic function of period T. Let us denote $\theta = \left(\frac{2\pi}{T}\right)t$. We have

$$a_0 = 2[A], \quad a_1 = 2\left[A \cos\left(\frac{2\pi}{T}t\right)\right] = 2[A \cos \theta], \quad b_1 = 2\left[A \sin\left(\frac{2\pi}{T}t\right)\right] = 2[A \sin \theta] \text{----- (1)}$$

We prepare the following table:

t	$\theta = \frac{2\pi t}{T}$	A	cos θ	sin θ	A cos θ	A sin θ
0	0	1.98	1	0	1.98	0
T/6	60 ⁰	1.30	0.5	0.866	0.65	1.1258
T/3	120 ⁰	1.05	-0.5	0.866	-0.525	0.9093
T/2	180 ⁰	1.30	-1	0	-1.30	0
2T/3	240 ⁰	-0.88	-0.5	-0.866	0.44	0.7621
5T/6	300 ⁰	-0.25	0.5	-0.866	-0.125	0.2165
Total		4.5			1.12	3.0137

Using the values of the table in (1), we get

$$a_0 = \frac{2\sum A}{6} = \frac{4.5}{3} = 1.5, \quad a_1 = \frac{2\sum A \cos \theta}{6} = \frac{1.12}{3} = 0.3733, \quad b_1 = \frac{2\sum A \sin \theta}{6} = \frac{3.0137}{3} = 1.0046$$

The Fourier expansion up to the first harmonic is

$$A = \frac{a_0}{2} + a_1 \cos\left(\frac{2\pi t}{T}\right) + b_1 \sin\left(\frac{2\pi t}{T}\right) = 0.75 + 0.3733 \cos\left(\frac{2\pi t}{T}\right) + 1.0046 \sin\left(\frac{2\pi t}{T}\right)$$

The expression shows that A has a constant part 0.75 in it. Also the amplitude of the first harmonic is $\sqrt{a_1^2 + b_1^2} = 1.0717$.

ASSIGNMENT:

1. The displacement y of a part of a mechanism is tabulated with corresponding angular movement x^0 of the crank. Express y as a Fourier series up to the third harmonic.

x^0	0	30	60	90	120	150	180	210	240	270	300	330
y	1.80	1.10	0.30	0.16	1.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00

2. Obtain the Fourier series of y up to the second harmonic using the following table:

x°	45	90	135	180	225	270	315	360
y	4.0	3.8	2.4	2.0	-1.5	0	2.8	3.4

3. Obtain the constant term and the coefficients of the first sine and cosine terms in the Fourier expansion of y as given in the following table:

x	0	1	2	3	4	5
y	9	18	24	28	26	20

4. Find the Fourier series of y up to the second harmonic from the following table:

x	0	2	4	6	8	10	12
Y	9.0	18.2	24.4	27.8	27.5	22.0	9.0

5. Obtain the first three coefficients in the Fourier cosine series for y, where y is given in the following table:

x	0	1	2	3	4	5
y	4	8	15	7	6	2

6. The turning moment T is given for a series of values of the crank angle $\theta^\circ = 75^\circ$.

θ°	0	30	60	90	120	150	180
T	0	5224	8097	7850	5499	2626	0

Obtain the first four terms in a series of sines to represent T and calculate T at $\theta = 75^\circ$.

Module-3

Fourier transforms & Z-transforms

Introduction

Fourier Transform is a technique employed to solve ODE's, PDE's, IVP's, BVP's and Integral equations.

Infinite Fourier Transform

Let $f(x)$ be a real valued, differentiable function that satisfies the following conditions:

- 1) $f(x)$ and its derivative $f'(x)$ are continuous, or have only a finite number of simple discontinuities in every finite interval, and
- 2) The integral $\int_{-\infty}^{\infty} |f(x)| dx$ exists.

Also, let s be non – zero real parameter. Then infinite Fourier Transform of $f(x)$ denoted by $\hat{f}(s)$ or $F[f(x)]$ or $F(s)$ is defined by

$$\hat{f}(s) = F(s) = F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{isx} dx, \text{ provided the integral exists.}$$

The infinite Fourier Transform is also called complex Fourier Transform or just the Fourier Transform. The Inverse Fourier Transform of $\hat{f}(s)$ denoted by $F^{-1}[\hat{f}(s)]$ is defined by

$$F^{-1}[\hat{f}(s)] = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(s)e^{-isx} ds$$

Note: The function $f(x)$ is said to be self reciprocal with respect to Fourier transform

$$\text{if } \hat{f}(s) = f(s).$$

Basic Properties:

Below we prove some basic properties of Fourier Transforms:

1. Linearity Property

For any two functions $f(x)$ and $\phi(x)$ (whose Fourier Transforms exist) and any two constants a and b , $F[af(x) + b\phi(x)] = aF[f(x)] + bF[\phi(x)]$

Proof: By definition, we have

$$\begin{aligned} F[af(x) + b\phi(x)] &= \int_{-\infty}^{\infty} [af(x) + b\phi(x)]e^{isx} dx = a \int_{-\infty}^{\infty} f(x)e^{isx} dx + b \int_{-\infty}^{\infty} \phi(x)e^{isx} dx \\ &= aF[f(x)] + bF[\phi(x)]. \text{ This is the desired property.} \end{aligned}$$

In particular, if $a = b = 1$, we get $F[f(x) + \phi(x)] = F[f(x)] + F[\phi(x)]$

Again if $a = -b = 1$, we get $F[f(x) - \phi(x)] = F[f(x)] - F[\phi(x)]$

2. Change of Scale Property

If $\hat{f}(s) = F[f(x)]$, then for any non-zero constant a , we have $F[f(ax)] = \frac{1}{a} \hat{f}\left(\frac{s}{a}\right)$, $a \neq 0$

Proof: By definition, we have $F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$\begin{aligned} \therefore F[f(ax)] &= \int_{-\infty}^{\infty} f(ax) e^{isx} dx \quad \text{put } ax = t \Rightarrow adx = dt \Rightarrow dx = \frac{dt}{a} \\ &= \int_{-\infty}^{\infty} f(t) e^{is\left(\frac{t}{a}\right)} \frac{dt}{a} = \frac{1}{a} \int_{-\infty}^{\infty} f(t) e^{i(s/a)t} dt = \frac{1}{a} \hat{f}\left(\frac{s}{a}\right) \end{aligned}$$

3. Shifting Properties:

For any real constant 'a', (i) $F[f(x-a)] = e^{isa} \hat{f}(s)$ (ii) $F[e^{iax} f(x)] = \hat{f}(s+a)$, Where $\hat{f}(s) = F[f(x)]$

Proof: (i) We have $F[f(x)] = \hat{f}(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx$

Hence, $F[f(x-a)] = \int_{-\infty}^{\infty} f(x-a) e^{isx} dx$, set $x-a = t \Rightarrow dx = dt$, then

$$F[f(x-a)] = \int_{-\infty}^{\infty} f(t) e^{is(t+a)} dt = e^{ias} \int_{-\infty}^{\infty} f(t) e^{ist} dt = e^{ias} \hat{f}(s)$$

$$\text{ii) We have } \hat{f}(s+a) = \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} dx = \int_{-\infty}^{\infty} [f(x) e^{iax}] e^{isx} dx$$

$$= \int_{-\infty}^{\infty} g(x) e^{isx} dx, \text{ where } g(x) = f(x) e^{iax}$$

$$= F[g(x)] = F[e^{iax} f(x)] \text{ This is the desired result.}$$

4. **Modulation Property:** If $F[f(x)] = \hat{f}(s)$, then $F[f(x) \cos ax] = \frac{1}{2} [\hat{f}(s+a) + \hat{f}(s-a)]$

where 'a' is a real constant.

Proof: We have $\cos ax = \frac{e^{iax} + e^{-iax}}{2}$

$$\begin{aligned} \text{Hence } F[f(x) \cos ax] &= F\left[f(x) \left(\frac{e^{iax} + e^{-iax}}{2}\right)\right] \\ &= \frac{1}{2} F[f(x) e^{iax} + f(x) e^{-iax}] = \frac{1}{2} [F[f(x) e^{iax}] + F[f(x) e^{-iax}]] \end{aligned}$$

$$= \frac{1}{2} [\hat{f}(s+a) + \hat{f}(s-a)] \text{ by using linearity and shift properties.}$$

This is desired property.

Note: Similarly $F[f(x)\sin ax] = \frac{1}{2} [\hat{f}(s+a) - \hat{f}(s-a)]$

Examples

1. Find the Fourier transform of $f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$ hence evaluate $\int_0^{\infty} \frac{\sin x}{x} dx$

<< The Fourier transform of f(x) is given by

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(x)e^{isx} dx = \int_{-\infty}^{-1} 0 \cdot e^{isx} dx + \int_{-1}^1 1 \cdot e^{isx} dx + \int_1^{\infty} 0 \cdot e^{isx} dx = \frac{e^{isx}}{is} \Big|_{-1}^1 = \frac{e^{is} - e^{-is}}{is} = \frac{2 \sin s}{s}, s \neq 0$$

& for $s = 0$, $\hat{f}(s) = 2$.

Now by the inverse Fourier transform we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(s)e^{-isx} ds \text{ Or } \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin s}{s} e^{-isx} ds = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

By putting $x = 0$ we get,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin s}{s} ds = 1 \Rightarrow \int_{-\infty}^{\infty} \frac{\sin s}{s} ds = \pi \Rightarrow 2 \int_0^{\infty} \frac{\sin s}{s} ds = \pi, \text{ as the integrand is even}$$

$$\Rightarrow \int_0^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2} \text{ Or } \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

2. Find the Fourier transform of $f(x) = \begin{cases} 1-x^2 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases}$ Hence evaluate

$$\int_0^{\infty} \frac{x \cos x - \sin x}{x^3} \cos \frac{x}{2} dx$$

<< The Fourier transform of f(x) is given by

$$\begin{aligned} \hat{f}(s) &= \int_{-\infty}^{\infty} f(x)e^{isx} dx = \int_{-\infty}^{-1} 0 \cdot e^{isx} dx + \int_{-1}^1 (1-x^2) \cdot e^{isx} dx + \int_1^{\infty} 0 \cdot e^{isx} dx \\ &= (1-x^2) \frac{e^{isx}}{is} \Big|_{-1}^1 - (-2x) \frac{e^{isx}}{(is)^2} \Big|_{-1}^1 + (-2) \frac{e^{isx}}{(is)^3} \Big|_{-1}^1 \\ &= 0 + \frac{2}{-s^2} (e^{is} + e^{-is}) - \frac{2}{-is^3} (e^{is} - e^{-is}) = \frac{-4 \cos s}{s^2} + \frac{4 \sin s}{s^3} = -\frac{4}{s^3} (s \cos s - \sin s) \end{aligned}$$

Now by inversion formula, we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(s)e^{-isx} ds$$

$$\text{Or } \frac{1}{2\pi} \int_{-\infty}^{\infty} -\frac{4}{s^3} (s \cos s - \sin s) e^{-isx} ds = \begin{cases} 1-x^2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

Putting $x = 1/2$, we get

$$-\frac{4}{2\pi} \int_{-\infty}^{\infty} \frac{1}{s^3} (s \cos s - \sin s) e^{-is/2} ds = \frac{3}{4}$$

$$\text{Or } \int_{-\infty}^{\infty} \frac{s \cos s - \sin s}{s^3} \left(\cos \frac{s}{2} - i \sin \frac{s}{2} \right) ds = -\frac{3\pi}{8}$$

$$\text{Or } \int_{-\infty}^{\infty} \frac{s \cos s - \sin s}{s^3} \cos \frac{s}{2} ds = -\frac{3\pi}{8} \Rightarrow \int_0^{\infty} \frac{s \cos s - \sin s}{s^3} \cos \frac{s}{2} ds = -\frac{3\pi}{16}, \text{ since the integrand is even}$$

$$\int_0^{\infty} \frac{x \cos x - \sin x}{x^3} \cdot \cos \frac{x}{2} dx = -\frac{3\pi}{16}$$

3. Find the Fourier Transform of the function $f(x) = e^{-a|x|}$ where $a > 0$

The Fourier transform of $f(x)$ is given by

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx = \int_{-\infty}^{\infty} e^{-a|x|} e^{isx} dx = \left[\int_{-\infty}^0 e^{-a|x|} e^{isx} dx + \int_0^{\infty} e^{-a|x|} e^{isx} dx \right]$$

Using the fact that $|x| = x, 0 \leq x < \infty$ & $|x| = -x, -\infty < x \leq 0$, we get

$$\begin{aligned} \hat{f}(s) &= \left[\int_{-\infty}^0 e^{ax} e^{isx} dx + \int_0^{\infty} e^{-ax} e^{isx} dx \right] = \left[\int_{-\infty}^0 e^{(a+is)x} dx + \int_0^{\infty} e^{-(a-is)x} dx \right] \\ &= \left[\left\{ \frac{e^{(a+is)x}}{(a+is)} \right\}_{-\infty}^0 + \left\{ \frac{e^{-(a-is)x}}{-(a-is)} \right\}_0^{\infty} \right] = \left[\frac{1}{(a+is)} + \frac{1}{(a-is)} \right] = \left[\frac{2a}{a^2 + s^2} \right] \end{aligned}$$

Exercises:

4. Find the Fourier Transform of the function $f(x) = \begin{cases} 1, & |x| \leq a \\ 0, & |x| > a \end{cases}$

where 'a' is a positive constant. Hence evaluate

$$(i) \int_{-\infty}^{\infty} \frac{\sin s a \cos s x}{s} ds \quad (ii) \int_0^{\infty} \frac{\sin s}{s} ds$$

<< For the given function, we have

$$\begin{aligned} F[f(x)] &= \left[\int_{-\infty}^{\infty} f(x) e^{isx} dx \right] \\ &= \left[\int_{-a}^a f(x) e^{isx} dx + \int_{-\infty}^{-a} f(x) e^{isx} dx + \int_a^{\infty} f(x) e^{isx} dx \right] \\ &= \left[\int_{-a}^a e^{isx} dx \right] = 2 \left[\frac{\sin sa}{s} \right] \end{aligned}$$

$$\text{Thus } F[f(x)] = \hat{f}(s) = 2 \left(\frac{\sin s a}{s} \right) \dots\dots\dots(1)$$

Inverting $\hat{f}(s)$ by employing inversion formula, we get

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2 \left[\frac{\sin sa}{s} \right] e^{-isx} ds \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin sa (\cos sx - i \sin sx)}{s} ds \\ &= \frac{1}{\pi} \left[\int_{-\infty}^{\infty} \frac{\sin sa (\cos s x)}{s} ds - i \int_{-\infty}^{\infty} \frac{\sin s a \sin sx}{s} ds \right] \end{aligned}$$

Here, the integrand in the first integral is even and the integrand in the second integral is odd. Hence using the relevant properties of integral here, we get

$$\begin{aligned} f(x) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin sa \cos sx}{s} ds \quad \text{or} \\ \int_{-\infty}^{\infty} \frac{\sin sa \cos sx}{s} ds &= \pi f(x) = \begin{cases} \pi, & |x| \leq a \\ 0, & |x| > a \end{cases} \end{aligned}$$

$$\text{For } x = 0, a = 1, \text{ this yields } \int_{-\infty}^{\infty} \frac{\sin s}{s} ds = \pi$$

$$\text{Since the integrand is even, we have } 2 \int_0^{\infty} \frac{\sin s}{s} ds = \pi$$

$$\text{Or } \int_0^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2}$$

5. Find the Fourier Transform of $f(x) = e^{-a^2 x^2}$ where 'a' is a positive constant.

Deduce that $f(x) = e^{-x^2/2}$ is self reciprocal with respect to Fourier transform.
Here

$$\begin{aligned} F[f(x)] &= \int_{-\infty}^{\infty} e^{-a^2 x^2} e^{isx} dx = \int_{-\infty}^{\infty} e^{-\left(a^2 x^2 - isx\right)} dx \\ &= \int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}\right]} dx \\ &= e^{-\left(\frac{s^2}{4a^2}\right)} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx \end{aligned}$$

$$\text{Setting } t = ax - \frac{is}{2a}, \text{ we get } F[f(x)] = e^{-\left(\frac{s^2}{4a^2}\right)} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a}$$

$$\begin{aligned}
&= \frac{1}{a} e^{-\left(\frac{s^2}{4a^2}\right)} 2 \int_0^\infty e^{-t^2} dt \quad \text{but } \int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2} \\
&= \frac{1}{a} e^{-\left(\frac{s^2}{4a^2}\right)} \sqrt{\pi}, \text{ using gamma function.} \\
\hat{f}(s) &= \frac{\sqrt{\pi}}{a} e^{-\left(\frac{s^2}{4a^2}\right)}
\end{aligned}$$

This is the desired Fourier Transform of $f(x)$.

$$\text{For } a^2 = 1/2 \text{ in } f(x) = e^{-a^2 x^2}$$

$$\text{we get } f(x) = e^{-x^2/2} \text{ and hence,}$$

$$\hat{f}(s) = \sqrt{2\pi} e^{-s^2/2}$$

$$\text{Also putting } x = s \text{ in } f(x) = e^{-x^2/2}, \text{ we get } f(s) = e^{-s^2/2}.$$

Hence, $f(s)$ and $\hat{f}(s)$ are same but for constant multiplication by $\sqrt{2\pi}$.

Thus $f(s) = \hat{f}(s)$

It follows that $f(x) = e^{-x^2/2}$ is self reciprocal

6) Find the inverse Fourier Transform of $\hat{f}(s) = e^{-s^2}$

Exercises:

Find the Complex Fourier Transforms of the following functions:

$$(1) f(x) = \begin{cases} x, & |x| \leq a \\ 0, & |x| > a \end{cases} \text{ where 'a' is a positive constant}$$

$$(2) f(x) = \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

$$(3) f(x) = \begin{cases} 0, & x < a \\ 1, & a \leq x \leq b \\ 0, & x > b \end{cases} \text{ where 'a' and 'b' are positive constants}$$

$$(4) f(x) = \begin{cases} a^2 - x^2, & |x| \leq a \\ 0, & |x| > a \end{cases}$$

$$(5) f(x) = x e^{-a|x|} \text{ where 'a' is a positive constant} \quad (6) f(x) = e^{-|x|} \quad (7) f(x) = \cos 2x^2$$

$$(8) f(x) = \sin 3x^2$$

FOURIER SINE TRANSFORMS:

Let $f(x)$ be defined for all positive values of x .

The integral $\int_0^{\infty} f(x) \sin sx dx$ is called the Fourier sine Transform of $f(x)$. This is denoted by

$$\hat{f}_s(s) \text{ or } F_s[f(x)], \text{ thus } \hat{f}_s(s) = F_s[f(x)] = \int_0^{\infty} f(x) \sin sx dx$$

The inverse Fourier sine Transform of $\hat{f}_s(s)$ is defined through the integral $\frac{2}{\pi} \int_0^{\infty} \hat{f}_s(s) \sin sx ds$.

This is denoted by $f(x)$ or $F_s^{-1}[\hat{f}_s(s)]$. Thus

$$f(x) = F_s^{-1}[\hat{f}_s(s)] = \frac{2}{\pi} \int_0^{\infty} \hat{f}_s(s) \sin sx ds$$

Properties

The following are the basic properties of Sine Transforms.

(1) LINEARITY PROPERTY

If 'a' and 'b' are two constants, then for two functions $f(x)$ and $\phi(x)$, we have

$$F_s[af(x) + b\phi(x)] = aF_s[f(x)] + bF_s[\phi(x)]$$

Proof : By definition, we have

$$\begin{aligned} F_s[af(x) + b\phi(x)] &= \int_0^{\infty} [af(x) + b\phi(x)] \sin sx dx = \int_0^{\infty} af(x) \sin sx dx + \int_0^{\infty} b\phi(x) \sin sx dx \\ &= a \int_0^{\infty} f(x) \sin sx dx + b \int_0^{\infty} \phi(x) \sin sx dx = aF_s[f(x)] + bF_s[\phi(x)] \end{aligned}$$

This is the desired result. In particular, we have

$$F_s[f(x) + \phi(x)] = F_s[f(x)] + F_s[\phi(x)] \text{ and } F_s[f(x) - \phi(x)] = F_s[f(x)] - F_s[\phi(x)]$$

(2) CHANGE OF SCALE PROPERTY

If $F_s[f(x)] = \hat{f}_s(s)$, then for $a \neq 0$, we have $F_s[f(ax)] = \frac{1}{a} \hat{f}_s\left(\frac{s}{a}\right)$

Proof : We have $F_s[f(ax)] = \int_0^{\infty} f(ax) \sin sx dx$

Setting $ax = t$, we get $F_s[f(ax)] = \int_0^{\infty} f(t) \sin\left(\frac{s}{a}t\right) t \left(\frac{dt}{a}\right)$

$$\frac{1}{a} \int_0^{\infty} f(t) \sin\left(\frac{s}{a}t\right) t dt = \frac{1}{a} \hat{f}_s\left(\frac{s}{a}\right)$$

(3) **MODULATION PROPERTY**

If $F_s[f(x)] = \hat{f}_s(s)$, then for $a \neq 0$, we have $F_s[f(x)\cos ax] = \frac{1}{2}[\hat{f}_s(s+a) + \hat{f}_s(s-a)]$

Proof: We have $F_s[f(x)\cos ax] = \int_0^\infty f(x)\cos ax \sin sx \, dx$

$$\begin{aligned} &= \frac{1}{2} \left[\int_0^\infty f(x) \{ \sin(s+a)x + \sin(s-a)x \} dx \right] \\ &= \frac{1}{2} \left[\int_0^\infty f(x) \sin(s+a)x \, dx + \int_0^\infty f(x) \sin(s-a)x \, dx \right] \\ &= \frac{1}{2} [\hat{f}_s(s+a) + \hat{f}_s(s-a)] \text{ by using Linearity property.} \end{aligned}$$

FOURIER COSINE TRANSFORMS:

Let $f(x)$ be defined for positive values of x . Then the integral $\int_0^\infty f(x) \cos sx \, dx$ is called the

Fourier Cosine Transform of $f(x)$ and is denoted by $\hat{f}_c(s)$ or $F_c[f(x)]$. Thus

$$\hat{f}_c(s) = F_c[f(x)] = \frac{2}{\pi} \int_0^\infty f(x) \cos sx \, dx$$

The inverse Fourier Cosine Transform of $\hat{f}_c(s)$ is defined through the integral

$$\frac{2}{\pi} \int_0^\infty \hat{f}_c(s) \cos sx \, ds. \text{ This is denoted by } f(x) \text{ or } F_c^{-1}[\hat{f}_c(s)]. \text{ Thus}$$

$$f(x) = F_c^{-1}[\hat{f}_c(s)] = \frac{2}{\pi} \int_0^\infty \hat{f}_c(s) \cos sx \, ds$$

Basic Properties:

The following are the basic properties of cosine transforms:

(1) **Linearity property:** If 'a' and 'b' are two constants, then for two functions $f(x)$ and

$$\phi(x), \text{ we have } F_c[af(x) + b\phi(x)] = aF_c(f(x)) + bF_c(\phi(x))$$

(2) **Change of scale property:** If $F_c\{f(x)\} = \hat{f}_c(s)$, then for $a \neq 0$, we have $F_c[f(ax)] = \frac{1}{a} \hat{f}_c\left(\frac{s}{a}\right)$

(3) **Modulation property:** If $F_c\{f(x)\} = \hat{f}_c(s)$, then for $a \neq 0$, we have

$$F_c[f(x)\cos ax] = \frac{1}{2} [\hat{f}_c(s+a) + \hat{f}_c(s-a)]$$

The proofs of these properties are similar to the proofs of the corresponding properties of Fourier Sine Transforms.

EXAMPLES

1. Find the Fourier Sine transform of $e^{-|x|}$. Hence show that $\int_0^{\infty} \frac{x \sin mx}{1+x^2} dx = \frac{\pi e^{-m}}{2}, m > 0$

$$\begin{aligned} << F_s[f(x)] &= \int_0^{\infty} f(x) \sin sxdx = \int_0^{\infty} e^{-|x|} \sin sxdx = \int_0^{\infty} e^{-x} \sin sxdx \quad [\text{since } |x| = -x \text{ in } (0, \infty)] \\ &= \frac{e^{-x}}{1+s^2} (-\sin sx - s \cos sx) \Big|_0^{\infty} = 0 - \frac{1}{1+s^2} (0 - s) = \frac{s}{1+s^2} \end{aligned}$$

By using inverse formula for Fourier sine transform, we get

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s[f(x)] \sin sxdx \Rightarrow e^{-x} = \frac{2}{\pi} \int_0^{\infty} \frac{s}{1+s^2} \sin sxdx, \text{ By changing } x \text{ to } m, \text{ we get}$$

$$e^{-m} = \frac{2}{\pi} \int_0^{\infty} \frac{s \sin ms}{1+s^2} ds = \frac{2}{\pi} \int_0^{\infty} \frac{x \sin mx}{1+x^2} dx \Rightarrow \int_0^{\infty} \frac{x \sin mx}{1+x^2} dx = \frac{\pi e^{-m}}{2}$$

2. Find the Fourier Cosine transform of $f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2-x & \text{for } 1 < x < 2 \\ 0 & \text{for } x > 2 \end{cases}$

$$\begin{aligned} << F_c[f(x)] &= \int_0^{\infty} f(x) \cos sxdx = \int_0^1 f(x) \cos sxdx + \int_1^2 f(x) \cos sxdx + \int_2^{\infty} f(x) \cos sxdx \\ &= \int_0^1 x \cos sxdx + \int_1^2 (2-x) \cos sxdx + \int_2^{\infty} 0 \cdot \cos sxdx \\ &= \left[x \cdot \frac{\sin sx}{s} - 1 \cdot \left(\frac{-\cos sx}{s^2} \right) \right] \Big|_0^1 + \left[(2-x) \frac{\sin sx}{s} - (-1) \left(\frac{-\cos sx}{s^2} \right) \right] \Big|_1^2 \\ &= \frac{\sin s}{s} + \frac{\cos s}{s^2} - \left(0 - \frac{1}{s^2} \right) - \frac{\cos 2s}{s^2} - \left(\frac{\sin s}{s} - \frac{\cos s}{s^2} \right) \\ &= \frac{2 \cos s}{s^2} - \frac{1}{s^2} - \frac{\cos 2s}{s^2} = \frac{1}{s^2} (2 \cos s - \cos 2s - 1) \end{aligned}$$

3. Find the Fourier Sine transform of e^{-ax}/x

<< Let $f(x) = e^{-ax}/x$, then its Fourier sine transform is

$$F_s[f(x)] = \int_0^{\infty} f(x) \sin sxdx = \int_0^{\infty} \frac{e^{-ax}}{x} \sin sxdx = F(s), \text{ say}$$

Differentiate both sides w.r.t. s we get,

$$\frac{d}{ds} \{F(s)\} = \int_0^{\infty} \frac{\partial}{\partial s} \left(\frac{e^{-ax}}{x} \sin sx \right) dx = \int_0^{\infty} \frac{x e^{-ax}}{x} \cos sxdx = \int_0^{\infty} e^{-ax} \cos sxdx$$

$$= \frac{e^{-ax}}{a^2 + s^2} (-a \cos sx + s \sin sx) \Big|_0^\infty$$

$$= \frac{1}{s^2 + a^2} [0 - (-a)] = \frac{a}{s^2 + a^2}$$

Integrating w.r.t. s, we get,

$$F(s) = \int \frac{a}{s^2 + a^2} ds = \tan^{-1} \frac{s}{a} + c$$

But $F(0) = 0$ when $s = 0 \Rightarrow c = 0$. Hence $F(s) = \tan^{-1} \frac{s}{a}$

4. Find the Fourier Cosine Transform of $f(x) = \frac{1}{1+x^2}$. Hence derive Fourier sine transform of

$$\phi(x) = \frac{x}{1+x^2}$$

$$<< F_c[f(x)] = \int_0^\infty f(x) \cos sx dx = \int_0^\infty \frac{1}{1+x^2} \cos sx dx = I(\text{say})$$

$$\therefore \frac{dI}{ds} = \int_0^\infty \frac{\partial}{\partial s} \left(\frac{\cos sx}{1+x^2} \right) dx = \int_0^\infty \frac{-x \sin sx}{1+x^2} dx = - \int_0^\infty \frac{x^2 \sin sx}{x(1+x^2)} dx = - \int_0^\infty \frac{[1+x^2-1] \sin sx}{x(1+x^2)} dx$$

$$= - \int_0^\infty \frac{\sin sx}{x} dx + \int_0^\infty \frac{\sin sx}{x(1+x^2)} dx = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)} dx \Rightarrow \frac{d^2 I}{ds^2} = \int_0^\infty \frac{x \cos sx}{x(1+x^2)} dx = I$$

$$\therefore \frac{d^2 I}{ds^2} - I = 0 \Rightarrow (D^2 - 1)I = 0, \text{ where } D = \frac{d}{ds}$$

$$\therefore I = c_1 e^s + c_2 e^{-s} \Rightarrow \frac{dI}{ds} = c_1 e^s - c_2 e^{-s}$$

$$\text{When } s = 0, \text{ we get } c_1 + c_2 = I = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2} \text{ \& } c_1 - c_2 = \frac{dI}{ds} = -\frac{\pi}{2} \Rightarrow c_1 = 0 \text{ \& } c_2 = \frac{\pi}{2}$$

$$\therefore F_c[f(x)] = I = \frac{\pi}{2} e^{-s}$$

$$\text{Now } F_s[\phi(x)] = \int_0^\infty \frac{x \sin sx}{1+x^2} dx = -\frac{dI}{ds} = \frac{\pi}{2} e^{-s}$$

5. Solve the integral equation $\int_0^\infty f(\theta) \cos \alpha \theta d\theta = \begin{cases} 1-\alpha, & 0 \leq \alpha \leq 1 \\ 0, & \alpha > 1 \end{cases}$. Hence evaluate $\int_0^\infty \frac{\sin^2 t}{t^2} dt$

$$<< \text{ We have } \int_0^\infty f(\theta) \cos \alpha \theta d\theta = F_c(\alpha)$$

$$\therefore F_c(\alpha) = \begin{cases} 1-\alpha, & 0 \leq \alpha \leq 1 \\ 0, & \alpha > 1 \end{cases} \text{----- (1)}$$

∴ by the inverse formula we have

$$f(\theta) = \frac{2}{\pi} \int_0^{\infty} F_c(\alpha) \cos \alpha \theta d\alpha = \frac{2}{\pi} \int_0^1 (1-\alpha) \cos \alpha \theta d\alpha = \frac{2}{\pi} \left[(1-\alpha) \frac{\sin \alpha \theta}{\theta} - (-1) \left(-\frac{\cos \alpha \theta}{\theta^2} \right) \right]_0^1$$

$$= \frac{2(1-\cos \theta)}{\pi \theta^2}$$

$$\text{Now } F_c(\alpha) = \int_0^{\infty} f(\theta) \cos \alpha \theta d\theta = \int_0^{\infty} \frac{2(1-\cos \theta)}{\pi \theta^2} \cos \alpha \theta d\theta \text{ ----- (2)}$$

$$\text{From (1) \& (2) we have } \frac{2}{\pi} \int_0^{\infty} \frac{1-\cos \theta}{\theta^2} \cos \alpha \theta d\theta = \begin{cases} 1-\alpha, & 0 \leq \alpha \leq 1 \\ 0, & \alpha > 1 \end{cases}$$

Now take limit as $\alpha \rightarrow 0$, we get

$$\frac{2}{\pi} \int_0^{\infty} \frac{1-\cos \theta}{\theta^2} d\theta = 1 \Rightarrow \int_0^{\infty} \frac{1-\cos \theta}{\theta^2} d\theta = \frac{\pi}{2} \Rightarrow \int_0^{\infty} \frac{2 \sin^2 \theta / 2}{\theta^2} d\theta = \frac{\pi}{2}, \text{ put } \frac{\theta}{2} = t, \text{ then } d\theta = 2dt$$

$$\Rightarrow \int_0^{\infty} \frac{2 \sin^2 t}{(t/2)^2} 2dt = \frac{\pi}{2} \Rightarrow \int_0^{\infty} \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$$

$$(6) \text{ Solve the integral equation } \int_0^{\infty} f(x) \cos \alpha x dx = e^{-a\alpha}$$

Let $\phi(\alpha)$ be defined by $\phi(\alpha) = e^{-a\alpha}$

$$\text{Given } \phi(\alpha) = \int_0^{\infty} f(x) \cos \alpha x dx = \hat{f}_c(\alpha)$$

Using this in the inversion formula, we get

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \phi(\alpha) \cos \alpha x d\alpha = \frac{2}{\pi} \int_0^{\infty} e^{-a\alpha} \cos \alpha x d\alpha$$

$$= \frac{2}{\pi} \left[\frac{e^{-a\alpha}}{a^2 + x^2} \{ -a \cos \alpha x + x \sin \alpha x \} \right]_0^{\infty} = \frac{2}{\pi} \left[\frac{a}{a^2 + x^2} \right] = \frac{2a}{\pi(a^2 + x^2)}$$

Exercises:

$$1. \text{ Find the Fourier sine transform of } f(x) = \begin{cases} 1, & 0 \leq x \leq a \\ 0, & x > a \end{cases} \quad \left[\text{Ans : } \frac{1 - \cos sa}{s} \right]$$

$$2. \text{ Find } f(x) \text{ from the integral equation } \int_0^{\infty} f(x) \sin \alpha x dx = \begin{cases} 1, & 0 \leq \alpha \leq 1 \\ 2, & 1 \leq \alpha < 2 \\ 0, & \alpha \geq 2 \end{cases}$$

$$\left[\text{Ans : } \frac{2}{\pi x} [1 + \cos x - 2 \cos 2x] \right]$$

3. Find the sine transforms of the following functions

$$(i) f(x) = \begin{cases} x, & 0 < x < 1 \\ a - x, & 1 < x < a \\ 0, & x > a \end{cases} \quad (ii) f(x) = xe^{-ax}, a > 0 \quad (iii) f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x > a \end{cases}$$

$$(4) \text{ Solve for } f(x) \text{ given } \int_0^{\infty} f(x) \sin \alpha x dx = \begin{cases} 1 - \alpha, & 0 \leq \alpha \leq 1 \\ 0, & \alpha > 1 \end{cases}$$

5) Find the inverse sine transforms of the following functions:

$$(i) \hat{f}_s(s) = \frac{e^{-as}}{s}, a > 0 \quad (ii) \hat{f}_s(s) = \frac{\pi}{2}$$

6. Find the Cosine transform of $f(x) = e^{-ax}, a > 0$. Hence evaluate $\int_0^{\infty} \frac{\cos kx}{x^2 + a^2} dx$

$$\left(\text{Ans: } \frac{a}{a^2 + \alpha^2} \& \int_0^{\infty} \frac{\cos kx}{x^2 + a^2} dx = \frac{\pi e^{-ax}}{2a} \right)$$

7. Find the Fourier Cosine Transforms of the following functions:

$$(i) f(x) = \begin{cases} 4x, & 0 < x < 1 \\ 4 - x, & 1 < x < 4 \\ 0, & x > 4 \end{cases} \quad (ii) f(x) = e^{-ax^2}, a > 0 \quad (iii) f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x > a \end{cases}$$

$$(iv) f(x) = xe^{-ax}, a > 0 \quad (v) f(x) = \frac{1}{1+x^2} \quad (vi) f(x) = \frac{\cos 2x}{1+x^2}$$

$$8. \text{ Solve for } f(x) \text{ given } \int_0^{\infty} f(x) \cos \alpha x dx = \begin{cases} 1 - \alpha, & 0 \leq \alpha \leq 1 \\ 0, & \alpha > 1 \end{cases}$$

Z – TRANSFORMS

Introduction

The Z-transform plays an important role in the study of communications, sample data control systems, discrete signal processing, solutions of difference equations etc.

Definition:

A difference equation is a relation between the differences of an unknown function at one or more general values of the argument.

Eg: $\Delta y_{(n+1)} + y_{(n)} = 2$ & $\Delta^2 y_{(n+1)} + \Delta y_{(n-1)} = 0$ are difference equations.

Another way of writing a difference equation is as follows:

We know that $\Delta y_{(n+1)} = y_{(n+2)} - y_{(n+1)}$

Hence the first equation can be written as $y_{(n+2)} - y_{(n+1)} + y_{(n)} = 2$ ----- (1)

Also $\Delta^2 y_{(n+1)} = y_{(n+3)} - 2y_{(n+2)} + y_{(n+1)}$

Hence the second equation can be written as

$$y_{(n+3)} - 2y_{(n+2)} + y_{(n+1)} + y_{(n)} - y_{(n-1)} = 0$$
 ----- (2)

Order of a difference equation:

Order of a difference equation is the difference between the largest and the smallest arguments occurring in the difference equation divided by the unit of increment.

Thus the order of the equation (1) is 2. Since

$$\frac{\text{Largest argument} - \text{smallest argument}}{\text{Unit of increment}} = \frac{(n+2) - n}{1} = 2$$

$$\text{\& that of (2) is } \frac{(n+3) - (n-1)}{1} = 4$$

Solution: Solution of a difference equation is an expression for $y_{(n)}$ which satisfies the given difference equation.

General solution: The general solution of a difference equation is that in which the number of arbitrary constants is equal to the order of the difference equation.

A Particular solution or Particular integral is that solution which is obtained from the general solution by giving particular values to the constants.

Z – TRANSFORMS:

Definition: Let $u_n = f(n)$ be a real-valued function defined for $n = 0, 1, 2, 3, \dots$ and $u_n = 0$ for $n < 0$. Then the Z-transform of u_n denoted by $Z(u_n)$ is defined by

$$U(z) = Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n} \text{ ----- (1) whenever the infinite series converges}$$

We can also write it as $Z^{-1}[U(z)] = u_n$ and is called the inverse Z – Transform.

Properties of Z-transform:

1. Linearity property:-

Consider the sequences $\{u_n\}$ & $\{v_n\}$ and constants a and b. Then

$$Z[au_n + bv_n] = aZ[u_n] + bZ[v_n]$$

Proof: By definition, we have $Z[au_n + bv_n] = \sum_{n=0}^{\infty} [au_n + bv_n] z^{-n}$

$$= a \sum_{n=0}^{\infty} u_n z^{-n} + b \sum_{n=0}^{\infty} v_n z^{-n} = aZ(u_n) + bZ(v_n)$$

In particular, for $a = b = 1$, we get $Z[u_n + v_n] = Z[u_n] + Z[v_n]$ and for $a = -b = 1$, we get

$$Z[u_n - v_n] = Z[u_n] - Z[v_n]$$

2. Damping property:-

Let $Z(u_n) = U(z)$. Then (i) $Z(a^n u_n) = U\left(\frac{z}{a}\right)$ (ii) $Z(a^{-n} u_n) = U(az)$

Proof: By definition, we have $Z(a^n u_n) = \sum_{n=0}^{\infty} (a^n u_n) z^{-n} = \sum_{n=0}^{\infty} u_n \left(\frac{z}{a}\right)^{-n} = U\left(\frac{z}{a}\right)$

$$\text{Thus } Z(a^n u_n) = U\left(\frac{z}{a}\right)$$

This is the result as desired. Here, we note that that if $Z(u_n) = U(z)$, then

$$Z(a^n u_n) = [U(z)]_{Z \rightarrow Z/a} = U\left(\frac{z}{a}\right)$$

Next, $Z(a^{-n} u_n) = \sum_{n=0}^{\infty} (a^{-n} u_n) z^{-n} = \sum_{n=0}^{\infty} u_n (az)^{-n} = U(az)$

Thus $Z(a^{-n} u_n) = U(az)$. This is the result as desired.

3. Shifting property:

(a) Right shifting rule:

If $Z(u_n) = U(z)$, then $Z(u_{n-k}) = z^{-k} U(z)$ where $k > 0$

Proof: By definition, we have $Z(u_{n-k}) = \sum_{n=0}^{\infty} u_{n-k} z^{-n}$

Since $u_n = 0$ for $n < 0$, we have $u_{n-k} = 0$ for $n = 0, 1, 2, \dots, (k-1)$

$$\begin{aligned} \text{Hence } Z(u_{n-k}) &= \sum_{n=k}^{\infty} u_{n-k} z^{-n} = u_0 z^{-k} + u_1 z^{-(k+1)} + \dots + \infty = z^{-k} [u_0 + u_1 z^{-1} + \dots + \infty] \\ &= z^{-k} \sum_{n=0}^{\infty} u_n z^{-n} = z^{-k} U(z) \end{aligned}$$

Thus $Z(u_{n-k}) = z^{-k} U(z)$

(b) Left shifting rule: $Z(u_{n+k}) = z^k [U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \dots - u_{k-1} z^{-(k-1)}]$

$$\begin{aligned} \text{Proof: } Z(u_{n+k}) &= \sum_{n=0}^{\infty} u_{n+k} z^{-n} = z^k \sum_{n=0}^{\infty} u_{n+k} z^{-(k+n)} = z^k \left\{ \sum_{n=0}^{\infty} u_{n+k} z^{-(k+n)} \right\} \\ &= z^k \left[\sum_{m=k}^{\infty} u_m z^{-m} \right], \text{ where } m = n+k \\ &= z^k \left[\sum_{n=k}^{\infty} u_n z^{-n} \right] = z^k \left[\sum_{n=0}^{\infty} u_n z^{-n} - \sum_{n=0}^{k-1} u_n z^{-n} \right] \\ &= z^k [U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \dots - u_{k-1} z^{-(k-1)}] \end{aligned}$$

Particular cases:

In particular, we have the following standard results:

1. $Z(u_{n+1}) = z[U(z) - u_0]$
2. $Z(u_{n+2}) = z^2[U(z) - u_0 - u_1 z^{-1}]$
3. $Z(u_{n+3}) = z^3[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}]$ etc.

Some Standard Z-Transforms:

1. Transform of a^n :

$$\text{By definition, we have } Z(a^n) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \dots + \infty$$

The series on the RHS is a Geometric series. Sum to infinity of the series is $\frac{1}{1 - a/z}$

$$\text{or } \frac{z}{z-a} \quad \text{Thus, } Z(a^n) = \frac{z}{z-a}$$

In particular, when $a=1$, we get $Z(1) = \frac{z}{z-1}$

2. Transform of e^{an} :

Here $Z(e^{an}) = Z(k^n)$ where $k = e^a$

$$= \frac{z}{z-k} = \frac{z}{z-e^a}$$

$$\text{Thus } Z(e^{an}) = \frac{z}{z-e^a}$$

3. Transform of n^p , p being a positive integer:

$$\text{We have, } Z(n^p) = \sum_{n=0}^{\infty} n^p z^{-n} = z \sum_{n=0}^{\infty} n^{p-1} z^{-(n+1)} \quad \text{----- (1)}$$

$$\text{By changing } p \text{ to } p-1, \text{ we get } Z(n^{p-1}) = \sum_{n=0}^{\infty} n^{p-1} z^{-n}$$

Differentiating with respect to z , we get

$$\frac{d}{dz} [Z(n^{p-1})] = \frac{d}{dz} \sum_{n=0}^{\infty} n^{p-1} z^{-n} = \sum_{n=0}^{\infty} n^{p-1} (-n) z^{-(n+1)}$$

$$\text{Using this in (1), we get } Z(n^p) = -z \frac{d}{dz} [Z(n^{p-1})]$$

Particular cases of $Z(n^p)$:-

$$1. \text{ For } p=1, \text{ we get } Z(n) = -z \frac{d}{dz} [Z(1)] = -z \frac{d}{dz} \left(\frac{z}{z-1} \right) = \frac{z}{(z-1)^2}$$

$$\text{Thus, } Z(n) = \frac{z}{(z-1)^2}$$

$$2. \text{ For } p=2, \text{ we get } Z(n^2) = -z \frac{d}{dz} [Z(n)] = -z \frac{d}{dz} \left(\frac{z}{(z-1)^2} \right) = \frac{z^2 + z}{(z-1)^3}$$

$$\text{Thus, } Z(n^2) = \frac{z^2 + z}{(z-1)^3}$$

$$3. \text{ For } p=3, \text{ we get } Z(n^3) = \frac{z^3 + 4z^2 + z}{(z-1)^4} \text{ \& soon}$$

4. Transform of na^n

By damping property, we have

$$Z(na^n) = [Z(n)]_{z \rightarrow z/a} = \left[\frac{z}{(z-1)^2} \right]_{z \rightarrow z/a} = \frac{\frac{z}{a}}{\left(\frac{z}{a} - 1 \right)^2} = \frac{az}{(z-a)^2}$$

$$\text{Thus, } Z(na^n) = \frac{az}{(z-a)^2}$$

5. Transform of $n^2 a^n$:

We have, $Z(n^2 a^n) = \left[Z(n^2) \right]_{Z \rightarrow \frac{Z}{a}} = \left[\frac{z^2 + z}{(z-1)^3} \right]_{Z \rightarrow \frac{Z}{a}} = \frac{(z/a)^2 + z/a}{(z/a-1)^3} = \frac{(z^2 + az)a^3}{a^2(z-a)^3}$

Thus, $Z(n^2 a^n) = \frac{az^2 + a^2 z}{(z-a)^3}$

6. Transforms of $\cosh n\theta$ and $\sinh n\theta$:

We have $\cosh n\theta = \frac{e^{n\theta} + e^{-n\theta}}{2} \Rightarrow Z(\cosh n\theta) = \frac{1}{2} Z(e^{n\theta} + e^{-n\theta})$
 $= \frac{1}{2} [Z(e^{n\theta}) + Z(e^{-n\theta})]$, by using the linearity property

$$= \frac{1}{2} \left[\frac{z}{z-e^\theta} + \frac{z}{z-e^{-\theta}} \right] = \frac{z}{2} \left[\frac{z-e^{-\theta} + z-e^\theta}{z^2 - z(e^\theta + e^{-\theta}) + 1} \right] = z \left[\frac{z - \frac{(e^\theta + e^{-\theta})}{2}}{z^2 - z(e^\theta + e^{-\theta}) + 1} \right]$$

$$= \frac{z[z - \cosh \theta]}{z^2 - 2z \cosh \theta + 1}$$

Next, $\sinh n\theta = \frac{e^{n\theta} - e^{-n\theta}}{2} \Rightarrow Z(\sinh n\theta) = \frac{z}{2} \left[\frac{1}{z-e^\theta} - \frac{1}{z-e^{-\theta}} \right]$

$$= \frac{z}{2} \left[\frac{e^\theta - e^{-\theta}}{z^2 - 2z \cosh \theta + 1} \right] = \frac{z \sinh \theta}{z^2 - 2z \cosh \theta + 1}$$

7. Transforms of $\cos n\theta$ and $\sin n\theta$

We have $\cos n\theta = \frac{e^{in\theta} + e^{-in\theta}}{2}$

$$Z(\cos n\theta) = \frac{1}{2} Z(e^{in\theta} + e^{-in\theta}) = \frac{1}{2} \left[\frac{z}{z-e^{i\theta}} + \frac{z}{z-e^{-i\theta}} \right] = \frac{z}{2} \left[\frac{2z - (e^{i\theta} + e^{-i\theta})}{z^2 - z(e^{i\theta} + e^{-i\theta}) + 1} \right]$$

$$= \frac{z[z - \cos \theta]}{z^2 - 2z \cos \theta + 1}$$

Next, $\sin n\theta = \frac{e^{in\theta} - e^{-in\theta}}{2i}$

$$Z(\sin n\theta) = \frac{1}{2i} \left[\frac{z}{z-e^{i\theta}} - \frac{z}{z-e^{-i\theta}} \right] = \frac{z}{2i} \left[\frac{e^{i\theta} - e^{-i\theta}}{z^2 - 2z \cos \theta + 1} \right] = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$$

8. Transforms of $a^n \cos n\theta$ and $a^n \sin n\theta$

We know that $Z(\cos n\theta) = \frac{z[z - \cos \theta]}{z^2 - 2z \cos \theta + 1} = U(z)$. Hence by using damping rule we get

$$Z(a^n \cos n\theta) = U(z/a) = \frac{z/a[z/a - \cos \theta]}{(z/a)^2 - 2(z/a)\cos \theta + 1} = \frac{z(z - a \cos \theta)}{z^2 - 2az \cos \theta + a^2}$$

Similarly we know that $Z(\sin n\theta) = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1} = U(z)$. Hence by using damping rule we

get

$$Z(a^n \sin n\theta) = U(z/a) = \frac{(z/a) \sin \theta}{(z/a)^2 - 2(z/a)\cos \theta + 1} = \frac{az \sin \theta}{z^2 - 2az \cos \theta + a^2}$$

Examples: Find the Z-transforms of the following:

1. $3n - 4 \sin(n\pi/4) + 5a$

$$<< Z(3n - 4 \sin(n\pi/4) + 5a) = 3Z(n) - 4Z(\sin(n\pi/4)) + 5aZ(1)$$

$$\begin{aligned} &= 3 \cdot \frac{z}{(z-1)^2} - 4 \cdot \frac{z \sin(\pi/4)}{z^2 - 2z \cos(\pi/4) + 1} + 5a \cdot \frac{z}{z-1} = \frac{3z}{(z-1)^2} - \frac{4z \left(\frac{1}{\sqrt{2}} \right)}{z^2 - 2z \left(\frac{1}{\sqrt{2}} \right) + 1} + \frac{5az}{z-1} \\ &= \frac{3z + 5az(z-1)}{(z-1)^2} - \frac{2\sqrt{2}z}{z^2 - \sqrt{2}z + 1} = \frac{(3-5a)z + 5az^2}{(z-1)^2} - \frac{2\sqrt{2}z}{z^2 - \sqrt{2}z + 1} \end{aligned}$$

2. $(n+1)^2$

$$\begin{aligned} << Z[(n+1)^2] = Z(n^2 + 2n + 1) = Z(n^2) + 2Z(n) + Z(1) = \frac{z^2 + z}{(z-1)^3} + 2 \cdot \frac{z}{(z-1)^2} + \frac{z}{z-1} \\ &= \frac{z^2 + z + 2z(z-1) + z(z-1)^2}{(z-1)^3} = \frac{z^2 + z + 2z^2 - 2z + z^3 - 2z^2 + z}{(z-1)^3} = \frac{z^3 + z^2}{(z-1)^3} \end{aligned}$$

3. $\sin(3n+5)$

$$\begin{aligned} << Z[\sin(3n+5)] &= Z(\sin 3n \cos 5 + \cos 3n \sin 5) = \cos 5 Z(\sin 3n) + \sin 5 Z(\cos 3n) \\ &= \cos 5 \cdot \frac{z \sin 3}{z^2 - 2z \cos 3 + 1} + \sin 5 \cdot \frac{z(z - \cos 3)}{z^2 - 2z \cos 3 + 1} = \frac{z \sin 3 \cos 5 + z^2 \sin 5 - z \sin 5 \cos 3}{z^2 - 2z \cos 3 + 1} \\ &= \frac{z(\sin 3 \cos 5 - \cos 3 \sin 5) + z^2 \sin 5}{z^2 - 2z \cos 3 + 1} = \frac{z \sin(-2) + z^2 \sin 5}{z^2 - 2z \cos 3 + 1} = \frac{z(z \sin 5 - \sin 2)}{z^2 - 2z \cos 3 + 1} \end{aligned}$$

4. ne^{an}

Let $u_n = n$, $e^{an} = k^n$ where $k = e^a$

Therefore $Z(ne^{an}) = Z(nk^n) = U(z/k)$, By damping rule.

$$\text{Where } U(z) = Z(n) = \frac{z}{(z-1)^2} \Rightarrow U(z/k) = \frac{z/k}{(z/k-1)^2} = \frac{kz}{(z-k)^2}, \text{ where } k = e^a$$

$$\text{Hence } Z(ne^{an}) = U(z/k) = \frac{e^a z}{(z - e^a)^2}$$

5. $n^2 e^{an}$

Let $u_n = n^2$, $e^{an} = k^n$ where $k = e^a$

Therefore $Z(n^2 e^{an}) = Z(n^2 k^n) = U(z/k)$, By damping rule.

Where $U(z) = Z(n^2) = \frac{z^2 + z}{(z-1)^3} \Rightarrow U(z/k) = \frac{(z/k)^2 + z/k}{(z/k-1)^3} = \frac{kz^2 + k^2 z}{(z-k)^3}$, where $k = e^a$

Hence $Z(n^2 e^{an}) = U(z/k) \frac{e^a z^2 + (e^a)^2 z}{(z - e^a)^3} = \frac{e^a z(z + e^a)}{(z - e^a)^3}$

6. i) $a^n \cosh n\theta$

ii) $a^n \sinh n\theta$

i) We know that $Z(\cosh n\theta) = \frac{z[z - \cosh \theta]}{z^2 - 2z \cosh \theta + 1} = U(z)$, Hence by using damping rule we

get $Z(a^n \cosh n\theta) = U(z/a) = \frac{(z/a)[(z/a) - \cosh \theta]}{(z/a)^2 - 2(z/a) \cosh \theta + 1} = \frac{z(z - a \cosh \theta)}{z^2 - 2az \cosh \theta + a^2}$

ii) We know that $Z(\sinh n\theta) = \frac{z \sinh \theta}{z^2 - 2z \cosh \theta + 1} = U(z)$, Hence by using damping rule we

get $Z(a^n \sinh n\theta) = U(z/a) = \frac{(z/a) \cosh \theta}{(z/a)^2 - 2(z/a) \cosh \theta + 1} = \frac{az \cosh \theta}{z^2 - 2az \cosh \theta + a^2}$

7. $e^t \sin 2t$

We know that $\sin 2t = \frac{z \sin 2}{z^2 - 2z \cos 2 + 1} = U(z)$

$\therefore Z(e^t \sin 2t) = U(z/e)$, By damping rule.

$= \frac{(z/e) \sin 2}{(z/e)^2 - 2(z/e) \cos 2 + 1} = \frac{ez \sin 2}{z^2 - 2ez \cos 2 + e^2}$

8. $e^k \cos k\alpha$ ($k \geq 0$)

We know that $Z(\cos k\alpha) = \frac{z(z - \cos \alpha)}{z^2 - 2z \cos \alpha + 1} = U(z)$

$\therefore Z(e^k \cos k\alpha) = Z((e^1)^k \cos k\alpha) = U(z/e) = \frac{(z/e)[(z/e) - \cos \alpha]}{(z/e)^2 - 2(z/e) \cos \alpha + 1} = \frac{z(z - e \cos \alpha)}{z^2 - 2ez \cos \alpha + e^2}$

9. $\cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)$

$\ll Z\left(\cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)\right) = Z\left(\cos \frac{n\pi}{2} \cos \frac{\pi}{4} - \sin \frac{n\pi}{2} \sin \frac{\pi}{4}\right) = \cos \frac{\pi}{4} Z\left(\cos \frac{n\pi}{2}\right) - \sin \frac{\pi}{4} Z\left(\sin \frac{n\pi}{2}\right)$
 $= \frac{1}{\sqrt{2}} \left(\frac{z(z - \cos \frac{\pi}{2})}{z^2 - 2z \cos \frac{\pi}{2} + 1} \right) - \frac{1}{\sqrt{2}} \left(\frac{z \sin \frac{\pi}{2}}{z^2 - 2z \cos \frac{\pi}{2} + 1} \right) = \frac{1}{\sqrt{2}} \left(\frac{z^2}{z^2 + 1} - \frac{z}{z^2 + 1} \right) = \frac{z(z-1)}{\sqrt{2}(z^2+1)}$

10. $\cosh\left(\frac{n\pi}{2} + \theta\right)$

$$\begin{aligned} \ll Z\left[\cosh\left(\frac{n\pi}{2} + \theta\right)\right] &= Z\left[\frac{e^{n\pi/2+\theta} + e^{-(n\pi/2+\theta)}}{2}\right] = \frac{1}{2}\left[e^\theta Z(e^{n\pi/2}) + e^{-\theta} Z(e^{-(n\pi/2)})\right] \\ &= \frac{1}{2}\left[e^\theta \frac{z}{z - e^{\pi/2}} + e^{-\theta} \frac{z}{z - e^{-\pi/2}}\right] \text{ Using } Z(e^{ax}) = \frac{z}{z - e^a} \\ &= \frac{z}{2}\left[\frac{e^\theta(z - e^{-\pi/2}) + e^{-\theta}(z - e^{\pi/2})}{z^2 - z(e^{\pi/2} + e^{-\pi/2}) + 1}\right] = \frac{z}{2}\left[\frac{z(e^\theta + e^{-\theta}) - e^{-(\pi/2-\theta)} - e^{\pi/2-\theta}}{z^2 - z(e^{\pi/2} + e^{-\pi/2}) + 1}\right] \\ &= \frac{z}{2}\left[\frac{z \cdot 2 \cosh \theta - [e^{(\pi/2-\theta)} + e^{-(\pi/2-\theta)}]}{z^2 - 2z \cosh(\pi/2) + 1}\right] = z\left[\frac{z \cosh \theta - \cosh(\pi/2 - \theta)}{z^2 - 2z \cosh(\pi/2) + 1}\right] \\ &= \frac{z^2 \cosh \theta - z \cosh(\pi/2 - \theta)}{z^2 - 2z \cosh(\pi/2) + 1} \end{aligned}$$

11. ${}^nC_p$ ($0 \leq p \leq n$)

$$\begin{aligned} \ll Z({}^nC_p) &= \sum_{p=0}^n ({}^nC_p z^{-p}) = 1 + {}^nC_1 z^{-1} + {}^nC_2 z^{-2} + \dots + {}^nC_n z^{-n} = (1 + z^{-1})^n \\ &= \left(1 + \frac{1}{z}\right)^n = \left(\frac{z+1}{z}\right)^n \end{aligned}$$

12. ${}^{n+p}C_n$

$$\begin{aligned} Z({}^{n+p}C_n) &= \sum_{p=0}^{\infty} {}^{n+p}C_n z^{-p} = 1 + {}^{n+1}C_n z^{-1} + {}^{n+2}C_n z^{-2} + {}^{n+3}C_n z^{-3} + \dots \\ &= 1 + {}^{n+1}C_1 z^{-1} + {}^{n+2}C_2 z^{-2} + {}^{n+3}C_3 z^{-3} + \dots, \quad \text{Using } {}^nC_{n-r} = {}^nC_r \\ &= 1 + (n+1)z^{-1} + \frac{(n+2)(n+1)}{2!} z^{-2} + \frac{(n+3)(n+2)(n+1)}{3!} z^{-3} + \dots \\ &= 1 + (-n-1)(-z)^{-1} + \frac{(-n-2)(-n-1)}{2!} (-z^{-1})^2 + \frac{(-n-3)(-n-2)(-n-1)}{3!} (-z^{-1})^3 + \dots \\ &= (1 - z^{-1})^{-n-1} = \left(1 - \frac{1}{z}\right)^{-(n+1)} = \left(\frac{z-1}{z}\right)^{-(n+1)} = \left(\frac{z}{z-1}\right)^{(n+1)} \end{aligned}$$

13. Unit impulse sequence $\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$

$$\ll Z[\delta(n)] = \sum_{n=0}^{\infty} \delta(n) z^{-n} = 1 + 0 + 0 + 0 + \dots = 1$$

14. Unit step sequence $u(n) = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$

$$<< Z[u(n)] = \sum_{n=0}^{\infty} u(n)z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + \dots = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

15. Show that $Z\left(\frac{1}{n!}\right) = e^{1/z}$. Hence evaluate $Z\left(\frac{1}{(n+1)!}\right) \& . Z\left(\frac{1}{(n+2)!}\right)$

$$<< Z\left(\frac{1}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \dots = e^{z^{-1}} = e^{1/z}$$

Shifting $\frac{1}{n!}$ one unit to the left gives $Z\left(\frac{1}{(n+1)!}\right) = z[U(z) - u_0] = z\left[Z\left(\frac{1}{n!}\right) - 1\right] = z(e^{1/z} - 1)$

Similarly $.Z\left(\frac{1}{(n+2)!}\right) = z^2[U(z) - u_0 - u_1 z^{-1}] = z^2[e^{1/z} - 1 - z^{-1}]$

16. $u_n = \left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n$

We have, $Z(u_n) = Z\left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right] = Z\left(\frac{1}{2}\right)^n + Z\left(\frac{1}{4}\right)^n = \frac{z}{z-\frac{1}{2}} + \frac{z}{z-\frac{1}{4}} = \frac{2z}{2z-1} + \frac{4z}{4z-1}$

$$= \frac{2z(8z-3)}{(2z-1)(4z-1)}$$

Multiplication by n: If $Z(u_n) = U(z)$, then $Z(nu_n) = -z \frac{d(U(z))}{dz}$

Proof: $Z(nu_n) = \sum_{n=0}^{\infty} nu_n z^{-n} = -z \sum_{n=0}^{\infty} u_n (-n) z^{-n-1} = -z \sum_{n=0}^{\infty} u_n \frac{d(z^{-n})}{dz} = -z \sum_{n=0}^{\infty} \frac{d(u_n z^{-n})}{dz}$

$$= -z \frac{d}{dz} \sum_{n=0}^{\infty} u_n z^{-n} = -z \frac{d(U(z))}{dz}$$

Note: In general $Z(n^m u_n) = (-z)^m \frac{d^m [U(z)]}{dz^m}$

Example: Find the z – transform of i) $n \sin n\theta$ ii) $n^2 e^{n\theta}$

i) We know that $Z(\sin n\theta) = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$, using $Z(nu_n) = -z \frac{d}{dz} U(z)$, we get

$$\begin{aligned} Z(n \sin n\theta) &= -z \frac{d}{dz} \left(\frac{z \sin \theta}{z^2 - 2z \cos \theta + 1} \right) = -z \left(\frac{(z^2 - 2z \cos \theta + 1) \sin \theta - z \sin \theta (2z - 2 \cos \theta)}{(z^2 - 2z \cos \theta + 1)^2} \right) \\ &= -z \left(\frac{z^2 \sin \theta - 2z \cos \theta \sin \theta + 1 \sin \theta - 2z^2 \sin \theta + 2z \sin \theta \cos \theta}{(z^2 - 2z \cos \theta + 1)^2} \right) \\ &= -z \left(\frac{\sin \theta - z^2 \sin \theta}{(z^2 - 2z \cos \theta + 1)^2} \right) = \frac{z(z^2 - 1) \sin \theta}{(z^2 - 2z \cos \theta + 1)^2} \end{aligned}$$

ii) We know that $Z(e^{n\theta}) = \frac{z}{z-e^\theta}$ & $Z(n^2 u_n) = (-z)^2 \frac{d^2}{dz^2} U(z)$

$$\begin{aligned} \therefore Z(n^2 e^{n\theta}) &= (-z)^2 \frac{d^2}{dz^2} \left(\frac{z}{z-e^\theta} \right) = z^2 \frac{d}{dz} \left[\frac{(z-e^\theta) \cdot 1 - z(1)}{(z-e^\theta)^2} \right] = z^2 \frac{d}{dz} \left[\frac{-e^\theta}{(z-e^\theta)} \right] \\ &= -z^2 e^\theta \left(\frac{-2}{(z-e^\theta)^3} \right) = \frac{2e^\theta z^2}{(z-e^\theta)^3} \end{aligned}$$

I. Initial value theorem: If $Z(u_n) = U(z)$, then $u_0 = \lim_{z \rightarrow \infty} z U(z)$

Proof: we know that $U(z) = Z(u_n) = u_0 + u_1 z^{-1} + u_2 z^{-2} + \dots$

Taking limit as $z \rightarrow \infty$, we get $\lim_{z \rightarrow \infty} z U(z) = u_0$

Similarly $u_1 = \lim_{z \rightarrow \infty} \{z[U(z) - u_0]\}$, $u_2 = \lim_{z \rightarrow \infty} \{z^2[U(z) - u_0 - u_1 z^{-1}]\}$ & so on

II. Final value theorem: If $Z(u_n) = U(z)$, then $\lim_{n \rightarrow \infty} u_n = \lim_{z \rightarrow 1} (z-1)U(z)$

Proof: By definition $Z(u_{n+1} - u_n) = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$ Or

$$Z(u_{n+1}) - Z(u_n) = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$$

Or $Z[U(z) - u_0] - U(z) = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$ Or $U(z)(z-1) - u_0 z = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$

Taking limits of both sides as $z \rightarrow 1$, we get

$$\lim_{z \rightarrow 1} [U(z)(z-1) - u_0 z] = \lim_{n \rightarrow \infty} \sum_{n=0}^{\infty} (u_{n+1} - u_n) = \lim_{n \rightarrow \infty} [(u_1 - u_0) + (u_2 - u_1) + (u_3 - u_2) + \dots + (u_{n+1} - u_n)]$$

$$= \lim_{n \rightarrow \infty} u_{n+1} - u_0 = u_{\infty} - u_0$$

Hence $u_{\infty} = \lim_{z \rightarrow 1} [U(z)(z-1)]$ Or $\lim_{n \rightarrow \infty} u_n = \lim_{z \rightarrow 1} [(z-1)U(z)]$

Example: If $U(z) = \frac{2z^2 + 5z + 14}{(z-1)^4}$, evaluate u_2 & u_3

$$\ll U(z) = \frac{2z^2 + 5z + 14}{(z-1)^4} = \frac{1}{z^2} \cdot \frac{2 + 5z^{-1} + 14z^{-2}}{(1-z^{-1})^4},$$

By Initial value theorem, $u_0 = \lim_{z \rightarrow \infty} z U(z) = \lim_{z \rightarrow \infty} \left[\frac{1}{z^2} \cdot \frac{2 + 5z^{-1} + 14z^{-2}}{(1-z^{-1})^4} \right] = 0$

Similarly $u_1 = \lim_{z \rightarrow \infty} \{z[U(z) - u_0]\} = \lim_{z \rightarrow \infty} z \left[\frac{1}{z^2} \cdot \frac{2 + 5z^{-1} + 14z^{-2}}{(1-z^{-1})^4} \right]$

$$= \lim_{z \rightarrow \infty} Lt \left[\frac{1}{z} \cdot \frac{2 + 5z^{-1} + 14z^{-2}}{(1 - z^{-1})^4} \right] = 0$$

$$\begin{aligned} \text{Now } u_2 &= \lim_{z \rightarrow \infty} Lt \left\{ z^2 [U(z) - u_0 - u_1 z^{-1}] \right\} = \lim_{z \rightarrow \infty} Lt \left\{ z^2 \left[\frac{1}{z^2} \cdot \frac{2 + 5z^{-1} + 14z^{-2}}{(1 - z^{-1})^4} \right] \right\} \\ &= \lim_{z \rightarrow \infty} Lt \left[\frac{2 + 5z^{-1} + 14z^{-2}}{(1 - z^{-1})^4} \right] = 2 \end{aligned}$$

$$\begin{aligned} \& u_3 &= \lim_{z \rightarrow \infty} Lt \left\{ z^3 [U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}] \right\} = \lim_{z \rightarrow \infty} Lt \left\{ z^3 \left[\frac{2z^2 + 5z + 14}{(z - 1)^4} - 0 - 0 - 2z^{-2} \right] \right\} \\ &= \lim_{z \rightarrow \infty} Lt \left\{ z^3 \left[\frac{2z^2 + 5z + 14}{(z - 1)^4} - \frac{2}{z^2} \right] \right\} = \lim_{z \rightarrow \infty} Lt \left\{ z^3 \left[\frac{z^2(2z^2 + 5z + 14) - 2(z - 1)^4}{z^2(z - 1)^4} \right] \right\} \\ &= \lim_{z \rightarrow \infty} Lt \left\{ z^3 \left[\frac{2z^4 + 5z^3 + 14z^2 - 2z^4 - 8z^2 - 2 + 8z^3 + 8z - 4z^2}{z^2(z - 1)^4} \right] \right\} \\ &= \lim_{z \rightarrow \infty} Lt \left\{ z^3 \left[\frac{13z^3 + 2z^2 + 8z - 2}{z^2(z - 1)^4} \right] \right\} = \lim_{z \rightarrow \infty} Lt \left\{ z^3 \left[\frac{z^3[13 + 2z^{-1} + 8z^{-2} - 2z^{-3}]}{z^6(1 - z^{-1})^4} \right] \right\} = 13 \end{aligned}$$

Exercises: Find the Z – Transforms of the following:

$$1) \frac{1}{(n+1)!} \quad 2) (\cos \theta + i \sin \theta)^n \quad 3) 2n + 5 \sin \frac{n\pi}{4} - 3a^4 \quad 4) (n-1)^2 \quad 5) a^{n+3}$$

$$6) \text{ Show that i) } Z(e^{-an} \cos n\theta) = \frac{ze^a(z e^a - \cos \theta)}{z^2 e^{2a} - 2ze^a \cos \theta + 1}$$

$$\text{ii) } Z(e^{-an} \sin n\theta) = \frac{ze^a \sin \theta}{z^2 e^{2a} - 2ze^a \cos \theta + 1}$$

$$7) \text{ Using } Z(n^2) = \frac{z^2 + z}{(z-1)^3}, \text{ Show that } Z(n+1)^2 = \frac{z^3 + z^2}{(z-1)^3}$$

$$8) \text{ If } Z(u_n) = \frac{z}{z-1} + \frac{z}{z^2+1}, \text{ find the Z – Transform of } u_{n+2}$$

$$9) \text{ If } U(z) = \frac{2z^2 + 3z + 12}{(z-1)^4}, \text{ find the value of } u_2 \& u_3$$

$$10) \text{ Given that } Z(u_n) = \frac{2z^2 + 3z + 4}{(z-3)^3}, |z| > 3, \text{ show that } u_1 = 2, u_2 = 21, u_3 = 139$$

11) Show that $Z\left(\frac{1}{n}\right) = z \log \frac{z}{z-1}$

12) Using $Z(n) = \frac{z}{(z-1)^2}$, show that $Z(n \cos \theta) = \frac{(z^3 + z) \cos \theta - 2z^2}{(z^2 - 2z \cos \theta + 1)^2}$

Answers: 1. $z(e^{1/z} - 1)$ 2. $\frac{z}{z - e^{i\theta}}$ 3. $\frac{2z}{(z-1)^2} + \frac{2/\sqrt{2}}{z^2 - \sqrt{2}z + 1} - \frac{3a^4 z}{z-1}$

4. $\frac{z^3 - 3z^2 + 4z}{(z-1)^3}$ 5) $\frac{za^3}{z-a}$ 8) $\frac{z^2(1+3z^2)}{(1-z)(1+z^2)}$

9) $u_2 = 2, u_3 = 11$

Inverse Z – Transform: If $U(z)$ is Z – Transform of u_n , then u_n is called Inverse Z – Transform of $U(z)$, Denoted by $u_n = Z^{-1}[U(z)]$.

Some Useful Inverse Z – transforms:

1. $Z^{-1}\left[\frac{z}{z-a}\right] = a^n$	2. $Z^{-1}\left[\frac{z}{z-1}\right] = 1$
3. $Z^{-1}\left[\frac{az}{(z-a)^2}\right] = a^n .n$	4. $Z^{-1}\left[\frac{z}{(z-1)^2}\right] = n$
5. $Z^{-1}\left[\frac{az^2 + a^2 z}{(z-a)^3}\right] = a^n .n^2$	6. $Z^{-1}\left[\frac{z^2 + z}{(z-1)^3}\right] = n^2$
7. $Z^{-1}\left[\frac{az^3 + 4a^2 z^2 + a^3 z}{(z-a)^4}\right] = a^n .n^3$	8. $Z^{-1}\left[\frac{z^3 + 4z^2 + z}{(z-1)^4}\right] = n^3$
9. $Z^{-1}\left[\frac{z}{z^2 + 1}\right] = \sin(n\pi/2)$	10. $Z^{-1}\left[\frac{z^2}{z^2 + 1}\right] = \cos(n\pi/2)$

Evaluation of inverse Z – Transforms: We have the following 3 methods –

1. Power series method:

Example: 1) Find the inverse Z – Transform of $\log \frac{z}{z+1}$, by power series method.

<< Putting $z = 1/y$, we get $U(z) = \log \frac{z}{z+1} = \log \frac{1/y}{1/y+1} = \log \frac{1}{1+y} = -\log(1+y)$

$$= -y + \frac{1}{2}y^2 - \frac{1}{3}y^3 + \dots$$

$$= -z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{3}z^{-3} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{-n}$$

$$\therefore u_n = \begin{cases} 0 & \text{for } n = 0 \\ (-1)^n / n & \text{other wise} \end{cases}$$

2) Find the inverse Z – Transform of $\frac{z}{(z+1)^2}$, by division method

$$z^{-1} - 2z^{-2} + 3z^{-3} - \dots$$

<< By Dividing we get
$$\begin{array}{r} z^2 + 2z + 1 \overline{) z} \\ \underline{z + 2 + z^{-1}} \\ -2 - z^{-1} \\ \underline{-2 - 4z^{-1} - 2z^{-3}} \\ 3z^{-1} + 2z^{-2} \\ \underline{3z^{-1} + 6z^{-2} + 3z^{-3}} \\ -4z^{-2} - 3z^{-3} \quad \& \text{so on} \end{array}$$

Hence $\frac{z}{(z+1)^2} = z^{-1} - 2z^{-2} + 3z^{-3} - 4z^{-4} + \dots$, which is an infinite series

$$i.e. U(z) = \sum_{n=0}^{\infty} (-1)^{n-1} n z^{-n} \Rightarrow u_n = (-1)^n n$$

II. Partial fraction method:

1. Find the inverse Z – Transform of i) $\frac{2z^2 + 3z}{(z+2)(z-4)}$ ii) $\frac{z^3 - 20z}{(z-2)^3(z-4)}$

<< i) Write $U(z) = \frac{2z^2 + 3z}{(z+2)(z-4)}$ as $\frac{U(z)}{z} = \frac{2z+3}{(z+2)(z-4)} = \frac{A}{z+2} + \frac{B}{z-4}$

Then $2z+3 = A(z-4) + B(z+2) \Rightarrow 11 = 6B \Rightarrow B = 11/6$ & $-1 = -6A \Rightarrow A = 1/6$

$$\frac{U(z)}{z} = \frac{1}{6} \frac{1}{z+2} + \frac{11}{6} \frac{1}{z-4} \Rightarrow U(z) = \frac{1}{6} \frac{z}{z+2} + \frac{11}{6} \frac{z}{z-4}$$

On inversion we get $u_n = \frac{1}{6}(-2)^n + \frac{11}{6}(4)^n$

ii) Write $U(z) = \frac{z^3 - 20z}{(z-2)^3(z-4)}$ as $\frac{U(z)}{z} = \frac{z^2 - 20}{(z-2)^3(z-4)} = \frac{A+Bz+Cz^2}{(z-2)^3} + \frac{D}{z-4}$

Then $(A+Bz+Cz^2)(z-4) + D(z-2)^3 = z^2 - 20$

If $z = 4$, then $8D = 16 - 20 \Rightarrow D = -1/2$, By putting $z = 0, 1$ & -1 , we get

$$-4A - 8D = -20 \Rightarrow A + 2D = 5 \Rightarrow A - 1 = 5 \Rightarrow A = 6$$

$$(A+B+C)(-3) - D = 1 - 20 \Rightarrow -3A - 3B - 3C - D = -19 \Rightarrow 3A + 3B + 3C - 1/2 = 19$$

$$6 + B + C = \frac{19 + 1/2}{3} = \frac{39}{6} = \frac{13}{2} \Rightarrow B + C = \frac{13}{2} - 6 = \frac{1}{2} \text{----- (1)}$$

$$\& (A - B + C)(-5) - 27D = -19 \Rightarrow -5(A - B + C) = -19 + 27(-1/2) = \frac{-38 - 27}{2} = \frac{-65}{2}$$

$$\Rightarrow A - B + C = \frac{-65}{2(-5)} = \frac{13}{2} \Rightarrow -B + C = \frac{13}{2} - 6 = \frac{1}{2} \text{----- (2)}$$

By adding (1) & (2) we get

$$2C = 1 \Rightarrow C = 1/2 \Rightarrow B = 0$$

$$\begin{aligned} \therefore \frac{U(z)}{z} &= \frac{6 + 1/2z^2}{(z-2)^3} - \frac{1}{2} \frac{1}{z-4} \Rightarrow U(z) = \frac{6z + 1/2z^3}{(z-2)^3} - \frac{1}{2} \frac{z}{z-4} = \frac{1}{2} \left[\frac{12z + z^3}{(z-2)^3} - \frac{z}{z-4} \right] \\ &= \frac{1}{2} \left[\frac{z(z-2)^2 + 4z^2 + 8z}{(z-2)^3} - \frac{z}{z-4} \right] = \frac{1}{2} \left[\frac{z}{z-2} + 2 \cdot \frac{2z^2 + 2^2z}{(z-2)^3} - \frac{z}{z-4} \right] \end{aligned}$$

$$\text{On inversion, we get } u_n = \frac{1}{2} [2^n + 2 \cdot 2^n n^2 - 4^n] = 2^{n-1} + n^2 2^n + 2^{2n-1}$$

2. Find the inverse Z – Transform of $\frac{2(z^2 - 5z + 6.5)}{(z-2)(z-3)^2}$, for $2 < |z| < 3$

$$<< U(z) = \frac{2(z^2 - 5z + 6.5)}{(z-2)(z-3)^2} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{(z-3)^2}, \text{ Then}$$

$$2(z^2 - 5z + 6.5) = A(z-3)^2 + B(z-2)(z-3) + C(z-2)$$

$$z = 2 \Rightarrow 2(4 - 10 + 6.5) = A \Rightarrow A = 1, \quad z = 3 \Rightarrow 2(9 - 15 + 6.5) = C \Rightarrow C = 1$$

$$z = 0 \Rightarrow 13 = 9A + 6B - 2C, \text{ i.e. } 13 = 9 + 6B - 2 \Rightarrow 6B = 6 \Rightarrow B = 1$$

$$\therefore U(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{(z-3)^2} = \frac{1}{z} \left(1 - \frac{2}{z}\right)^{-1} - \frac{1}{3} \left(1 - \frac{z}{3}\right)^{-1} + \frac{1}{9} \left(1 - \frac{z}{3}\right)^{-2}$$

(so that $2/z < 1$ & $z/3 < 1$)

$$= \frac{1}{z} \left[1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \dots \right] - \frac{1}{3} \left[1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \dots \right] + \frac{1}{9} \left[1 + \frac{2z}{3} + \frac{3z^2}{9} + \frac{4z^3}{27} + \dots \right]$$

Where $2 < |z| < 3$

$$= \left(\frac{1}{z} + \frac{2}{z^2} + \frac{2^2}{z^3} + \frac{2^3}{z^4} + \dots \right) - \left(\frac{1}{3} + \frac{z}{3^2} + \frac{z^2}{3^3} + \frac{z^3}{3^4} + \dots \right) + \left(\frac{1}{3^2} + \frac{2z}{3^3} + \frac{3z^2}{3^4} + \frac{4z^3}{3^5} + \dots \right)$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} \left(\frac{1}{3} \right)^{n+1} z^n + \sum_{n=0}^{\infty} (n+1) \left(\frac{1}{3} \right)^{n+2} z^n$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} \left[(n+1) \left(\frac{1}{3} \right)^{n+2} - \left(\frac{1}{3} \right)^{n+1} \right] z^n$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} (n-2) \left(\frac{1}{3}\right)^{n+2} z^n$$

On inversion, we get

$$u_n = 2^{n-1}, n \geq 1 \text{ \& } u_n = -(n+2)3^{n-2}, n \leq 0$$

Exercises: Find the inverse Z – Transform of the following:

1. $\frac{z^2}{(z-1)(z-3)}$
2. $\frac{4z}{z-a}, |z| > |a|$
3. $\frac{5z}{(2-z)(3z-1)}$
4. $\frac{z}{(z-1)^2}$
5. $\frac{3z^2 + z}{(5z-1)(5z+2)}$
6. $\frac{8z - z^3}{(4-z)^3}$
7. $\frac{4z^2 - 2z}{z^3 - 5z^2 + 8z - 4}$
8. $\frac{z+3}{(z+1)(z-2)}$

Answers:

1. $\frac{1}{2}(3^{n+1} - 1)$
2. $4a^n$
3. $\left(\frac{1}{3}\right)^n - 2^n$
4. n
5. $\frac{13}{75}(0.2)^n + \frac{4}{75}(-0.4)^n$
6. $(n^2 + 7n + 4)(4)^{n-1}$
7. $2 + 2^n + 3(n-1)2^n, (n \geq 1)$
8. $2(-i)^{n-1} - (-2)^{n-1}$

Application to difference equations:

Procedure to solve a linear difference equation with constant coefficients by Z 0

Transforms:

1. Take the Z – transforms of both sides of the difference equation.
2. Transpose all terms without $U(z)$ to the right.
3. Divide by the coefficient of $U(z)$, getting $U(z)$ as a function of z .
4. Find inverse Z – Transform to get u_n .

Example: 1) Solve $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$ with $u_0 = 0, u_1 = 1$

Solution: If $Z(u_n) = U(z)$, Then $Z(u_{n+1}) = z[U(z) - u_0]$ & $Z(u_{n+2}) = z^2[U(z) - u_0 - u_1 z^{-1}]$

$$\text{Also } Z(3^n) = \frac{z}{z-3}$$

$\therefore$ Taking the Z – Transform on both sides, we get

$$Z[u_{n+2}] + 4Z[u_{n+1}] + 3Z[u_n] = Z[3^n]$$

$$\Rightarrow z^2[U(z) - u_0 - u_1 z^{-1}] + 4z[U(z) - u_0] + 3U(z) = \frac{z}{z-3}$$

$$\Rightarrow z^2[U(z) - 0 - 1 \cdot z^{-1}] + 4z[U(z) - 0] + 3U(z) = \frac{z}{z-3}, (\text{Using the given conditions}).$$

$$\text{i.e. } (z^2 + 4z + 3)U(z) = z + \frac{z}{z-3} = \frac{z^2 - 3z + z}{z-3} = \frac{z^2 - 2z}{z-3}$$

$$\therefore U(z) = \frac{z^2 - 2z}{(z^2 + 4z + 3)(z-3)} = \frac{z^2 - 2z}{(z+1)(z+3)(z-3)}$$

$$\text{Let } \frac{U(z)}{z} = \frac{z-2}{(z+1)(z+3)(z-3)} = \frac{A}{z+1} + \frac{B}{z+3} + \frac{C}{z-3}$$

$$\Rightarrow z-2 = A(z+3)(z-3) + B(z+1)(z-3) + C(z+1)(z+3)$$

$$z = -3 \Rightarrow -5 = B(-2)(-6) \Rightarrow B = -\frac{5}{12}, \quad z = 3 \Rightarrow 1 = C(4)(6) \Rightarrow C = \frac{1}{24},$$

$$z = -1 \Rightarrow -3 = A(2)(-4) \Rightarrow A = \frac{3}{8}.$$

$$\therefore \frac{U(z)}{z} = \frac{3}{8} \frac{1}{z+1} - \frac{5}{12} \frac{1}{z+3} + \frac{1}{24} \frac{1}{z-3} \Rightarrow U(z) = \frac{3}{8} \frac{z}{z+1} - \frac{5}{12} \frac{z}{z+3} + \frac{1}{24} \frac{z}{z-3},$$

On inversion we get,

$$u_n = \frac{3}{8} Z^{-1} \left[\frac{z}{z+1} \right] - \frac{5}{12} Z^{-1} \left[\frac{z}{z+3} \right] + \frac{1}{24} Z^{-1} \left[\frac{z}{z-3} \right] = \frac{3}{8} (-1)^n - \frac{5}{12} (-3)^n + \frac{1}{24} 3^n$$

2) Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ with $y_0 = y_1 = 0$, using Z - Transform.

Solution: Let $Z(y_n) = U(z)$, then $Z(y_{n+1}) = z[U(z) - y_0]$ & $Z(y_{n+2}) = z^2[U(z) - y_0 - y_1 z^{-1}]$

Also $Z(2^n) = \frac{z}{z-2}$. $\therefore$ by taking Z - Transform we get,

$$Z[y_{n+2}] + 6Z[y_{n+1}] + 9Z[y_n] = Z[2^n] \Rightarrow z^2[U(z) - y_0 - y_1 z^{-1}] + 6z[U(z) - y_0] + 9U(z) = \frac{z}{z-2}$$

$$\text{i.e. } z^2[U(z) - 0 - 0 \cdot z^{-1}] + 6z[U(z) - 0] + 9U(z) = \frac{z}{z-2} \quad (\text{Using the condition } y_0 = y_1 = 0)$$

$$\Rightarrow (z^2 + 6z + 9)U(z) = \frac{z}{z-2} \Rightarrow U(z) = \frac{z}{(z^2 + 6z + 9)(z-2)}$$

$$\text{Let } \frac{U(z)}{z} = \frac{1}{(z^2 + 6z + 9)(z-2)} = \frac{1}{(z+3)^2(z-2)} = \frac{A}{z+3} + \frac{B}{(z+3)^2} + \frac{C}{z-2}$$

$$\therefore 1 = A(z+3)(z-2) + B(z-2) + C(z+3)^2$$

$$z = 2 \Rightarrow 1 = C(25) \Rightarrow C = \frac{1}{25}, \quad z = -3 \Rightarrow 1 = B(-5) \Rightarrow B = -\frac{1}{5} \quad \& \quad z = 0 \Rightarrow 1 = -6A - 2B + 9C$$

$$\Rightarrow 1 = -6A - 2\left(-\frac{1}{5}\right) + 9\left(\frac{1}{25}\right) \Rightarrow 6A = \frac{9}{25} + \frac{2}{5} - 1 = \frac{9+10-25}{25} = -\frac{6}{25} \Rightarrow A = -\frac{1}{25}$$

$$\therefore \frac{U(z)}{z} = -\frac{1}{25} \frac{1}{z+3} - \frac{1}{5} \frac{1}{(z+3)^2} + \frac{1}{25} \frac{1}{z-2} = \frac{1}{25} \left[\frac{1}{z-2} - \frac{1}{z+3} - \frac{5}{(z+3)^2} \right]$$

$$\Rightarrow U(z) = \frac{1}{25} \left[\frac{z}{z-2} - \frac{z}{z+3} - \frac{5z}{(z+3)^2} \right], \text{ On taking inverse Z - transform we get,}$$

$$y_n = \frac{1}{25} \left[Z^{-1} \left[\frac{z}{z-2} \right] - Z^{-1} \left[\frac{z}{z+3} \right] + \frac{5}{3} Z^{-1} \left[\frac{-3z}{(z+3)^2} \right] \right]$$

$$= \frac{1}{25} \left[2^n - (-3)^n + \frac{5}{3} (n(-3)^n) \right] \left[\because Z^{-1} \left[\frac{az}{(z-a)^2} \right] = na^n \right]$$

3) Find the response of the system $y_{n+2} - 5y_{n+1} + 6y_n = u_n$ with $y_0 = 0$, $y_1 = 1$ & $u_n = 1$ for $n = 0, 1, 2, \dots$ by Z - Transform method.

Solution: Taking Z - Transform of both sides we get,

$$Z[y_{n+2}] - 5Z[y_{n+1}] + 6Z[y_n] = Z[u_n] = Z[1]$$

$$\text{i.e. } z^2[U(z) - y_0 - y_1 z^{-1}] - 5z[U(z) - y_0] + 6U(z) = \frac{z}{z-1}, \text{ Using } y_0 = 0 \text{ \& } y_1 = 1 \text{ we get,}$$

$$z^2[U(z) - 0 - 1 \cdot z^{-1}] - 5z[U(z) - 0] + 6U(z) = \frac{z}{z-1}$$

$$\Rightarrow (z^2 - 5z + 6)U(z) - z = \frac{z}{z-1} \Rightarrow (z^2 - 5z + 6)U(z) = z + \frac{z}{z-1} = \frac{z^2 - z + z}{z-1} = \frac{z^2}{z-1}$$

$$\Rightarrow U(z) = \frac{z^2}{(z-1)(z^2 - 5z + 6)} = \frac{z^2}{(z-1)(z-2)(z-3)}$$

$$\text{Let } \frac{U(z)}{z} = \frac{z}{(z-1)(z-2)(z-3)} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3}$$

$$\Rightarrow z = A(z-2)(z-3) + B(z-1)(z-3) + C(z-1)(z-2)$$

$$z = 1 \Rightarrow 1 = A(-1)(-2) \Rightarrow A = \frac{1}{2}, \quad z = 2 \Rightarrow 2 = B(1)(-1) \Rightarrow B = -2,$$

$$z = 3 \Rightarrow 3 = C(2)(1) \Rightarrow C = \frac{3}{2}$$

$$\therefore \frac{U(z)}{z} = \frac{1}{2} \frac{1}{z-1} - \frac{2}{z-2} + \frac{3}{2} \frac{1}{z-3} \Rightarrow U(z) = \frac{1}{2} \frac{z}{z-1} - 2 \frac{z}{z-2} + \frac{3}{2} \frac{z}{z-3}, \text{ inversion we get}$$

$$y_n = \frac{1}{2} Z^{-1} \left[\frac{z}{z-1} \right] - 2 Z^{-1} \left[\frac{z}{z-2} \right] + \frac{3}{2} Z^{-1} \left[\frac{z}{z-3} \right] = \frac{1}{2} (1)^n - 2 \cdot 2^n + \frac{3}{2} 3^n = 2^{-1} - 2^{n+1} + 3^{n+1} 2^{-1}$$

4. Using the Z - Transform, solve $u_{n+2} - 2u_{n+1} + u_n = 3n + 5$

Solution: Taking Z - Transform of both sides we get,

$$Z[u_{n+2}] - 2Z[u_{n+1}] + Z[u_n] = Z[3n + 5]$$

$$\begin{aligned}
&\Rightarrow z^2[U(z) - u_0 - u_1 z^{-1}] - 2z[U(z) - u_0] + U(z) = 3 \frac{z}{(z-1)^2} + 5 \frac{z}{z-1} \\
&\Rightarrow (z^2 - 2z + 1)U(z) - z^2 u_0 - u_1 z + 2z u_0 = \frac{3z}{(z-1)^2} + \frac{5z}{z-1} \\
&\Rightarrow (z^2 - 2z + 1)U(z) = \frac{3z + 5z^2 - 5z}{(z-1)^2} + (z^2 - 2z)u_0 + u_1 z = \frac{5z^2 - 2z}{(z-1)^2} + (z^2 - 2z)u_0 + u_1 z \\
&\Rightarrow U(z) = \frac{5z^2 - 2z}{(z-1)^4} + \frac{(z^2 - 2z)}{(z-1)^2} u_0 + u_1 \frac{z}{(z-1)^2}, \text{ On inversion we get,} \\
&u_n = Z^{-1} \left[\frac{5z^2 - 2z}{(z-1)^4} \right] + u_0 Z^{-1} \left[\frac{(z^2 - 2z)}{(z-1)^2} \right] + u_1 Z^{-1} \left[\frac{z}{(z-1)^2} \right] \text{----- (I)}
\end{aligned}$$

Now we know that $Z^{-1}[1] = \frac{z}{z-1}$, $Z^{-1}[n] = \frac{z}{(z-1)^2}$, $Z^{-1}[n^2] = \frac{z^2 + z}{(z-1)^3}$ &

$$Z^{-1}[n^3] = \frac{z^3 + 4z^2 - z}{(z-1)^4}.$$

$\therefore$ We can write $\frac{5z^2 - 2z}{(z-1)^4}$ as

$$\begin{aligned}
\frac{5z^2 - 2z}{(z-1)^4} &= A \frac{z^3 + 4z - z}{(z-1)^4} + B \frac{z^2 + z}{(z-1)^3} + C \frac{z}{(z-1)^2} + D \frac{z}{z-1} \\
\Rightarrow 5z^2 - 2z &= A(z^3 + 4z^2 - z) + B(z^2 + z)(z-1) + Cz(z-1)^2 + Dz(z-1)^3 \\
\text{i.e. } 5z^2 - 2z &= A(z^3 + 4z^2 - z) + B(z^3 - z) + C(z^3 - 2z^2 + z) + D(z^4 - 3z^3 + 3z^2 - z)
\end{aligned}$$

By equating coefficient of like powers of z, we get

$$D = 0, \quad A + B + C - 3D = 0, \quad 4A - 2C + 3D = 5, \quad -A - B + C - D = -2$$

$$D = 0 \Rightarrow A + B + C = 0 \text{----- (1)}$$

$$4A - 2C = 5 \text{----- (2)}$$

$$-A - B + C = -2 \text{----- (3)}$$

Adding (1) & (3) we get $2C = -2 \Rightarrow C = -1$

From (2) we get $4A - 2(-1) = 5 \Rightarrow 4A = 5 - 2 = 3 \Rightarrow A = \frac{3}{4}$.

From (3) we get $B = -A + C + 2 = -\frac{3}{4} - 1 + 2 = \frac{-3 - 4 + 8}{4} = \frac{1}{4}$.

$$\therefore \frac{5z^2 - 2z}{(z-1)^4} = \frac{3}{4} \frac{z^3 + 4z - z}{(z-1)^4} + \frac{1}{4} \frac{z^2 + z}{(z-1)^3} - \frac{z}{(z-1)^2}$$

$$\text{Hence } Z^{-1}\left[\frac{5z^2-2z}{(z-1)^4}\right] = \frac{3}{4}Z^{-1}\left[\frac{z^3+4z-z}{(z-1)^4}\right] + \frac{1}{4}Z^{-1}\left[\frac{z^2+z}{(z-1)^3}\right] - Z^{-1}\left[\frac{z}{(z-1)^2}\right]$$

$$= \frac{3}{4}n^3 + \frac{1}{4}n^2 - n = \frac{1}{4}(3n^3 + n^2 - 4n) = \frac{n}{4}(3n^2 + n - 4) = \frac{1}{4}n(n-1)(3n+4)$$

$$\frac{z-2}{(z-1)^2} = \frac{A}{z-1} + \frac{B}{(z-1)^2} \Rightarrow z-2 = A(z-1) + B$$

$$z=1 \Rightarrow -1 = B \Rightarrow B = -1 \text{ \& } z=0 \Rightarrow -2 = -A + B \Rightarrow A = B + 2 = -1 + 2 = 1$$

$$\therefore \frac{z-2}{(z-1)^2} = \frac{1}{z-1} - \frac{1}{(z-1)^2} \Rightarrow \frac{z^2-2z}{(z-1)^2} = \frac{z}{z-1} - \frac{z}{(z-1)^2}$$

$$\therefore Z^{-1}\left[\frac{z^2-2z}{(z-1)^2}\right] = Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z}{(z-1)^2}\right] = 1 - n \text{ \& } Z^{-1}\left[\frac{z}{(z-1)^2}\right] = n$$

Substituting these values in (I) we get

$$u_n = \frac{1}{4}n(n-1)(3n+4) + u_0(1-n) + u_1n = \frac{1}{4}n(n-1)(3n+4) + u_0 + (u_1 - u_0)n$$

$$\frac{1}{4}n(n-1)(3n+4) + c_0 + c_1n, \text{ Where } c_0 = u_0 \text{ \& } c_1 = u_1 - u_0$$

Exercises: Solve the following difference equations using Z – Transforms

1. $y_{n+2} - 4y_n = 0$ given that $y_0 = 0, y_1 = 2$
2. $u_{n+2} - 5u_{n+1} + 6u_n = 0$
3. $x(n+2) - 3x(n+1) + 2x(n) = 0$, given that $x(0) = 0, x(1) = 1$
4. $f(n) = 3f(n-1) - 4f(n-2) = 0, n \geq 2$, given that $f(0) = 3, f(1) = -2$
5. $u_{n+2} - 5u_{n+1} - 6u_n = 2^n$
6. $y_{n+2} - 6y_{n+1} + 9y_n = 3^n$
7. $y_{n+1} + \frac{1}{4}y_n = \left(\frac{1}{4}\right)^n$ ($n \geq 0$), $y_0 = 0$
8. $y_{n+2} + 2y_{n+1} + y_n = n$ with $y_0 = y_1 = 0$
9. $u_{k+2} - 2u_{k+1} + u_k = 2^k$ with $u_0 = 2, u_1 = 1$
10. $y_{n+2} - 5y_{n+1} - 6y_n = 4^n$

$$\text{Answers: 1. } y_n = 2^{n-1} + (-2)^{n-1} \quad 2. u_n = c_1.2^n + c_2.3^n \quad 3. x(n) = 1 - 2^n$$

$$4. f(n) = 2 + (-4)^n \quad 5. u_n = c_1(-1)^n + c_2(6)^n - \frac{1}{12}(2)^n$$

$$6. y_n = (c_1 + c_2n)3^n + \frac{1}{2}n(n-1)3^{n-2} \quad 7. y_n = 2\left[\left(\frac{1}{4}\right)^n - \left(-\frac{1}{4}\right)^n\right]$$

$$8. y_n = \frac{3}{4}n \quad 9. u_k = 1 - 2k + 2^k \quad 10. y_n = c_1(6)^n + c_2(-1)^n - \frac{1}{10}4^n$$

MODULE- 4

PROBABILITY DISTRIBUTIONS

Introduction

The three main components of statistics are collection of data, analysis of data and inference. The scope of statistics is very vast and finds its applications in almost all spheres. Statistical methods guides us in any scientific enquiry and leans heavily on probability theory.

Basic Definitions and concepts

Experiment is a process by which measurement or observation is obtained.

Random experiment : An experiment which does not result in the same out come when performed under the same conditions.

Eg : 1. Tossing of a fair coin.

2. Throwing of a fair die.

Sample space : The set of all possible outcomes of a random experiment is called a sample space usually denoted by S.

Eg 1. $S = \{H,T\}$ is the sample space in tossing of a single coin.

2. $S = \{HH,TH,HT,TT\}$ is the sample space obtained in tossing of two coins.

3. $S = \{1,2,3,4,5,6\}$ in throwing of a die.

Event : Any event is a subset of an appropriate sample space.

Exhaustive event : An event consisting of all the various possibilities is called an exhaustive event.

Mutually exclusive events : Events are said to be mutually exclusive if and only if one of them can take place at a time, in other words happening of one event prevent the simultaneous happening of the others.

Eg: In the tossing of a coin, either heads or tails may turn up but not both.

Independent events: Two or more events are said to be independent if the happening or non happening of one event does not prevent the happening or non happening of the others.

Eg: 1. When two coins are tossed the event of getting head is an independent event as both the coins can turn out heads.

Probability is the measure of the chance with which we can expect the event to occur. If an event A can occur in **m** different ways out of number of **n** possible ways, all of which are equally likely then the probability of the event A denoted by P(A) is given by $P(A) = \frac{m}{n}$.

Note: 1) Since $0 \leq m \leq n \Rightarrow P(A)$ is non negative and $0 \leq P(A) \leq 1$

2) If $P(A) = 0$ then A is called a null event.

3) If $P(A) = 1$ then A is called a sure event.

4) $P(A) + P(\bar{A}) = 1$, the set of all unfavourable events is denoted by $\bar{A}$.

Addition theorem of probability

The probability of the happening of one or the other mutually exclusive events is equal to the sum of the probabilities of the two events.

That is, if A, B are two mutually exclusive events then,

$$P(A \text{ or } B) = P(A) + P(B)$$

Product theorem of probability

If a compound event is made up of a number of independent events, the probability of the happening of the compound event is equal to the product of the probabilities of the independent events.

That is, if A, B are independent events then,

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

RANDOM VARIABLE & PROBABILITY DISTRIBUTION

Introduction

When we consider an experiment of tossing two coins. The sample points or out comes are qualitative and are described by their attributes though the probabilities attached to each of these outcomes are quantitative. We now associate to each sample point a real number called the **random variable**.

Random variable

Definition: Let S be a sample space then $X : S \rightarrow R$ defined over the elements of S is called a **random variable**. A random variable is a function that assigns a real number to every sample point in the sample space of a random experiment. Random variables are usually denoted by X, Y, Z ... and it may be noted that different random variables may be associated with the sample space S. The set of all real numbers of a random variable X is called the range of X.

Eg:

1. While tossing a coin, suppose that the value 1 is associated for the outcome 'head' and 0 for the outcome tail. We have the sample space $S = \{H, T\}$ and if X is the random variable, then $X(H) = 1$ & $X(T) = 0$.

Range of X = $\{0, 1\}$

2. Suppose a coin is tossed twice, we shall associate two different random variables X, Y as follows, where we have the sample space

$$S = \{HH, HT, TH, TT\}$$

X = Number of 'heads' in the outcome.

The association of the elements in S to X is as follows.

Out come	HH	HT	TH	TT
Random variable X	2	1	1	0

Range of $X = \{0,1,2\}$

Suppose $Y =$ Number of 'tails' in the outcome, we have

Out come	HH	HT	TH	TT
Random variable Y	0	1	1	2

Range of $Y = \{0,1,2\}$

Discrete and Continuous random variables

If a random variable takes finite or countably infinite number of values then it is called a discrete random variable.

Discrete random variable will have finite or countably infinite range.

Eg: 1. Tossing of a coin and observing the outcome.

2. Throwing a 'die' and observing the numbers on the face.

If a random variable takes non countable number of values then it is called a non discrete or continuous random variable.

Eg:

1. Weight of articles

2. Length of nails produced by machine.

Generally counting problems correspond to discrete random variables and measuring problems lead to continuous random variables.

Probability function and Discrete probability distribution

If for each value x_i of a discrete random variable X , we assign a real number $p(x_i)$ such that

i) $p(x_i) \geq 0$ ii) $\sum_i p(x_i) = 1$ then the function $p(x)$ is called a probability function.

If the probability that X take the values x_i is p_i , then $P(X = x_i) = p_i$ or $p(x_i)$.

The set of values $[x_i, p(x_i)]$ is called a discrete (finite) probability distribution of the discrete random variable X . The function $P(X)$ is called the probability density function (p.d.f) or the probability mass function (p.m.f)

The distribution function $f(x)$ defined by $f(x) = P(X \leq x) = \sum_{i=1}^n P(x_i)$, x being an integer, is called the cumulative distribution function (c.d.f).

The mean and variance of the discrete probability distribution are defined as follows

Mean $\mu = \sum_i x_i \cdot p(x_i)$

Variance $V = \sum_i (x_i - \mu)^2 \cdot p(x_i)$

$$V = \sum_i x_i^2 \cdot p(x_i) - [\sum_i x_i \cdot p(x_i)]^2$$

$$V = \sum_i x_i^2 \cdot p(x_i) - \mu^2$$

Standard deviation $\sigma = \sqrt{V}$

Problems

1. A coin is tossed twice. A random variable X represent the number of heads turning up. Find the discrete probability distribution for X . Also find its mean and variance.

Sol: Here $S = \{HH, HT, TH, TT\}$

$$X = \{0, 1, 2\}$$

$$\text{Now, } P(HH) = \frac{1}{4} \quad P(HT) = \frac{1}{4} \quad P(TH) = \frac{1}{4} \quad P(TT) = \frac{1}{4}$$

$$P(X = 0, \text{ i.e., no head turning up}) = P(TT) = \frac{1}{4}$$

$$P(X = 1, \text{ i.e., one head turning up}) = P(HT \cup TH) = P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P(X = 2, \text{ i.e., two heads turning up}) = P(HH) = \frac{1}{4}$$

The discrete probability distribution for X is as follows

$X=x_i$	0	1	2
$p(x_i)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

We observe that, $p(x_i) > 0$ and $\sum_i p(x_i) = 1$.

$$\text{Mean, } \mu = \sum_i x_i \cdot p(x_i) = (0) \cdot \left(\frac{1}{4}\right) + (1) \cdot \left(\frac{1}{2}\right) + (2) \cdot \left(\frac{1}{4}\right) = \frac{1}{2} + \frac{1}{2}$$

$$\mu = 1$$

$$\text{Variance, } V = \sum_i (x_i - \mu)^2 \cdot p(x_i)$$

$$V = (0 - 1)^2 \cdot \frac{1}{4} + (1 - 1)^2 \cdot \frac{1}{2} + (2 - 1)^2 \cdot \frac{1}{4}$$

$$V = \frac{1}{4} + 0 + \frac{1}{4}$$

$$V = \frac{1}{2}$$

2. A random experiment of tossing a 'die' twice is performed. Random variable X on this sample space is defined to be the sum of the two numbers turning up on the toss. Find the discrete probability distribution for the random variable X and compute the corresponding mean and standard deviation.

Sol: Here $S = \{(x, y) \text{ where } x = 1, 2, \dots, 6 ; y = 1, 2, \dots, 6\}$

i.e., $S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), \dots, (6, 5), (6, 6)\}$

Number of elements in $S=36$.

The set of values of the random variable X defined as sum of two numbers on the face of the 'die' are $\{2,3,4,5, \dots, 10,11,12\}$

Now, $p(x_1) = \frac{n(E)}{n(S)} = \frac{1}{36}$; $p(x_2) = \frac{2}{36}$; ...

The discrete probability distribution for X is as follows

X = x_i	2	3	4	5	6	7	8	9	10	11	12
$p(x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Here $p(x_i) > 0$ and $\sum_i p(x_i) = 1$.

Mean $\mu = \sum_i x_i \cdot p(x_i)$

$$\mu = (2) \cdot \left(\frac{1}{36}\right) + (3) \cdot \left(\frac{2}{36}\right) + \dots + (11) \cdot \left(\frac{2}{36}\right) + (12) \cdot \left(\frac{1}{36}\right)$$

$$\mu = \frac{252}{36} = 7$$

Variance, $V = \sum_i (x_i - \mu)^2 \cdot p(x_i)$

$$V = (2 - 7)^2 \cdot \frac{1}{36} + (3 - 7)^2 \cdot \frac{2}{36} + \dots + (12 - 7)^2 \cdot \frac{1}{36}$$

$$V = \frac{210}{36} = \frac{35}{6}$$

Std. deviation , $\sigma = \sqrt{V}$

$$\sigma = \sqrt{\frac{35}{6}} = 2.42$$

3. Show that the following distribution represents a discrete probability distribution. Find the mean and variance

x	10	20	30	40
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Sol: We observe that $p(x) > 0$ and $\sum p(x) = 1$.

Mean $\mu = \sum x \cdot p(x)$

$$\mu = (10) \cdot \frac{1}{8} + (20) \cdot \frac{3}{8} + (30) \cdot \frac{3}{8} + (40) \cdot \frac{1}{8}$$

$$\mu = \frac{10+60+90+40}{8} = \frac{200}{8}$$

$$\mu = 25$$

$$\text{Variance, } V = \sum (x - \mu)^2 \cdot p(x)$$

$$V = (10 - 25)^2 \cdot \frac{1}{8} + (20 - 25)^2 \cdot \frac{3}{8} + (30 - 25)^2 \cdot \frac{3}{8} + (40 - 25)^2 \cdot \frac{1}{8}$$

$$V = \frac{600}{8} = 75$$

4. Find the value of k such that the following distribution represents a finite probability distribution. Hence find its mean and standard deviation. Also find $p(x \leq 1)$, $p(x > 1)$ and $p(-1 < x \leq 2)$

x	-3	-2	-1	0	1	2	3
p(x)	k	2k	3k	4k	3k	2k	k

Sol: We must have $p(x) \geq 0$ for all x and $\sum p(x) = 1$. The first condition is satisfied if $k \geq 0$

$$\text{Since } \sum p(x) = 1$$

$$k + 2k + 3k + 4k + 3k + 2k + k = 1$$

$$16k = 1$$

$$\therefore k = \frac{1}{16}$$

The discrete probability distribution is as follows

x	-3	-2	-1	0	1	2	3
p(x)	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

$$\text{Mean } \mu = \sum x \cdot p(x)$$

$$\mu = (-3) \cdot \frac{1}{16} + (-2) \cdot \frac{2}{16} + (-1) \cdot \frac{3}{16} + (0) \cdot \frac{4}{16} + (1) \cdot \frac{3}{16} + (2) \cdot \frac{2}{16} + (3) \cdot \frac{1}{16}$$

$$\mu = 0$$

Variance, $V = \sum (x - \mu)^2 \cdot p(x)$

$$V = (-3 - 0)^2 \cdot \frac{1}{16} + (-2 - 0)^2 \cdot \frac{2}{16} + (-1 - 0)^2 \cdot \frac{3}{16} + (0 - 0)^2 \cdot \frac{4}{16} + (1 - 0)^2 \cdot \frac{3}{16} + (2 - 0)^2 \cdot \frac{2}{16} + (3 - 0)^2 \cdot \frac{1}{16}$$

$$V = \frac{1}{16} (9 + 8 + 3 + 0 + 3 + 8 + 9) = \frac{40}{16} = \frac{5}{2}$$

Standard deviation $\sigma = \sqrt{V}$

$$\sigma = \sqrt{\frac{5}{2}} = 1.58$$

Now, $p(x \leq 1) = p(-3) + p(-2) + p(-1) + p(0) + p(1)$

$$= \frac{1}{16} + \frac{2}{16} + \frac{3}{16} + \frac{4}{16} + \frac{3}{16}$$

$$p(x \leq 1) = \frac{13}{16}$$

$p(x > 1) = p(2) + p(3)$

$$= \frac{2}{16} + \frac{1}{16}$$

$$p(x > 1) = \frac{3}{16}$$

$p(-1 < x \leq 2) = p(0) + p(1) + p(2)$

$$= \frac{4}{16} + \frac{3}{16} + \frac{2}{16}$$

$$p(-1 < x \leq 2) = \frac{9}{16}$$

5. A random variable X has the following probability function for various values of x.

i) Find k ii) Evaluate $P(x < 6)$, $P(x \geq 6)$ and $P(3 < x \leq 6)$. Also find the probability

x	0	1	2	3	4	5	6	7
P(x)	0	k	2k	2k	3k	k ²	2k ²	7k ² + k

distribution and the distribution function of X.

Sol: We must have $P(x) \geq 0$ and $\sum P(x) = 1$.

The first condition is satisfied for $k \geq 0$ and we have to find k such that

$$\sum P(x) = 1$$

$$\text{i.e., } 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + (7k^2 + k) = 1$$

$$10k^2 + 9k - 1 = 0$$

$$(10k - 1)(k + 1) = 0$$

$$k = \frac{1}{10} \quad \text{and} \quad k = -1 \text{ (Neglect)}$$

Hence we have the following table

x	0	1	2	3	4	5	6	7
P(x)	0	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{1}{50}$	$\frac{17}{100}$

Now, $P(x < 6) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$

$$= 0 + \frac{1}{10} + \frac{1}{5} + \frac{1}{5} + \frac{3}{10} + \frac{1}{100} = \frac{81}{100}$$

$$\mathbf{P(x < 6) = 0.81}$$

$$P(x \geq 6) = P(6) + P(7)$$

$$P(x \geq 6) = \frac{1}{50} + \frac{17}{100} = \frac{19}{100}$$

$$\mathbf{P(x \geq 6) = 0.19}$$

$$P(3 < x \leq 6) = P(4) + P(5) + P(6)$$

$$= \frac{3}{10} + \frac{1}{100} + \frac{1}{50} = \frac{33}{100}$$

$$\mathbf{P(3 < x \leq 6) = 0.33}$$

The probability distribution is as follows

x	0	1	2	3	4	5	6	7
P(x)	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

The distribution function of X is $f(x) = P(X \leq x) = \sum_{i=1}^n p(x_i)$ is called cumulative distribution function and same is as follows

x	0	1	2	3	4	5	6	7
f(x)	0	0+0.1 =0.1	0.1+0.2 =0.3	0.3+0.2 =0.5	0.5+0.3 =0.8	0.8+0.01 =0.81	0.81+0.02=0.83	0.83+0.17=1

6. A random variable X take the values $-3, -2, -1, 0, 1, 2, 3$ such hat $P(X = 0) = P(X < 0)$ and $P(X = -3) = P(X = -2) = P(X = -1) = P(X = 1) = P(X = 2) = P(X = 3)$.Find the probability distribution.

Sol: Let the distribution be as follows

X	-3	-2	-1	0	1	2	3
P(X)	p_1	p_2	p_3	p_4	p_5	p_6	p_7

By data $P(X = 0) = P(X < 0)$

$$\Rightarrow P(X = 0) = P(X = -1) + P(X = -2) + P(X = -3)$$

$$\text{i.e., } p_4 = p_3 + p_2 + p_1 \quad \text{--- (1)}$$

Also we have

$$P(X = -3) = P(X = -2) = P(X = -1) = P(X = 1) = P(X = 2) = P(X = 3)$$

$$\text{i.e., } p_1 = p_2 = p_3 = p_5 = p_6 = p_7 \quad \text{--- (2)}$$

Further we must have

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 = 1 \quad \text{--- (3)}$$

$$\text{Using (2) in (1), we get } p_4 = p_1 + p_1 + p_1$$

$$\mathbf{p_4 = 3p_1}$$

$$\text{Using (2) in (3), we get } p_1 + p_1 + p_1 + p_4 + p_1 + p_1 + p_1 = 1$$

$$6p_1 + p_4 = 1$$

$$6p_1 + 3p_1 = 1 \quad (\text{since } p_4 = 3p_1)$$

$$9p_1 = 1$$

$$\mathbf{p_1 = \frac{1}{9}}$$

$$\text{Hence } \mathbf{p_4 = \frac{3}{9}}$$

$$\text{Thus, } p_2 = p_3 = p_5 = p_6 = p_7 = \frac{1}{9}$$

Thus the probability distribution is as follows

X	-3	-2	-1	0	1	2	3
P(X)	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

7. The p.d.f. of variate X is given by the following table.

x	0	1	2	3	4	5	6
P(x)	K	3K	5K	7K	9K	11K	13K

For what value of K, this represents a valid probability distribution? Also find $P(x \geq 5)$ and $P(3 < x \leq 6)$.

Sol: The probability distribution is valid if $P(x) \geq 0$ and $\sum P(x) = 1$

Hence we must have $K \geq 0$ and $K + 3K + 5K + 7K + 9K + 11K + 13K = 1$

$$49K = 1 \Rightarrow K = \frac{1}{49}$$

Also, $P(x \geq 5) = P(5) + P(6)$

$$P(x \geq 5) = 11K + 13K = 24K$$

$$P(x \geq 5) = \frac{24}{49}$$

Now, $P(3 < x \leq 6) = P(4) + P(5) + P(6)$

$$P(3 < x \leq 6) = 9K + 11K + 13K$$

$$P(3 < x \leq 6) = 33K$$

$$P(3 < x \leq 6) = \frac{33}{49}$$

Binomial Distribution

If p is the probability of success and q is the probability of failure, the probability of x success out of n trials is given by $P(x) = {}^nC_x p^x q^{n-x}$

Mean and Standard deviation of the Binomial Distribution

$$\text{Mean, } \mu = np$$

$$\text{Variance, } V = npq$$

$$\text{Std. Deviation, } \sigma = \sqrt{npq}$$

Problems

1. Find the binomial probability which has mean 2 and variance $\frac{4}{3}$.

Sol: Given Mean, $\mu = 2$ and Variance, $V = \frac{4}{3}$

For binomial distribution, $\mu = np$

$$np = 2$$

$$npq = V$$

$$2q = \frac{4}{3}$$

$$q = \frac{2}{3}$$

Since $p = 1 - q = 1 - \frac{2}{3}$

$$p = \frac{1}{3}$$

Since $np = 2$

$$\frac{n}{3} = 2$$

$$n = 6$$

The binomial probability function is $P(x) = {}^nC_x p^x q^{n-x}$ becomes

$$P(x) = {}^6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$$

The distribution of probability is as follows

x	0	1	2	3	4	5	6
P(x)	$\left(\frac{2}{3}\right)^6$	${}^6C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^5$	${}^6C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^4$	${}^6C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3$	${}^6C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2$	${}^6C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^1$	$\left(\frac{1}{3}\right)^6$

2. When a coin is tossed 4 times, find the probability of getting i) exactly one head
ii) atmost 3 heads iii) atleast two heads.

Sol: Given $n = 4$, $p = P(H) = \frac{1}{2}$, $q = P(T) = \frac{1}{2}$ (since $p + q = 1$)

From binomial distribution, the probability of x successes out of n trials is given by

$$P(x) = {}^nC_x p^x q^{n-x}$$

$$P(x) = {}^4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}$$

$$P(x) = {}^4C_x \left(\frac{1}{2}\right)^4$$

$$P(x) = \frac{1}{16} \cdot {}^4C_x$$

$$i) P(1 \text{ head}) = P(x = 1) = \frac{1}{16} \cdot {}^4C_1 = \frac{1}{16} (4)$$

$$P(x = 1) = \frac{1}{4}$$

$$P(x = 1) = 0.25$$

$$ii) P(\text{atmost 3 heads}) = P(x \leq 3)$$

$$P(x \leq 3) = P(x = 0) + P(x = 1) + P(x = 2) + P(x = 3)$$

$$P(x \leq 3) = \frac{1}{16} \cdot {}^4C_0 + \frac{1}{16} \cdot {}^4C_1 + \frac{1}{16} \cdot {}^4C_2 + \frac{1}{16} \cdot {}^4C_3$$

$$P(x \leq 3) = \frac{1}{16} [1 + 4 + 6 + 4]$$

$$P(x \leq 3) = \frac{15}{16}$$

$$\mathbf{P(x \leq 3) = 0.9375}$$

$$\text{iii) } P(\text{atleast 2 heads}) = P(x \geq 2)$$

$$P(x \geq 2) = 1 - P(x < 2)$$

$$P(x \geq 2) = 1 - [P(x = 0) + P(x = 1)]$$

$$P(x \geq 2) = 1 - \left[\frac{1}{16} \cdot {}^4C_0 + \frac{1}{16} \cdot {}^4C_1 \right]$$

$$P(x \geq 2) = 1 - \frac{1}{16} [1 + 4]$$

$$P(x \geq 2) = 1 - \frac{5}{16}$$

$$P(x \geq 2) = \frac{11}{16}$$

$$\mathbf{P(x \geq 2) = 0.6875}$$

3. In a consignment of electric lamps 5% are defective. If a random sample of 8 lamps are inspected. What is the probability that one or more lamps are defective?

Sol: Given $n = 8$, Probability of defective lamp $= p = 5\% = \frac{5}{100} = 0.05$

$$\therefore q = 1 - p = 1 - 0.05$$

$$\mathbf{q = 0.95}$$

From binomial distribution,

$$P(x) = {}^nC_x p^x q^{n-x}$$

$$P(x) = {}^8C_x (0.05)^x (0.95)^{8-x}$$

Probability that one or more lamps are defective

$$P(x \geq 1) = 1 - P(x < 1)$$

$$P(x \geq 1) = 1 - P(x = 0)$$

$$P(x \geq 1) = 1 - {}^8C_0 (0.05)^0 (0.95)^{8-0}$$

$$\mathbf{P(x \geq 1) = 0.3366.}$$

Thus the required probability is 0.3366

4. The probability that a person aged 60 years will live upto 70 is 0.65. What is the probability that out of 10 person aged 60 atleast 7 of them will live upto 70.

Sol: Given $n = 10$, $p = 0.65$

$$\therefore q = 1 - p = 1 - 0.65$$

$$\mathbf{q = 0.35}$$

Let x denote the number of person aged 60 years living upto 70 years.

From binomial distribution,

$$P(x) = {}^n C_x p^x q^{n-x}$$

$$P(x) = {}^{10} C_x (0.65)^x (0.35)^{10-x}$$

Probability that out of 10 person aged 60 atleast 7 of them will live upto 70

$$P(x \geq 7) = P(7) + P(8) + P(9) + P(10)$$

$$P(x \geq 7) = {}^{10} C_7 (0.65)^7 (0.35)^3 + {}^{10} C_8 (0.65)^8 (0.35)^2 + {}^{10} C_9 (0.65)^9 (0.35)^1 + {}^{10} C_{10} (0.65)^{10} (0.35)^0$$

$$\mathbf{P(x \geq 7) = 0.5138}$$

5. The number of telephone lines busy at an instant of time is a binomial variate with probability 0.1 that a line is busy. If 10 such lines are chosen at random, what is the probability that i) no line is busy ii) all lines are busy iii) atleast one line is busy iv) atmost 2 lines are busy.

Sol: Given $n = 10$, $p = 0.1$

$$\therefore q = 1 - p = 1 - 0.1$$

$$\mathbf{q = 0.9}$$

Let x denote the number of telephone lines busy.

From binomial distribution,

$$P(x) = {}^n C_x p^x q^{n-x}$$

$$\mathbf{P(x) = {}^{10} C_x (0.1)^x (0.9)^{10-x}}$$

i) Probability that no line is busy

$$P(x = 0) = {}^{10} C_0 (0.1)^0 (0.9)^{10-0} = (0.9)^{10} = \mathbf{0.3487}$$

ii) Probability that all lines are busy

$$P(x = 10) = {}^{10} C_{10} (0.1)^{10} (0.9)^0 = \mathbf{(0.1)^{10}}$$

iii) Probability that atleast one line is busy

$$P(x \geq 1) = 1 - P(x < 1)$$

$$P(x \geq 1) = 1 - P(x = 0)$$

$$P(x \geq 1) = 1 - {}^{10} C_0 (0.1)^0 (0.9)^{10}$$

$$P(x \geq 1) = 1 - 0.3487$$

$$\mathbf{P(x \geq 1) = 0.6513}$$

iv) Probability that atmost 2 lines are busy

$$P(x \leq 2) = P(x = 0) + P(x = 1) + P(x = 2)$$

$$P(x \leq 2) = (0.9)^{10} + 10c_1(0.1)^1(0.9)^2 + 10c_2(0.1)^2(0.9)^8$$

$$P(x \leq 2) = \mathbf{0.9298}$$

6. In a quiz contest of answering 'Yes' or 'No' what is the probability of guessing atleast 6 answers correctly out of 10 questions asked ? Also find the probability of the same if there are 4 options for a correct answer.

Sol: Given $n = 10$

Let x denote the correct answer .

$$\text{i) } p = \frac{1}{2} = \mathbf{0.5}$$

$$\therefore q = 1 - p = 1 - 0.5$$

$$\mathbf{q = 0.5}$$

From binomial distribution,

$$P(x) = {}^n C_x p^x q^{n-x}$$

$$P(x) = {}^{10} C_x (0.5)^x (0.5)^{10-x}$$

$$\mathbf{P(x) = {}^{10} C_x (0.5)^{10}}$$

Probability of guessing atleast 6 answers correctly out of 10 questions

$$P(x \geq 6) = P(x = 6) + P(x = 7) + P(x = 8) + P(x = 9) + P(x = 10)$$

$$P(x \geq 6) = {}^{10} C_6 (0.5)^{10} + {}^{10} C_7 (0.5)^{10} + {}^{10} C_8 (0.5)^{10} + {}^{10} C_9 (0.5)^{10} + {}^{10} C_{10} (0.5)^{10}$$

$$P(x \geq 6) = (0.5)^{10} [{}^{10} C_6 + {}^{10} C_7 + {}^{10} C_8 + {}^{10} C_9 + {}^{10} C_{10}]$$

$$P(x \geq 6) = (0.5)^{10} [210 + 120 + 45 + 10 + 1]$$

$$\mathbf{P(x \geq 6) = 0.3770}$$

$$\text{ii) } p = \frac{1}{4}$$

$$\therefore q = 1 - p = 1 - \frac{1}{4}$$

$$\mathbf{q = \frac{3}{4}}$$

From binomial distribution,

$$P(x) = {}^n C_x p^x q^{n-x}$$

$$P(x) = {}^{10} C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{10-x}$$

$$P(x) = 10c_x \frac{3^{10-x}}{4^{x+10-x}}$$

$$P(x) = 10c_x \frac{3^{10-x}}{4^{10}}$$

$$P(x) = \frac{1}{4^{10}} [3^{10-x} \cdot 10c_x]$$

Probability of guessing atleast 6 answers correctly out of 10 questions

$$P(x \geq 6) = P(x = 6) + P(x = 7) + P(x = 8) + P(x = 9) + P(x = 10)$$

$$P(x \geq 6) = \frac{1}{4^{10}} [3^4 \cdot 10c_6 + 3^3 \cdot 10c_7 + 3^2 \cdot 10c_8 + 3 \cdot 10c_9 + 10c_{10}]$$

$$P(x \geq 6) = \frac{1}{4^{10}} [81 \cdot 210 + 27 \cdot 120 + 9 \cdot 45 + 3 \cdot 10 + 1]$$

$$P(x \geq 6) = \mathbf{0.0197}$$

7. In sampling a large number of parts manufactured by a company, the mean number of defectives in samples of 20 is 2. Out of 1000 such samples how many would be expected to contain atleast 3 defective parts.

Sol: Given Mean, $\mu = 2$ and $n = 20$

$$np = 2$$

$$20p = 2$$

$$p = \frac{1}{10} = \mathbf{0.1}$$

$$q = 1 - p$$

$$q = 1 - \frac{1}{10} \quad q = \frac{9}{10} = \mathbf{0.9}$$

Let x denote the defective part.

From binomial distribution,

$$P(x) = {}^n c_x p^x q^{n-x}$$

$$P(x) = \mathbf{{}^{20} c_x (0.1)^x (0.9)^{20-x}}$$

Probability of atleast 3 defective parts

$$P(x \geq 3) = 1 - P(x < 3)$$

$$P(x \geq 3) = 1 - [P(0) + P(1) + P(2)]$$

$$P(x \geq 3) = 1 - [{}^{20} c_0 (0.1)^0 (0.9)^{20} + {}^{20} c_1 (0.1)^1 (0.9)^{19} + {}^{20} c_2 (0.1)^2 (0.9)^{18}]$$

$$P(x \geq 3) = 1 - 0.6769$$

$$P(x \geq 3) = \mathbf{0.3231}$$

Thus, the number of defectives in 1000 such samples = $1000 \cdot 0.3231 = 323.1 \cong \mathbf{323}$

8. If the mean and standard deviation of the number of correctly answered questions in a test given to 4096 students are 2.5 and $\sqrt{1.875}$. Find an estimate the number of candidates answering correctly i) 5 questions ii) 2 or less iii) 8 or more.

Sol: Given Mean = 2.5 $\sigma = \sqrt{1.875}$

Wkt, $\mu = np$ and $\sigma = \sqrt{V}$

Thus, $np = 2.5$ and $V = npq = 1.875$

Now, $\frac{npq}{np} = \frac{1.875}{2.5}$

We get $q = 0.75$, and $p = 1 - q = 0.25$

$n = 10$

Let x denote the number of correctly answered questions

$$P(x) = {}^nC_x p^x q^{n-x} = {}^{10}C_x (0.25)^x (0.75)^{10-x}$$

i) Probability that 5 questions are answered correctly

$$P(x = 5) = {}^{10}C_5 (0.25)^5 (0.75)^{10-5}$$

$P(x = 5) = 0.0584$

Estimation for 4096 students is $= 4096 * 0.0584 \cong 239$

Number of students correctly answering 5 questions is **239**

ii) Probability that 2 or less questions are answered correctly

$$P(x \leq 2) = P(x = 0) + P(x = 1) + P(x = 2)$$

$$= {}^{10}C_0 (0.25)^0 (0.75)^{10} + {}^{10}C_1 (0.25)^1 (0.75)^9 + {}^{10}C_2 (0.25)^2 (0.75)^8$$

$P(x \leq 2) = 0.5256$

Estimation for 4096 students is $= 4096 * 0.5256 \cong 2153$

Number of students correctly answering 2 or less than 2 questions is **2153**

iii) Probability that 8 or more questions are answered correctly

$$P(x \geq 8) = P(x = 8) + P(x = 9) + P(x = 10) \\ = {}^{10}C_8 (0.25)^8 (0.75)^2 + {}^{10}C_9 (0.25)^9 (0.75)^1 + {}^{10}C_{10} (0.25)^{10} (0.75)^0$$

$P(x \geq 8) = 0.0005$

Estimation for 4096 students is $= 4096 * 0.0005 \cong 2$

Number of students correctly answering 8 or more than 8 questions is **2**

. In 800 families with 5 children each how many families would be expected to have

i) 3 boys ii) 5 girls iii) either 2 or 3 boys iv) atmost 2 girls by assuming probabilities for boys and girls to be equal.

10. An airline knows that 5% of the people making reservations on a certain flight will not turn up. Consequently, their policy is to sell 52 tickets for a flight that can only hold 50 people. What is the probability that there will be a seat for every passenger who turns up?

11. 4 coins are tossed 100 times and the following results were obtained. Fit a binomial distribution for the data and calculate the theoretical frequencies.

Number of heads	0	1	2	3	4
Frequency	5	29	36	25	5

Sol: Let x denote the number of heads and f the corresponding frequency. Since the data is in the form of a frequency distribution we shall first calculate the mean.

$$\text{Mean, } \mu = \frac{\sum fx}{\sum f} = \frac{(0)(5)+(1)(29)+(2)(36)+(3)(25)+(4)(5)}{5+29+36+25+5} = \frac{196}{100} = 1.96$$

From binomial distribution, $\mu = np$. Here $n=4$

$$\text{Hence } 4p = 1.96$$

$$p = 0.49$$

$$q = 0.51$$

From binomial distribution,

$$P(x) = {}^n C_x p^x q^{n-x} \quad P(x) = {}^4 C_x (0.49)^x (0.51)^{4-x}$$

Since 4 coins were tossed, expected(theoretical) frequencies are obtained from

$$F(x) = 100P(x)$$

$$F(x) = 100.4 C_x (0.49)^x (0.51)^{4-x} \quad \text{where } x = 0, 1, 2, 3, 4.$$

$$F(0) = 100.4 C_0 (0.49)^0 (0.51)^4 = 6.765 \cong 7$$

$$F(1) = 100.4 C_1 (0.49)^1 (0.51)^3 = 25.999 \cong 26$$

$$F(2) = 100.4 C_2 (0.49)^2 (0.51)^2 = 37.47 \cong 37$$

$$F(3) = 100.4 C_3 (0.49)^3 (0.51)^1 = 24.0004 \cong 24$$

$$F(4) = 100.4 C_4 (0.49)^4 (0.51)^0 = 5.765 \cong 6$$

Thus the required theoretical frequencies are **7,26,37,24,6**.

Poisson Distribution

Poisson distribution is regarded as the limiting form of the binomial distribution when n is very large ($n \rightarrow \infty$) and p the probability of success is very small ($p \rightarrow 0$) so that np tends to a fixed finite constant say m .

Let m be a positive constant. Consider the infinite discrete probability distribution of a

random variable x is given by $P(x) = \frac{m^x}{x!} e^{-m}$

The above probability density function is called the **poisson probability** function.

Mean and Standard deviation of the Poisson Distribution

$$\text{Mean, } \mu = m$$

$$\text{Variance, } V = m$$

$$\text{Std. Deviation, } \sigma = \sqrt{m}$$

Problems

1. The probabilities of a poisson variate taking the values 3 and 4 are equal. Find the PDF. Also, calculate the probabilities of the variate taking the values 0 and 1.

Sol: Given $P(x = 3) = P(x = 4)$

$$\frac{m^3}{3!} e^{-m} = \frac{m^4}{4!} e^{-m}$$

$$\frac{4!}{3!} = \frac{m^4}{m^3}$$

$$\therefore m = 4.$$

From poisson distribution,

$$P(x) = \frac{m^x}{x!} e^{-m} = \frac{4^x}{x!} e^{-4}$$

$$P(x = 0) = \frac{4^0}{0!} e^{-4} = \mathbf{0.0183}$$

$$P(x = 1) = \frac{4^1}{1!} e^{-4} = \mathbf{0.0733}$$

2. In a certain factory turning out razor blades there is a small probability of $\frac{1}{500}$ for any blade to be defective. The blades are supplied in packets of 10. Use poisson distribution to calculate the approximate number of packets containing i) no defective ii) one defective iii) two defective blades in a consignment of 10,000 packets.

Sol: Probability of a defective blade = $p = \frac{1}{500} = 0.002$

In a packet of 10, the mean number of defective blades is

Mean, $m = np$

$$m = 10(0.002)$$

$$\mathbf{m = 0.02}$$

From poisson distribution,

$$P(x) = \frac{m^x}{x!} e^{-m}$$

$$P(x) = \frac{(0.02)^x}{x!} e^{-(0.02)}$$

$$P(x) = (0.9802) \frac{(0.02)^x}{x!} \quad (\text{Since } e^{-0.02} = 0.9802)$$

i) Probability of no defective

$$P(x = 0) = (0.9802) \frac{(0.02)^0}{0!} = 0.9802$$

$$\text{For consignment of 10,000 packets} = 10,000 * 0.9802 = 9802$$

ii) Probability of one defective

$$P(x = 1) = (0.9802) \frac{(0.02)^1}{1!} = 0.0196$$

$$\text{For consignment of 10,000 packets} = 10,000 * 0.0196 = 196$$

iii) Probability of one defective

$$P(x = 2) = (0.9802) \frac{(0.02)^2}{2!} = 0.0002$$

$$\text{For consignment of 10,000 packets} = 10,000 * 0.0002 = 2$$

3. The number of accidents in a year to taxi drivers in a city follows a Poisson distribution with mean 3. Out of 1000 taxi drivers find approximately the number of the drivers with
i) no accident in a year ii) more than 3 accidents in a year.

Sol: Given Mean=3

Mean, $\mu = m$

$$m = 3$$

From poisson distribution,

$$P(x) = \frac{m^x}{x!} e^{-m}$$

$$P(x) = \frac{(3)^x}{x!} e^{-3}$$

$$P(x) = (0.05) \frac{(3)^x}{x!} \quad (\text{Since } e^{-3} = 0.05)$$

i) Probability of number of drivers with no accident in a year

$$P(x = 0) = (0.05) \frac{(3)^0}{0!} = 0.05$$

$$\text{Number of drivers out of 1000 with no accident in a year} = 1,000 * 0.05 = 50$$

ii) Probability of more than 3 accidents in a year

$$P(x > 3) = 1 - P(x \leq 3)$$

$$P(x > 3) = 1 - [P(0) + P(1) + P(2) + P(3)]$$

$$P(x > 3) = 1 - \left[(0.05) \frac{(3)^0}{0!} + (0.05) \frac{(3)^1}{1!} + (0.05) \frac{(3)^2}{2!} + (0.05) \frac{(3)^3}{3!} \right]$$

$$P(x > 3) = 1 - (0.05)[1 + 3 + 4.5 + 4.5]$$

$$P(x > 3) = 0.35$$

Number of drivers out of 1000 with more than 3 accidents in a year = $1,000 * 0.35 = 350$

4. Given that 2% of the fuses manufactured by a firm are defective, find, by using poisson distribution, the probability that a box containing 200 fuses has i) no defective fuse ii) at least one defective fuse iii) exactly 3 defective fuses iv) 3 or more defective fuses.

Sol: The probability that a fuses is defective $p = 2/100 = 0.02$.

And $n = 200$

Mean, $\mu = np = 200 * 0.02$; **$m = 4$**

From poisson probability function $P(x) = \frac{m^x}{x!} e^{-m}$

$$P(x) = \frac{4^x}{x!} e^{-4}$$

Let x denotes the number of defective fuses in a box containing 200 fuses

i) Probability that the box contains no defective fuse

$$P(x = 0) = \frac{4^0}{0!} e^{-4} = \mathbf{0.0183}$$

ii) Probability that the box contains at least one defective fuse is

$$\begin{aligned} P(x \geq 1) &= 1 - P(x < 1) \\ &= 1 - P(x = 0) \\ &= 1 - 0.013 \end{aligned}$$

$$\mathbf{P(x \geq 1) = 0.9817}$$

iii) Probability that the box contains exactly 3 defective fuse

$$P(x = 3) = \frac{4^3}{3!} e^{-4}$$

$$\mathbf{P(x = 3) = 0.1952}$$

iv) Probability that the box contains 3 or more defective fuses

$$P(x \geq 3) = 1 - P(x < 3)$$

$$P(x \geq 3) = 1 - \left[\frac{4^0}{0!} e^{-4} + \frac{4^1}{1!} e^{-4} + \frac{4^2}{2!} e^{-4} \right]$$

$$\mathbf{P(x \geq 3) = 0.7621}$$

5. A certain screw making machine has a chance of producing 2 defectives out of 1000. The screws are packed in boxes of 100. Using poisson distribution, find the approximate number of boxes containing i) no defective screw ii) one defective screw iii) two defective screw, in a consignment of 5000 boxes.

Sol: The probability that a screw is defective is given as $p = 2/1000 = 0.002$.

Here $n=100$

Mean = $np=100 \times 0.002=0.2$; **$m = 0.2$**

From poisson probability function $P(x) = \frac{m^x}{x!} e^{-m}$

$$P(x) = \frac{(0.2)^x}{x!} e^{-0.2}$$

Let x denotes the number of defective fuses in a box containing 100 screws

i) Probability that the box contains no defective screw

$$P(x = 0) = \frac{0.2^0}{0!} e^{-0.2} = \mathbf{0.8187}$$

Number of boxes containing no defective screw in 5000 boxes = $5000 \times 0.8187 = 4094$

ii) Probability that the box contains one defective screw is

$$\begin{aligned} P(x = 1) &= \frac{(0.2)^1}{1!} e^{-0.2} \\ &= 0.2 \times 0.8187 \end{aligned}$$

$$P(x = 1) = \mathbf{0.1637}$$

Number of boxes containing one defective screw in 5000 boxes = $5000 \times 0.1637 = 819$

iii) Probability that the box contains two defective screw is

$$\begin{aligned} P(x = 2) &= \frac{(0.2)^2}{2!} e^{-0.2} \\ &= 0.02 \times 0.8187 \end{aligned}$$

$$P(x = 2) = \mathbf{0.0163}$$

Number of boxes containing two defective screw in 5000 boxes = $5000 \times 0.0163 = 82$

6. A communication channel receives independent pulses at the rate of 12 pulses per micro second. The probability of transmission error is 0.001 for each micro second. Compute the probabilities of i) no error during a micro second ii) one error per micro second iii) atleast one per micro second iv) two errors v) atmost two errors.

Sol: Given $n = 12$ and $p = 0.001$

Mean number of errors in one micro second

$$\mu = np$$

$$m = 12 \times 0.001 \quad (\text{Since } \mu = m, \text{ from poisson distribution})$$

$$\mathbf{m = 0.012}$$

From Poisson distribution

$$P(x) = \frac{m^x}{x!} \cdot e^{-m}$$

$$P(x) = \frac{(0.012)^x}{x!} \cdot e^{-(0.012)}$$

$$P(x) = (0.9880) \frac{(0.012)^x}{x!} \quad (\text{Since } e^{-(0.012)} = 0.9880)$$

i) Probability of no error during a micro second

$$P(x = 0) = (0.9880) \frac{(0.012)^0}{0!}$$

$$\mathbf{P(x = 0) = 0.9880}$$

ii) Probability of one error during a micro second

$$P(x = 1) = (0.9880) \frac{(0.012)^1}{1!}$$

$$\mathbf{P(x = 1) = 0.0119}$$

iii) Probability of atleast one error during a micro second

$$P(x \geq 1) = 1 - P(x < 1)$$

$$P(x \geq 1) = 1 - P(x = 0)$$

$$P(x \geq 1) = 1 - P(x = 0)$$

$$P(x \geq 1) = 1 - 0.9880$$

$$\mathbf{P(x \geq 1) = 0.012}$$

iv) Probability of two error during a micro second

$$P(x = 2) = (0.9880) \frac{(0.012)^2}{2!}$$

$$\mathbf{P(x = 2) = 0.00007}$$

v) Probability of atmost two error during a micro second

$$P(x \leq 2) = P(x = 0) + P(x = 1) + P(x = 2)$$

$$P(x \leq 2) = 0.9880 + 0.0119 + 0.00007$$

$$\mathbf{P(x \leq 2) = 0.99997}$$

Continuous Probability Distribution

If for every x belonging to the range of a continuous random variable X, we assign a real number f(x) satisfying the conditions,

i) $f(x) \geq 0$ ii) $\int_{-\infty}^{\infty} f(x)dx = 1$, then f(x) is called a Continuous probability function or probability density function (p.d.f)

Cumulative distribution function

If X is a continuous random variable with probability density function f(x) then the function F(x) defined by $F(x) = P(X \leq x) = \int_{-\infty}^x f(x)dx$ is called the Cumulative distribution function (c.d.f) of X.

NOTE: If r is any real number, then

$$1. P(x \geq r) = \int_r^{\infty} f(x) dx$$

$$P(x < r) = 1 - P(x \geq r) = 1 - \int_r^{\infty} f(x) dx$$

Mean and Variance

If X is a continuous random variable with probability density function $f(x)$ where $-\infty < x < \infty$, the mean (μ) or expectation $E(X)$ and the variance (σ^2) of X is defined as follows.

$$\text{Mean, } \mu = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

$$\text{Variance, } \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 \cdot f(x) dx \quad \text{or} \quad V = \int_{-\infty}^{\infty} x^2 \cdot f(x) dx - \mu^2$$

Problems

1. Find which of the following is a probability density function

$$a) f_1(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \quad b) f_2(x) = \begin{cases} |x|, & 0 < |x| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$c) f_3(x) = \begin{cases} 2x, & 0 < x \leq 1 \\ 4 - 4x, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Sol: Conditions for p.d.f are $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x) dx = 1$

a) Clearly $f_1(x) \geq 0$

$$\int_{-\infty}^{\infty} f_1(x) dx = \int_0^1 f_1(x) dx = \int_0^1 2x dx = 2 \cdot \left[\frac{x^2}{2} \right]_0^1 = [1 - 0] = 1$$

$\therefore f_1(x)$ is a p.d.f

b) Evidently $f_2(x) = |x| \geq 0$

$$\int_{-\infty}^{\infty} f_2(x) dx = \int_{-1}^1 f_2(x) dx = \int_{-1}^1 |x| dx$$

$$\text{But, } |x| = \begin{cases} -x, & -1 < x < 0 \\ +x, & 0 < x < 1 \end{cases}$$

$$\therefore \int_{-\infty}^{\infty} f_2(x) dx = \int_{-1}^0 (-x) dx + \int_0^1 x dx$$

$$\int_{-\infty}^{\infty} f_2(x) dx = - \left[\frac{x^2}{2} \right]_{-1}^0 + \left[\frac{x^2}{2} \right]_0^1$$

$$\int_{-\infty}^{\infty} f_2(x) dx = - \left[0 - \frac{1}{2} \right] + \left[\frac{1}{2} - 0 \right] = \frac{1}{2} + \frac{1}{2} = 1$$

$\therefore f_2(x)$ is a p.d.f

c) Given $f_3(x) = 2x > 0$ in $0 < x \leq 1$

But $f_3(x) = 4 - 4x$ is negative in $1 < x < 2$

The first condition is not satisfied.

$\therefore f_3(x)$ is not a p.d.f

2. Find the value of c such that $f(x) = \begin{cases} \frac{x}{6} + c, & 0 \leq x \leq 3 \\ 0, & \text{elsewhere} \end{cases}$ is a p.d.f. Also find $P(1 \leq x \leq 2)$.

Sol: Here $f(x) \geq 0$ if $c \geq 0$; Also we must have $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\text{i.e., } \int_0^3 \left(\frac{x}{6} + c \right) dx = 1$$

$$\text{i.e., } \left[\frac{x^2}{12} + cx \right]_0^3 = 1$$

$$\text{i.e., } \frac{1}{12}(9 - 0) + c(3 - 0) = 1$$

$$\text{i.e., } \frac{9}{12} + 3c = 1$$

$$\text{i.e., } 3c = 1 - \frac{9}{12}$$

$$3c = \frac{3}{12}$$

$$\therefore c = \frac{1}{12}$$

$$\text{Now, } P(1 \leq x \leq 2) = \int_1^2 f(x)dx$$

$$= \int_1^2 \left(\frac{x}{6} + \frac{1}{12} \right) dx$$

$$= \left[\frac{x^2}{12} + \frac{x}{12} \right]_1^2$$

$$= \frac{1}{12}(4 - 1) + \frac{1}{12}(2 - 1)$$

$$= \frac{3}{12} + \frac{1}{12} = \frac{4}{12}$$

$$P(1 \leq x \leq 2) = \frac{1}{3}$$

3. Find the constant k such that $f(x) = \begin{cases} kx^2, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$ is a p.d.f. Also compute

i) $P(1 < x < 2)$ ii) $P(x \leq 1)$ iii) $P(x > 1)$ iv) Mean v) Variance

Sol: Here $f(x) \geq 0$ if $k \geq 0$; Also we must have $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\text{i.e., } \int_0^3 kx^2 \, dx = 1$$

$$\text{i.e., } \left[\frac{kx^3}{3} \right]_0^3 = 1$$

$$\text{i.e., } \frac{k}{3}(27 - 0) = 1$$

$$\text{i.e., } 9k = 1$$

$$\text{i.e., } \mathbf{k} = \frac{1}{9}$$

$$\therefore f(x) = \begin{cases} \frac{x^2}{9}, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} \text{i) } P(1 < x < 2) &= \int_1^2 f(x) \, dx \\ &= \int_1^2 \frac{x^2}{9} \, dx \\ &= \left[\frac{x^3}{27} \right]_1^2 \\ &= \frac{1}{27}(8 - 1) \end{aligned}$$

$$\mathbf{P(1 < x < 2) = \frac{7}{27}}$$

$$\text{ii) } P(x \leq 1) = \int_0^1 f(x) \, dx$$

$$\begin{aligned} P(x \leq 1) &= \int_0^1 \frac{x^2}{9} \, dx \\ &= \left[\frac{x^3}{27} \right]_0^1 \\ &= \frac{1}{27}(1 - 0) \end{aligned}$$

$$\mathbf{P(x \leq 2) = \frac{1}{27}}$$

$$\text{iii) } P(x > 1) = \int_1^3 f(x) \, dx$$

$$= \int_1^3 \frac{x^2}{9} \, dx$$

$$\begin{aligned} P(x > 1) &= \left[\frac{x^3}{27} \right]_1^3 \\ &= \frac{1}{27}(27 - 1) \end{aligned}$$

$$\mathbf{P(x > 1) = \frac{26}{27}}$$

$$\text{iv) Mean, } \mu = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

$$= \int_0^3 x \cdot \frac{x^2}{9} dx$$

$$= \int_0^3 \frac{x^3}{9} dx$$

$$\mu = \left[\frac{x^4}{36} \right]_0^3$$

$$\mu = \frac{1}{36} (81 - 0)$$

$$\mu = \frac{81}{36}$$

$$\mu = \frac{9}{4}$$

$$\text{v) Variance, } V = \int_{-\infty}^{\infty} x^2 \cdot f(x) dx - \mu^2$$

$$= \int_0^3 x^2 \cdot \frac{x^2}{9} dx - \left(\frac{9}{4} \right)^2$$

$$= \int_0^3 \frac{x^4}{9} dx - \frac{81}{16}$$

$$= \left[\frac{x^5}{45} \right]_0^3 - \frac{81}{16}$$

$$V = \frac{1}{45} [3^5 - 0] - \frac{81}{16}$$

$$V = \frac{243}{45} - \frac{81}{16}$$

$$V = \frac{81}{240}$$

$$V = \frac{27}{80}$$

Exponential Distribution

The continuous probability distribution having the probability density function $f(x)$ given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & \text{for } x > 0 \\ 0, & \text{otherwise} \end{cases}, \text{ where } \alpha > 0 \text{ is known as the } \mathbf{\text{exponential distribution}}.$$

Evidently $f(x) \geq 0$ and we have $\int_{-\infty}^{\infty} f(x) dx = 1$.

$f(x)$ satisfy both the conditions required for a continuous probability function.

Mean and standard deviation of the Exponential Distribution

$$\mathbf{\text{Mean, } } \mu = \frac{1}{\alpha}$$

Std. deviation , $\sigma = \frac{1}{\alpha}$

Problems

1. If x is an exponential variate with mean 3 find i) $P(x > 1)$ ii) $P(x < 3)$

Sol: The p.d.f of the exponential distribution is given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & \text{for } 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

The mean of the distribution is given by $\frac{1}{\alpha}$.

By data, $\mu = \frac{1}{\alpha} = 3$

$$\therefore \alpha = \frac{1}{3}$$

Hence , $f(x) = \begin{cases} \frac{1}{3} e^{-\frac{x}{3}}, & \text{for } 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$

i) $P(x > 1) = 1 - P(x \leq 1)$

$$\begin{aligned} P(x > 1) &= 1 - P(x \leq 1) \\ &= 1 - \int_0^1 f(x) dx \\ &= 1 - \int_0^1 \frac{1}{3} e^{-\frac{x}{3}} dx \\ &= 1 - \frac{1}{3} \int_0^1 e^{-\frac{x}{3}} dx \\ &= 1 - \frac{1}{3} \left[\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}} \right]_0^1 \\ &= 1 + \left[e^{-\frac{1}{3}} - e^{-0} \right] \\ &= 1 + e^{-\frac{1}{3}} - 1 \\ &= e^{-\frac{1}{3}} \end{aligned}$$

$P(x > 1) = 0.7165$

$$\begin{aligned} \text{ii) } P(x < 3) &= \int_0^3 f(x) dx \\ &= \int_0^3 \frac{1}{3} e^{-\frac{x}{3}} dx \\ &= \frac{1}{3} \int_0^3 e^{-\frac{x}{3}} dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3} \left[\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}} \right]_0^3 \\
&= -[e^{-1} - e^{-0}] \\
&= -e^{-1} + 1 = 1 - \frac{1}{e}
\end{aligned}$$

$$\mathbf{P(x < 3) = 0.6321}$$

2. The length of telephone conversation in a booth has been an exponential distribution and found on an average to be 5 minutes. Find the probability that a random call made from this booth i) ends less than 5 minutes ii) between 5 and 10 minutes.

Sol: We have, $f(x) = \alpha e^{-\alpha x}$, $0 < x < \infty$

$$\text{By data, } \mu = \frac{1}{\alpha} = 5$$

$$\therefore \alpha = \frac{1}{5}$$

$$f(x) = \frac{1}{5} e^{-\frac{x}{5}}, \quad 0 < x < \infty$$

$$\begin{aligned}
\text{i) } P(x < 5) &= \int_0^5 f(x) dx \\
&= \int_0^5 \frac{1}{5} e^{-\frac{x}{5}} dx \\
&= \frac{1}{5} \int_0^5 e^{-\frac{x}{5}} dx
\end{aligned}$$

$$\begin{aligned}
P(x < 5) &= \frac{1}{5} \left[\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}} \right]_0^5 \\
&= -[e^{-1} - e^{-0}] \\
&= -e^{-1} + 1 = 1 - \frac{1}{e}
\end{aligned}$$

$$\mathbf{P(x < 5) = 0.6321}$$

$$\begin{aligned}
\text{ii) } P(5 < x < 10) &= \int_5^{10} f(x) dx \\
&= \int_5^{10} \frac{1}{5} e^{-\frac{x}{5}} dx \\
&= \frac{1}{5} \int_5^{10} e^{-\frac{x}{5}} dx
\end{aligned}$$

$$\begin{aligned}
 P(5 < x < 10) &= \frac{1}{5} \left[\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}} \right]_5^{10} \\
 &= -[e^{-2} - e^{-1}] \\
 &= -e^{-2} + e^{-1} = \frac{1}{e} - \frac{1}{e^2}
 \end{aligned}$$

$$P(5 < x < 10) = 0.2325$$

3. In a certain town the duration of a shower is exponentially distributed with mean 5 minutes. What is the probability that a shower will last for: (i) 10 minutes or more (ii) less than 10 minutes (iii) between 10 and 12 minutes.

Sol: We have, $f(x) = \alpha e^{-\alpha x}$, $0 < x < \infty$

$$\text{By data, } \mu = \frac{1}{\alpha} = 5$$

$$\therefore \alpha = \frac{1}{5}$$

$$f(x) = \frac{1}{5} e^{-\frac{x}{5}}, \quad 0 < x < \infty$$

$$\begin{aligned}
 \text{i) } P(x \geq 10) &= \int_{10}^{\infty} f(x) dx \\
 &= \int_{10}^{\infty} \frac{1}{5} e^{-\frac{x}{5}} dx \\
 &= \frac{1}{5} \int_{10}^{\infty} e^{-\frac{x}{5}} dx \\
 &= \frac{1}{5} \left[\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}} \right]_{10}^{\infty} \\
 &= -[e^{-\infty} - e^{-2}] \\
 &= -e^{-\infty} + e^{-2} = -0 + e^{-2}
 \end{aligned}$$

$$P(x \geq 10) = 0.1353$$

$$\text{ii) } P(x < 10) = 1 - P(x \geq 10)$$

$$P(x < 10) = 1 - 0.1353$$

$$P(x < 10) = 0.8647$$

$$\begin{aligned}
 \text{iii) } P(10 < x < 12) &= \int_{10}^{12} f(x) dx \\
 &= \int_{10}^{12} \frac{1}{5} e^{-\frac{x}{5}} dx
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{5} \int_{10}^{12} e^{-\frac{x}{5}} dx \\
&= \frac{1}{5} \left[\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}} \right]_{10}^{12} \\
&= -[e^{-2.4} - e^{-2}]
\end{aligned}$$

$$P(10 < x < 12) = e^{-2} - e^{-2.4}$$

$$P(10 < x < 12) = 0.0446$$

4. The life of a compressor manufactured by a company is known to be 200 months on an average following an exponential distribution. Find the probability that the life of a compressor of that company is i) less than 200 months ii) between 100 months and 25 years.

Sol: We have, $f(x) = \alpha e^{-\alpha x}$, $0 < x < \infty$

$$\text{By data, } \mu = \frac{1}{\alpha} = 200$$

$$\therefore \alpha = \frac{1}{200}$$

$$f(x) = \frac{1}{200} e^{-\frac{x}{200}}, \quad 0 < x < \infty$$

$$\begin{aligned}
\text{i) } P(x < 200) &= \int_0^{200} f(x) dx \\
&= \int_0^{200} \frac{1}{200} e^{-\frac{x}{200}} dx \\
&= \frac{1}{200} \int_0^{200} e^{-\frac{x}{200}} dx \\
&= \frac{1}{200} \left[\frac{e^{-\frac{x}{200}}}{-\frac{1}{200}} \right]_0^{200} \\
&= -[e^{-1} - e^{-0}] \\
&= -e^{-1} + 1 = 1 - \frac{1}{e}
\end{aligned}$$

$$P(x < 200) = 0.6321$$

Now, 25 years = 25 * 12 months = 300 months

$$\begin{aligned}
\text{ii) } P(100 < x < 300) &= \int_{100}^{300} f(x) dx \\
&= \int_{100}^{300} \frac{1}{200} e^{-\frac{x}{200}} dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{200} \int_{100}^{300} e^{-\frac{x}{200}} dx \\
&= \frac{1}{200} \left[\frac{e^{-\frac{x}{200}}}{-\frac{1}{200}} \right]_{100}^{300} \\
&= -[e^{-1.5} - e^{-2}]
\end{aligned}$$

$$\mathbf{P(100 < x < 300) = 0.3834}$$

5. Determine k if random variable x has the density function $f(x) = \frac{k}{1+x^2}$, $-\infty < x < \infty$
hence find i) $P(x \geq 0)$ ii) $P(0 < x < 1)$ iii) c. d. f

Sol: Given f(x) is a probability density function.

$$\therefore f(x) \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\text{i.e., } \int_{-\infty}^{\infty} \frac{k}{1+x^2} dx = 1$$

$$2k \int_0^{\infty} \frac{1}{1+x^2} dx = 1$$

$$2k[\tan^{-1}x]_0^{\infty} = 1 \quad ; \quad 2k[\tan^{-1}(\infty) - \tan^{-1}(0)] = 1$$

$$2k\left[\frac{\pi}{2} - 0\right] = 1$$

$$k \cdot \pi = 1 \quad k = \frac{1}{\pi}$$

$$f(x) = \frac{\frac{1}{\pi}}{1+x^2}$$

$$\text{i) } P(x \geq 0) = \int_0^{\infty} f(x) dx$$

$$= \int_0^{\infty} \frac{\frac{1}{\pi}}{1+x^2} dx$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{1}{1+x^2} dx$$

$$= \frac{1}{\pi} [\tan^{-1}x]_0^{\infty}$$

$$P(x \geq 0) = \frac{1}{\pi} [\tan^{-1}(\infty) - \tan^{-1}(0)]$$

$$P(x \geq 0) = \frac{1}{\pi} \left[\frac{\pi}{2} - 0 \right]$$

$$\mathbf{P(x \geq 0) = \frac{1}{2}}$$

$$\text{ii) } P(0 < x < 1) = \int_0^1 f(x) dx$$

$$= \int_0^1 \frac{\frac{1}{\pi}}{1+x^2} dx$$

$$= \frac{1}{\pi} \int_0^1 \frac{1}{1+x^2} dx$$

$$P(0 < x < 1) = \frac{1}{\pi} [\tan^{-1} x]_0^1$$

$$= \frac{1}{\pi} [\tan^{-1}(1) - \tan^{-1}(0)]$$

$$= \frac{1}{\pi} \left[\frac{\pi}{4} - 0 \right]$$

$$P(0 < x < 1) = \frac{1}{4}$$

$$\text{iii) c. d. f} = \int_{-\infty}^x f(x) dx$$

$$= \int_{-\infty}^x \frac{\frac{1}{\pi}}{1+x^2} dx$$

$$= \frac{1}{\pi} \int_{-\infty}^x \frac{1}{1+x^2} dx$$

$$\text{c. d. f} = \frac{1}{\pi} [\tan^{-1} x]_{-\infty}^x$$

$$= \frac{1}{\pi} [\tan^{-1}(x) - \tan^{-1}(-\infty)]$$

$$= \frac{1}{\pi} \left[\tan^{-1} x - \left(-\frac{\pi}{2} \right) \right]$$

$$\text{c. d. f} = \frac{1}{\pi} \left[\tan^{-1} x + \frac{\pi}{2} \right] \quad \text{if } -\infty < x < \infty.$$

6. The daily turn over in a medical shop is exponentially distributed with Rs.6000 as the average with a net profit of 8%. Find the probability that the net profit exceeds Rs.500 on a randomly chosen day.

Sol: Let 'x' denote the turn over per day.

∴ By exponential distribution

$$f(x) = \alpha e^{-\alpha x} ; x > 0.$$

Given, By data, $\mu = \frac{1}{\alpha} = 6000$

$$\therefore \alpha = \frac{1}{6000}$$

$$f(x) = \frac{1}{6000} e^{-\frac{x}{6000}}, \quad 0 < x < \infty$$

Let 'A' be the turn over for which the net profit is Rs.500.

Since the net profit is 8% of the turn over,

$$A * \frac{8}{100} = 500$$

$$\Rightarrow A = 6250$$

$\therefore$ The probability of profit exceeding Rs.500 is

$$P(x > 6250) = \int_{6250}^{\infty} f(x) dx$$

$$P(x > 6250) = \int_{6250}^{\infty} \frac{1}{6000} e^{-\frac{x}{6000}} dx$$

$$= \frac{1}{6000} \int_{6250}^{\infty} e^{-\frac{x}{6000}} dx$$

$$= \frac{1}{6000} \left[\frac{e^{-\frac{x}{6000}}}{-\frac{1}{6000}} \right]_{6250}^{\infty}$$

$$= - \left[e^{-\infty} - e^{-\frac{6250}{6000}} \right]$$

$$= e^{-\frac{6250}{6000}}$$

$$P(x > 6250) = 0.3529$$

Normal Distribution

The continuous probability distribution having the probability density function $f(x)$ given

by $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ where $-\infty < x < \infty$, $-\infty < \mu < \infty$ and $\sigma > 0$ is known as the normal distribution.

Evidently $f(x) \geq 0$ and we have $\int_{-\infty}^{\infty} f(x) dx = 1$.

$f(x)$ satisfy both the conditions required for a continuous probability function.

Mean and standard deviation of the Normal Distribution

$$\text{Mean} = \mu$$

$$\text{Variance} = \sigma^2$$

Geometrically we can write

$$\text{i) } \int_{-\infty}^{\infty} \phi(z) dz = 1 \quad \text{ii) } \int_{-\infty}^0 \phi(z) dz = \int_0^{\infty} \phi(z) dz = \frac{1}{2}$$

Note

$$1. P(-\infty \leq z \leq \infty) = 1$$

$$2. P(-\infty \leq z \leq 0) = \frac{1}{2}$$

$$3. P(0 \leq z \leq \infty) \text{ or } P(z \geq 0) = \frac{1}{2}$$

$$\text{Also, } P(-\infty < z < z_1) = P(-\infty < z \leq 0) + P(0 \leq z < z_1)$$

$$\text{i.e., } P(z < z_1) = 0.5 + \phi(z_1)$$

$$\text{Also, } P(z > z_2) = P(z \geq 0) - P(0 \leq z < z_2)$$

$$\text{i.e., } P(z > z_2) = 0.5 - \phi(z_2)$$

Problems

1. Evaluate the following probabilities with the help of normal probability tables

i) $P(z \geq 0.85)$ ii) $P(-1.64 \leq z \leq -0.88)$ iii) $P(z \leq -2.43)$ iv) $P(|z| \leq 1.94)$

$$\text{Sol: i) } P(z \geq 0.85) = P(z \geq 0) - P(0 \leq z \leq 0.85)$$

$$= 0.5 - \phi(0.85)$$

$$= 0.5 - 0.3023$$

$$\mathbf{P(z \geq 0.85) = 0.1977}$$

$$\text{ii) } P(-1.64 \leq z \leq -0.88)$$

$$\text{By Symmetry, } P(-1.64 \leq z \leq -0.88) = P(0.88 \leq z \leq 1.64)$$

$$= P(0 \leq z \leq 1.64) - P(0 \leq z \leq 0.88)$$

$$= \phi(1.64) - \phi(0.88)$$

$$= 0.4495 - 0.3106$$

$$\mathbf{P(-1.64 \leq z \leq -0.88) = 0.1389}$$

$$\text{iii) } P(z \leq -2.43) = P(z \geq 2.43)$$

$$P(z \leq -2.43) = P(z \geq 0) - P(0 \leq z \leq 2.43)$$

$$= 0.5 - \phi(2.43)$$

$$= 0.5 - 0.4925$$

$$\mathbf{P(z \geq -2.43) = 0.0075}$$

$$\text{iv) } P(|z| \leq 1.94) = P(-1.94 \leq z \leq 1.94)$$

$$= 2 P(0 \leq z \leq 1.94)$$

$$= 2. \phi(1.94)$$

$$= 2. (0.4738)$$

$$P(|z| \leq 1.94) = 0.9476$$

2. If x is a normal variate with mean 30 and standard deviation 5 find the probability that
i) $26 \leq x \leq 40$ ii) $x \geq 45$

Sol: We have standard normal variate, $z = \frac{x-\mu}{\sigma} = \frac{x-30}{5}$ [By data $\mu = 30, \sigma = 5$]

i) To find $P(26 \leq x \leq 40)$

$$\text{If } x = 26, z = \frac{26-30}{5} = -0.8 \quad ; \quad \text{If } x = 40, z = \frac{40-30}{5} = 2$$

Hence we need to find, $P(-0.8 \leq z \leq 2)$

$$\begin{aligned} P(-0.8 \leq z \leq 2) &= P(-0.8 \leq z \leq 0) + P(0 \leq z \leq 2) \\ &= P(0 \leq z \leq 0.8) + P(0 \leq z \leq 2) \\ &= \phi(0.8) + \phi(2) \\ &= 0.2881 + 0.4772 \end{aligned}$$

$$P(-0.8 \leq z \leq 2) = 0.7653$$

ii) To find $P(x \geq 45)$

$$\text{If } x = 45, z = \frac{45-30}{5} = 3$$

Hence we need to find, $P(z \geq 3)$

$$\begin{aligned} P(z \geq 3) &= P(z \geq 0) - P(0 \leq z \leq 3) \\ &= 0.5 - \phi(3) \\ &= 0.5 - 0.4987 \end{aligned}$$

$$P(z \geq 3) = 0.0013$$

3. The marks of 1000 students in an examination follows a normal distribution with mean 70 and standard deviation 5. Find the number of students whose marks will be
i) less than 65 ii) more than 75 iii) 65 to 75

Sol: Let x represent the marks of students

We have standard normal variate, $z = \frac{x-\mu}{\sigma} = \frac{x-70}{5}$ [By data $\mu = 70, \sigma = 5$]

i) To find $P(x < 65)$

$$\text{If } x = 65, z = \frac{65-70}{5} = -1$$

Hence we need to find, $P(z < -1)$

$$\begin{aligned} P(z < -1) &= P(z > 1) \\ &= P(z \geq 0) - P(0 \leq z \leq 1) \end{aligned}$$

$$= 0.5 - \phi(1)$$

$$= 0.5 - 0.3413$$

$$P(z < -1) = 0.1587$$

Number of students scoring less than 65 marks = $1000 * 0.1587 = 158.7 \cong 159$

ii) To find $P(x > 75)$

$$\text{If } x = 75, z = \frac{75-70}{5} = 1$$

Hence we need to find, $P(z > 1)$

$$P(z > 1) = P(z \geq 0) - P(0 \leq z \leq 1)$$

$$= 0.5 - \phi(1)$$

$$= 0.5 - 0.3413$$

$$P(z > 1) = 0.1587$$

Number of students scoring more than 75 marks = $1000 * 0.1587 = 158.7 \cong 159$

iii) To find $P(65 < x < 75)$

Hence we need to find, $P(-1 < z < 1)$

$$P(-1 < z < 1) = 2P(0 < z < 1)$$

$$= 2\phi(1)$$

$$= 2(0.3413)$$

$$P(-1 < z < 1) = 0.6826$$

Number of students scoring marks between 65 to 75 marks = $1000 * 0.6826 = 682.6 \cong 683$

4. In a test on electric bulbs, it was found that the life time of a particular brand was distributed normally with an average life of 2000 hours and S.D of 60 hours. If a firm purchases 2500 bulbs, find the number of bulbs that are likely to last for i) more than 2100 hours ii) less than 1950 hours iii) between 1900 to 2100 hours.

Sol: Given $\mu = 2000$, $\sigma = 60$

We have standard normal variate, $z = \frac{x-\mu}{\sigma} = \frac{x-2000}{60}$

i) To find $P(x > 2100)$

$$\text{If } x = 2100, z = \frac{2100-2000}{60} = 1.67$$

Hence we need to find, $P(z > 1.67)$

$$P(z > 1.67) = P(z \geq 0) - P(0 < z < 1.67)$$

$$= 0.5 - \phi(1.67)$$

$$= 0.5 - 0.4525$$

$$P(z > 1.67) = 0.0475$$

$$\begin{aligned}\therefore \text{Number of bulbs that are likely to last for more than 2100 hours} &= 2500 * 0.0475 \\ &= 118.75 \cong 119\end{aligned}$$

ii) To find $P(x < 1950)$

$$\text{If } x = 1950, z = \frac{1950-2000}{60} = -0.83$$

Hence we need to find, $P(z < -0.83)$

$$\begin{aligned}P(z < -0.83) &= P(z > 0.83) \\ &= P(z \geq 0) - P(0 < z < 0.83) \\ &= 0.5 - \phi(0.83) \\ &= 0.5 - 0.2967\end{aligned}$$

$$P(z < -0.83) = 0.2033$$

$$\begin{aligned}\therefore \text{Number of bulbs that are likely to last for more than 1950 hours} &= 2500 * 0.2033 \\ &= 508.25 \cong 508\end{aligned}$$

iii) To find $P(1900 < x < 2100)$

$$\text{If } x = 1900, z = \frac{1900-2000}{60} = -1.67 \quad ; \quad \text{If } x = 2100, z = \frac{2100-2000}{60} = 1.67$$

Hence we need to find, $P(-1.67 < z < 1.67)$

$$\begin{aligned}P(-1.67 < z < 1.67) &= 2P(0 < z < 1.67) \\ &= 2 * \phi(1.67) \\ &= 2 * 0.4525\end{aligned}$$

$$P(-1.67 < z < 1.67) = 0.905$$

$$\begin{aligned}\therefore \text{Number of bulbs that are likely to last between 1900 and 2100 hours} \\ &= 2500 * 0.905 \\ &= 2262.5 \cong 2263\end{aligned}$$

5. In a normal distribution 31% of the items are under 45 and 8% of the items are over 64.
Find the mean and S.D of the distribution.

Sol: Let μ and σ be the mean and S.D of the normal distribution.

By data, $P(x < 45) = 0.31$ and $P(x > 64) = 0.08$

We have standard normal variate, $z = \frac{x-\mu}{\sigma}$

When $x = 45$, $z = \frac{45-\mu}{\sigma} = z_1$ (Say)

$x = 64$, $z = \frac{64-\mu}{\sigma} = z_2$ (Say)

So we have ,

$$P(z < z_1) = 0.31 \quad \text{and} \quad P(z > z_2) = 0.08$$

$$\text{i.e.,} \quad 0.5 + \phi(z_1) = 0.31 \quad \text{and} \quad 0.5 - \phi(z_2) = 0.08$$

$$\phi(z_1) = 0.31 - 0.5 \quad \text{and} \quad \phi(z_2) = 0.5 - 0.08$$

$$\phi(z_1) = -0.19 \quad \text{and} \quad \phi(z_2) = 0.42$$

Referring to the normal probability tables, we have

$$0.1915 (\cong 0.19) = \phi(0.5) \quad \text{and} \quad 0.4192 (\cong 0.42) = \phi(1.4)$$

$$\phi(z_1) = -\phi(0.5) \quad \text{and} \quad \phi(z_2) = \phi(1.4)$$

$$\Rightarrow \quad z_1 = -0.5 \quad \text{and} \quad z_2 = 1.4$$

$$\text{i.e.,} \quad \frac{45-\mu}{\sigma} = -0.5 \quad \text{and} \quad \frac{64-\mu}{\sigma} = 1.4$$

$$45 - \mu = -0.5\sigma \quad \text{and} \quad 64 - \mu = 1.4\sigma$$

$$\mu - 0.5\sigma = 45 \quad \text{and} \quad \mu + 1.4\sigma = 64$$

Solving above equations, we get

$$\mu = 50 \quad \text{and} \quad \sigma = 10$$

Thus, Mean=50 and S.D=10

6. The mean weight of 500 students during a medical examination was found to be 50 kgs and S.D weight 6 kgs. Assuming that the weights are normally distributed, find the number of student having weight i) between 40 and 50 kgs ii) more than 60 kgs. [$\phi(1.67) = 0.4525$]

Sol: Given $\mu = 50$, $\sigma = 6$

We have standard normal variate, $z = \frac{x-\mu}{\sigma} = \frac{x-50}{6}$

i) **To find $P(40 < x < 50)$**

$$\text{If } x = 40, z = \frac{40-50}{6} = -1.67 \quad ; \quad \text{If } x = 50, z = \frac{50-50}{6} = 0$$

Hence we need to find , **$P(-1.67 < z < 0)$**

$$P(-1.67 < z < 0) = P(0 < z < 1.67)$$

$$= \phi(1.67)$$

$$\mathbf{P(-1.67 < z < 0) = 0.4525}$$

$$\begin{aligned}\therefore \text{Number of students having weight between 40 and 50 kgs} &= 500 * 0.4525 \\ &= 226.25 \cong 226\end{aligned}$$

ii) To find $P(x > 60)$

$$\text{If } x = 60, z = \frac{60-50}{6} = 1.67$$

Hence we need to find, $P(z > 1.67)$

$$\begin{aligned}P(z > 1.67) &= P(z \geq 0) - P(0 < z < 1.67) \\ &= 0.5 - \phi(1.67) \\ &= 0.5 - 0.4525\end{aligned}$$

$$P(z > 1.67) = 0.0475$$

$$\begin{aligned}\therefore \text{Number of students having weight more than 60 kgs} &= 500 * 0.0475 \\ &= 23.75 \cong 24\end{aligned}$$

7. In an examination 7% of students score less than 35% marks and 89% of students score less than 60% of marks. Find the mean and S.D of the distribution if the marks are normally distributed. It is given that $P(1.2263)=0.39$ and $P(1.4757)=0.43$.

Sol: Let μ and σ be the mean and S.D of the normal distribution.

$$\text{By data, } P(x < 35) = 0.07 \quad \text{and} \quad P(x < 60) = 0.89$$

$$\text{We have standard normal variate, } z = \frac{x-\mu}{\sigma}$$

$$\text{When } x = 35, z = \frac{35-\mu}{\sigma} = z_1 \text{ (Say)}$$

$$x = 60, z = \frac{60-\mu}{\sigma} = z_2 \text{ (Say)}$$

So we have ,

$$P(z < z_1) = 0.07 \quad \text{and} \quad P(z < z_2) = 0.89$$

$$\text{i.e., } 0.5 + \phi(z_1) = 0.07 \quad \text{and} \quad 0.5 + \phi(z_2) = 0.89$$

$$\phi(z_1) = 0.07 - 0.5 \quad \text{and} \quad \phi(z_2) = 0.89 - 0.5$$

$$\phi(z_1) = -0.43 \quad \text{and} \quad \phi(z_2) = 0.39$$

Using the given data in the RHS of these, we have

$$\phi(z_1) = -\phi(1.4757) \quad \text{and} \quad \phi(z_2) = \phi(1.2263)$$

$$\Rightarrow \quad z_1 = -1.4757 \quad \text{and} \quad z_2 = 1.2263$$

$$\text{i.e., } \frac{35-\mu}{\sigma} = -1.4757 \quad \text{and} \quad \frac{60-\mu}{\sigma} = 1.2263$$

$$35 - \mu = -1.4757\sigma \quad \text{and} \quad 60 - \mu = 1.2263\sigma$$

$$\mu - 1.4757\sigma = 35 \quad \text{and} \quad \mu + 1.2263\sigma = 60$$

Solving above equations, we get

$$\mu = 48.65 \quad \text{and} \quad \sigma = 9.25$$

Thus, Mean = 48.65 and S.D = 9.25

PROBABILITY DISTRIBUTION AND SAMPLING THEORY

Introduction

The three main components of statistics are collection of data, analysis of data and inference. The scope of statistics is very vast and finds its applications in almost all spheres. Statistical methods guides us in any scientific enquiry and leans heavily on probability theory.

Basic Definitions and concepts

Experiment is a process by which measurement or observation is obtained.

Random experiment : An experiment which does not result in the same out come when performed under the same conditions.

Eg : 1. Tossing of a fair coin.

2. Throwing of a fair die.

Sample space : The set of all possible outcomes of a random experiment is called a sample space usually denoted by S.

Eg 1. $S = \{H,T\}$ is the sample space in tossing of a single coin.

2. $S = \{HH,TH,HT,TT\}$ is the sample space obtained in tossing of two coins.

3. $S = \{1,2,3,4,5,6\}$ in throwing of a die.

Event : Any event is a subset of an appropriate sample space.

Exhaustive event : An event consisting of all the various possibilities is called an exhaustive event.

Mutually exclusive events : Events are said to be mutually exclusive if and only if one of them can take place at a time, in other words happening of one event prevent the simultaneous happening of the others.

Eg: In the tossing of a coin, either heads or tails may turn up but not both.

Independent events: Two or more events are said to be independent if the happening or non happening of one event does not prevent the happening or non happening of the others.

Eg: 1. When two coins are tossed the event of getting head is an independent event as both the coins can turn out heads.

Probability is the measure of the chance with which we can expect the event to occur. If an event A can occur in **m** different ways out of number of **n** possible ways, all of which are equally likely then the probability of the event A denoted by P(A) is given by $P(A) = \frac{m}{n}$.

Note: 1) Since $0 \leq m \leq n \Rightarrow P(A)$ is non negative and $0 \leq P(A) \leq 1$

2) If $P(A) = 0$ then A is called a null event.

3) If $P(A) = 1$ then A is called a sure event.

4) $P(A) + P(\bar{A}) = 1$, the set of all unfavourable events is denoted by $\bar{A}$.

Addition theorem of probability

The probability of the happening of one or the other mutually exclusive events is equal to the sum of the probabilities of the two events.

That is, if A, B are two mutually exclusive events then,

$$P(A \text{ or } B) = P(A) + P(B)$$

Product theorem of probability

If a compound event is made up of a number of independent events, the probability of the happening of the compound event is equal to the product of the probabilities of the independent events .

That is, if A, B are independent events then,

$$P(A \text{ and } B) = P(A).P(B)$$

RANDOM VARIABLE & PROBABILITY DISTRIBUTION

Introduction

When we consider an experiment of tossing two coins. The sample points or out comes are qualitative and are described by their attributes though the probabilities attached to each of these outcomes are quantitative. We now associate to each sample point a real number called the **random variable**.

Random variable

Definition: Let S be a sample space then $X : S \rightarrow R$ defined over the elements of S is called a **random variable**. A random variable is a function that assigns a real number to every sample point in the sample space of a random experiment. Random variables are usually denoted by X, Y, Z ... and it may be noted that different random variables may be associated with the sample space S. The set of all real numbers of a random variable X is called the range of X.

Eg:

1. While tossing a coin, suppose that the value 1 is associated for the outcome 'head' and 0 for the outcome tail. We have the sample space $S = \{H, T\}$ and if X is the random variable , then $X(H) = 1$ & $X(T) = 0$.

Range of X = $\{0,1\}$

2. Suppose a coin is tossed twice, we shall associate two different random variables X, Y as follows, where we have the sample space

$S = \{HH, HT, TH, TT\}$

X = Number of 'heads' in the outcome.

The association of the elements in S to X is as follows.

Out come	HH	HT	TH	TT
Random variable X	2	1	1	0

Range of $X = \{0,1,2\}$

Suppose $Y =$ Number of 'tails' in the outcome, we have

Out come	HH	HT	TH	TT
Random variable Y	0	1	1	2

Range of $Y = \{0,1,2\}$

Discrete and Continuous random variables

If a random variable takes finite or countably infinite number of values then it is called a discrete random variable.

Discrete random variable will have finite or countably infinite range.

Eg: 1. Tossing of a coin and observing the outcome.

2. Throwing a 'die' and observing the numbers on the face.

If a random variable takes non countable number of values then it is called a non discrete or continuous random variable.

Eg:

1. Weight of articles

2. Length of nails produced by machine.

Generally counting problems correspond to discrete random variables and measuring problems lead to continuous random variables.

Probability function and Discrete probability distribution

If for each value x_i of a discrete random variable X , we assign a real number $p(x_i)$ such that

i) $p(x_i) \geq 0$ ii) $\sum_i p(x_i) = 1$ then the function $p(x)$ is called a probability function.

If the probability that X take the values x_i is p_i , then $P(X = x_i) = p_i$ or $p(x_i)$.

The set of values $[x_i, p(x_i)]$ is called a discrete (finite) probability distribution of the discrete random variable X . The function $P(X)$ is called the probability density function (p.d.f) or the probability mass function (p.m.f)

The distribution function $f(x)$ defined by $f(x) = P(X \leq x) = \sum_{i=1}^n P(x_i)$, x being an integer, is called the cumulative distribution function (c.d.f).

The mean and variance of the discrete probability distribution are defined as follows

Mean $\mu = \sum_i x_i \cdot p(x_i)$

Variance $V = \sum_i (x_i - \mu)^2 \cdot p(x_i)$

$$V = \sum_i x_i^2 \cdot p(x_i) - [\sum_i x_i \cdot p(x_i)]^2$$

$$V = \sum_i x_i^2 \cdot p(x_i) - \mu^2$$

Standard deviation $\sigma = \sqrt{V}$

Problems

1. A coin is tossed twice. A random variable X represent the number of heads turning up. Find the discrete probability distribution for X . Also find its mean and variance.

Sol: Here $S = \{HH, HT, TH, TT\}$

$$X = \{0, 1, 2\}$$

$$\text{Now, } P(HH) = \frac{1}{4} \quad P(HT) = \frac{1}{4} \quad P(TH) = \frac{1}{4} \quad P(TT) = \frac{1}{4}$$

$$P(X = 0, \text{ i.e., no head turning up}) = P(TT) = \frac{1}{4}$$

$$P(X = 1, \text{ i.e., one head turning up}) = P(HT \cup TH) = P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P(X = 2, \text{ i.e., two heads turning up}) = P(HH) = \frac{1}{4}$$

The discrete probability distribution for X is as follows

$X=x_i$	0	1	2
$p(x_i)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

We observe that, $p(x_i) > 0$ and $\sum_i p(x_i) = 1$.

$$\text{Mean, } \mu = \sum_i x_i \cdot p(x_i) = (0) \cdot \left(\frac{1}{4}\right) + (1) \cdot \left(\frac{1}{2}\right) + (2) \cdot \left(\frac{1}{4}\right) = \frac{1}{2} + \frac{1}{2}$$

$$\mu = 1$$

$$\text{Variance, } V = \sum_i (x_i - \mu)^2 \cdot p(x_i)$$

$$V = (0 - 1)^2 \cdot \frac{1}{4} + (1 - 1)^2 \cdot \frac{1}{2} + (2 - 1)^2 \cdot \frac{1}{4}$$

$$V = \frac{1}{4} + 0 + \frac{1}{4}$$

$$V = \frac{1}{2}$$

2. A random experiment of tossing a 'die' twice is performed. Random variable X on this sample space is defined to be the sum of the two numbers turning up on the toss. Find the discrete probability distribution for the random variable X and compute the corresponding mean and standard deviation.

Sol: Here $S = \{(x, y) \text{ where } x = 1, 2, \dots, 6 ; y = 1, 2, \dots, 6\}$

i.e., $S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), \dots, (6, 5), (6, 6)\}$

Number of elements in $S=36$.

The set of values of the random variable X defined as sum of two numbers on the face of the 'die' are $\{2,3,4,5, \dots, 10,11,12\}$

Now, $p(x_1) = \frac{n(E)}{n(S)} = \frac{1}{36}$; $p(x_2) = \frac{2}{36}$; ...

The discrete probability distribution for X is as follows

X = x_i	2	3	4	5	6	7	8	9	10	11	12
$p(x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Here $p(x_i) > 0$ and $\sum_i p(x_i) = 1$.

Mean $\mu = \sum_i x_i \cdot p(x_i)$

$$\mu = (2) \cdot \left(\frac{1}{36}\right) + (3) \cdot \left(\frac{2}{36}\right) + \dots + (11) \cdot \left(\frac{2}{36}\right) + (12) \cdot \left(\frac{1}{36}\right)$$

$$\mu = \frac{252}{36} = 7$$

Variance, $V = \sum_i (x_i - \mu)^2 \cdot p(x_i)$

$$V = (2 - 7)^2 \cdot \frac{1}{36} + (3 - 7)^2 \cdot \frac{2}{36} + \dots + (12 - 7)^2 \cdot \frac{1}{36}$$

$$V = \frac{210}{36} = \frac{35}{6}$$

Std. deviation , $\sigma = \sqrt{V}$

$$\sigma = \sqrt{\frac{35}{6}} = 2.42$$

3. Show that the following distribution represents a discrete probability distribution. Find the mean and variance

x	10	20	30	40
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Sol: We observe that $p(x) > 0$ and $\sum p(x) = 1$.

Mean $\mu = \sum x \cdot p(x)$

$$\mu = (10) \cdot \frac{1}{8} + (20) \cdot \frac{3}{8} + (30) \cdot \frac{3}{8} + (40) \cdot \frac{1}{8}$$

$$\mu = \frac{10+60+90+40}{8} = \frac{200}{8}$$

$$\mu = 25$$

$$\text{Variance, } V = \sum (x - \mu)^2 \cdot p(x)$$

$$V = (10 - 25)^2 \cdot \frac{1}{8} + (20 - 25)^2 \cdot \frac{3}{8} + (30 - 25)^2 \cdot \frac{3}{8} + (40 - 25)^2 \cdot \frac{1}{8}$$

$$V = \frac{600}{8} = 75$$

4. Find the value of k such that the following distribution represents a finite probability distribution. Hence find its mean and standard deviation. Also find $p(x \leq 1)$, $p(x > 1)$ and $p(-1 < x \leq 2)$

x	-3	-2	-1	0	1	2	3
p(x)	k	2k	3k	4k	3k	2k	k

Sol: We must have $p(x) \geq 0$ for all x and $\sum p(x) = 1$. The first condition is satisfied if $k \geq 0$

$$\text{Since } \sum p(x) = 1$$

$$k + 2k + 3k + 4k + 3k + 2k + k = 1$$

$$16k = 1$$

$$\therefore k = \frac{1}{16}$$

The discrete probability distribution is as follows

x	-3	-2	-1	0	1	2	3
p(x)	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

$$\text{Mean } \mu = \sum x \cdot p(x)$$

$$\mu = (-3) \cdot \frac{1}{16} + (-2) \cdot \frac{2}{16} + (-1) \cdot \frac{3}{16} + (0) \cdot \frac{4}{16} + (1) \cdot \frac{3}{16} + (2) \cdot \frac{2}{16} + (3) \cdot \frac{1}{16}$$

$$\mu = 0$$

Variance, $V = \sum (x - \mu)^2 \cdot p(x)$

$$V = (-3 - 0)^2 \cdot \frac{1}{16} + (-2 - 0)^2 \cdot \frac{2}{16} + (-1 - 0)^2 \cdot \frac{3}{16} + (0 - 0)^2 \cdot \frac{4}{16} + (1 - 0)^2 \cdot \frac{3}{16} + (2 - 0)^2 \cdot \frac{2}{16} + (3 - 0)^2 \cdot \frac{1}{16}$$

$$V = \frac{1}{16} (9 + 8 + 3 + 0 + 3 + 8 + 9) = \frac{40}{16} = \frac{5}{2}$$

Standard deviation $\sigma = \sqrt{V}$

$$\sigma = \sqrt{\frac{5}{2}} = 1.58$$

Now, $p(x \leq 1) = p(-3) + p(-2) + p(-1) + p(0) + p(1)$

$$= \frac{1}{16} + \frac{2}{16} + \frac{3}{16} + \frac{4}{16} + \frac{3}{16}$$

$$p(x \leq 1) = \frac{13}{16}$$

$p(x > 1) = p(2) + p(3)$

$$= \frac{2}{16} + \frac{1}{16}$$

$$p(x > 1) = \frac{3}{16}$$

$p(-1 < x \leq 2) = p(0) + p(1) + p(2)$

$$= \frac{4}{16} + \frac{3}{16} + \frac{2}{16}$$

$$p(-1 < x \leq 2) = \frac{9}{16}$$

5. A random variable X has the following probability function for various values of x.

i) Find k ii) Evaluate $P(x < 6)$, $P(x \geq 6)$ and $P(3 < x \leq 6)$. Also find the probability

x	0	1	2	3	4	5	6	7
P(x)	0	k	2k	2k	3k	k ²	2k ²	7k ² + k

distribution and the distribution function of X.

Sol: We must have $P(x) \geq 0$ and $\sum P(x) = 1$.

The first condition is satisfied for $k \geq 0$ and we have to find k such that

$$\sum P(x) = 1$$

$$\text{i.e., } 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + (7k^2 + k) = 1$$

$$10k^2 + 9k - 1 = 0$$

$$(10k - 1)(k + 1) = 0$$

$$k = \frac{1}{10} \quad \text{and} \quad k = -1 \text{ (Neglect)}$$

Hence we have the following table

x	0	1	2	3	4	5	6	7
P(x)	0	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{1}{50}$	$\frac{17}{100}$

Now, $P(x < 6) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$

$$= 0 + \frac{1}{10} + \frac{1}{5} + \frac{1}{5} + \frac{3}{10} + \frac{1}{100} = \frac{81}{100}$$

$$\mathbf{P(x < 6) = 0.81}$$

$$P(x \geq 6) = P(6) + P(7)$$

$$P(x \geq 6) = \frac{1}{50} + \frac{17}{100} = \frac{19}{100}$$

$$\mathbf{P(x \geq 6) = 0.19}$$

$$P(3 < x \leq 6) = P(4) + P(5) + P(6)$$

$$= \frac{3}{10} + \frac{1}{100} + \frac{1}{50} = \frac{33}{100}$$

$$\mathbf{P(3 < x \leq 6) = 0.33}$$

The probability distribution is as follows

x	0	1	2	3	4	5	6	7
P(x)	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

The distribution function of X is $f(x) = P(X \leq x) = \sum_{i=1}^n p(x_i)$ is called cumulative distribution function and same is as follows

x	0	1	2	3	4	5	6	7
f(x)	0	0+0.1 =0.1	0.1+0.2 =0.3	0.3+0.2 =0.5	0.5+0.3 =0.8	0.8+0.01 =0.81	0.81+0.02=0.83	0.83+0.17=1

6. A random variable X take the values $-3, -2, -1, 0, 1, 2, 3$ such hat $P(X = 0) = P(X < 0)$ and $P(X = -3) = P(X = -2) = P(X = -1) = P(X = 1) = P(X = 2) = P(X = 3)$.Find the probability distribution.

Sol: Let the distribution be as follows

X	-3	-2	-1	0	1	2	3
P(X)	p_1	p_2	p_3	p_4	p_5	p_6	p_7

By data $P(X = 0) = P(X < 0)$

$$\Rightarrow P(X = 0) = P(X = -1) + P(X = -2) + P(X = -3)$$

$$\text{i.e., } p_4 = p_3 + p_2 + p_1 \quad \text{--- (1)}$$

Also we have

$$P(X = -3) = P(X = -2) = P(X = -1) = P(X = 1) = P(X = 2) = P(X = 3)$$

$$\text{i.e., } p_1 = p_2 = p_3 = p_5 = p_6 = p_7 \quad \text{--- (2)}$$

Further we must have

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 = 1 \quad \text{--- (3)}$$

$$\text{Using (2) in (1), we get } p_4 = p_1 + p_1 + p_1$$

$$\mathbf{p_4 = 3p_1}$$

$$\text{Using (2) in (3), we get } p_1 + p_1 + p_1 + p_4 + p_1 + p_1 + p_1 = 1$$

$$6p_1 + p_4 = 1$$

$$6p_1 + 3p_1 = 1 \quad (\text{since } p_4 = 3p_1)$$

$$9p_1 = 1$$

$$\mathbf{p_1 = \frac{1}{9}}$$

$$\text{Hence } \mathbf{p_4 = \frac{3}{9}}$$

$$\text{Thus, } p_2 = p_3 = p_5 = p_6 = p_7 = \frac{1}{9}$$

Thus the probability distribution is as follows

X	-3	-2	-1	0	1	2	3
P(X)	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

7. The p.d.f. of variate X is given by the following table.

x	0	1	2	3	4	5	6
P(x)	K	3K	5K	7K	9K	11K	13K

For what value of K , this represents a valid probability distribution? Also find $P(x \geq 5)$ and $P(3 < x \leq 6)$.

Sol: The probability distribution is valid if $P(x) \geq 0$ and $\sum P(x) = 1$

Hence we must have $K \geq 0$ and $K + 3K + 5K + 7K + 9K + 11K + 13K = 1$

$$49K = 1 \Rightarrow K = \frac{1}{49}$$

Also, $P(x \geq 5) = P(5) + P(6)$

$$P(x \geq 5) = 11K + 13K = 24K$$

$$P(x \geq 5) = \frac{24}{49}$$

Now, $P(3 < x \leq 6) = P(4) + P(5) + P(6)$

$$P(3 < x \leq 6) = 9K + 11K + 13K$$

$$P(3 < x \leq 6) = 33K$$

$$P(3 < x \leq 6) = \frac{33}{49}$$

Binomial Distribution

If p is the probability of success and q is the probability of failure, the probability of x success out of n trials is given by $P(x) = {}^nC_x p^x q^{n-x}$

Mean and Standard deviation of the Binomial Distribution

$$\text{Mean, } \mu = np$$

$$\text{Variance, } V = npq$$

$$\text{Std. Deviation, } \sigma = \sqrt{npq}$$

Problems

1. Find the binomial probability which has mean 2 and variance $\frac{4}{3}$.

Sol: Given Mean, $\mu = 2$ and Variance, $V = \frac{4}{3}$

For binomial distribution, $\mu = np$

$$np = 2$$

$$npq = V$$

$$2q = \frac{4}{3}$$

$$q = \frac{2}{3}$$

Since $p = 1 - q = 1 - \frac{2}{3}$

$$p = \frac{1}{3}$$

Since $np = 2$

$$\frac{n}{3} = 2$$

$$n = 6$$

The binomial probability function is $P(x) = {}^nC_x p^x q^{n-x}$ becomes

$$P(x) = {}^6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$$

The distribution of probability is as follows

x	0	1	2	3	4	5	6
P(x)	$\left(\frac{2}{3}\right)^6$	${}^6C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^5$	${}^6C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^4$	${}^6C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3$	${}^6C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2$	${}^6C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^1$	$\left(\frac{1}{3}\right)^6$

2. When a coin is tossed 4 times, find the probability of getting i) exactly one head
ii) atmost 3 heads iii) atleast two heads.

Sol: Given $n = 4$, $p = P(H) = \frac{1}{2}$, $q = P(T) = \frac{1}{2}$ (since $p + q = 1$)

From binomial distribution, the probability of x successes out of n trials is given by

$$P(x) = {}^nC_x p^x q^{n-x}$$

$$P(x) = {}^4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}$$

$$P(x) = {}^4C_x \left(\frac{1}{2}\right)^4$$

$$P(x) = \frac{1}{16} \cdot {}^4C_x$$

$$i) P(1 \text{ head}) = P(x = 1) = \frac{1}{16} \cdot {}^4C_1 = \frac{1}{16} (4)$$

$$P(x = 1) = \frac{1}{4}$$

$$P(x = 1) = 0.25$$

$$ii) P(\text{atmost 3 heads}) = P(x \leq 3)$$

$$P(x \leq 3) = P(x = 0) + P(x = 1) + P(x = 2) + P(x = 3)$$

$$P(x \leq 3) = \frac{1}{16} \cdot {}^4C_0 + \frac{1}{16} \cdot {}^4C_1 + \frac{1}{16} \cdot {}^4C_2 + \frac{1}{16} \cdot {}^4C_3$$

$$P(x \leq 3) = \frac{1}{16} [1 + 4 + 6 + 4]$$

$$P(x \leq 3) = \frac{15}{16}$$

$$\mathbf{P(x \leq 3) = 0.9375}$$

iii) P(atleast 2 heads) = $P(x \geq 2)$

$$P(x \geq 2) = 1 - P(x < 2)$$

$$P(x \geq 2) = 1 - [P(x = 0) + P(x = 1)]$$

$$P(x \geq 2) = 1 - \left[\frac{1}{16} \cdot {}^4C_0 + \frac{1}{16} \cdot {}^4C_1 \right]$$

$$P(x \geq 2) = 1 - \frac{1}{16} [1 + 4]$$

$$P(x \geq 2) = 1 - \frac{5}{16}$$

$$P(x \geq 2) = \frac{11}{16}$$

$$\mathbf{P(x \geq 2) = 0.6875}$$

3. In a consignment of electric lamps 5% are defective. If a random sample of 8 lamps are inspected. What is the probability that one or more lamps are defective?

Sol: Given $n = 8$, Probability of defective lamp = $p = 5\% = \frac{5}{100} = 0.05$

$$\therefore q = 1 - p = 1 - 0.05$$

$$\mathbf{q = 0.95}$$

From binomial distribution,

$$P(x) = {}^nC_x p^x q^{n-x}$$

$$P(x) = {}^8C_x (0.05)^x (0.95)^{8-x}$$

Probability that one or more lamps are defective

$$P(x \geq 1) = 1 - P(x < 1)$$

$$P(x \geq 1) = 1 - P(x = 0)$$

$$P(x \geq 1) = 1 - {}^8C_0 (0.05)^0 (0.95)^{8-0}$$

$$\mathbf{P(x \geq 1) = 0.3366.}$$

Thus the required probability is 0.3366

4. The probability that a person aged 60 years will live upto 70 is 0.65. What is the probability that out of 10 person aged 60 atleast 7 of them will live upto 70.

Sol: Given $n = 10$, $p = 0.65$

$$\therefore q = 1 - p = 1 - 0.65$$

$$\mathbf{q = 0.35}$$

Let x denote the number of person aged 60 years living upto 70 years.

From binomial distribution,

$$P(x) = {}^n C_x p^x q^{n-x}$$

$$P(x) = {}^{10} C_x (0.65)^x (0.35)^{10-x}$$

Probability that out of 10 person aged 60 atleast 7 of them will live upto 70

$$P(x \geq 7) = P(7) + P(8) + P(9) + P(10)$$

$$P(x \geq 7) = {}^{10} C_7 (0.65)^7 (0.35)^3 + {}^{10} C_8 (0.65)^8 (0.35)^2 + {}^{10} C_9 (0.65)^9 (0.35)^1 + {}^{10} C_{10} (0.65)^{10} (0.35)^0$$

$$\mathbf{P(x \geq 7) = 0.5138}$$

5. The number of telephone lines busy at an instant of time is a binomial variate with probability 0.1 that a line is busy. If 10 such lines are chosen at random, what is the probability that i) no line is busy ii) all lines are busy iii) atleast one line is busy iv) atmost 2 lines are busy.

Sol: Given $n = 10$, $p = 0.1$

$$\therefore q = 1 - p = 1 - 0.1$$

$$\mathbf{q = 0.9}$$

Let x denote the number of telephone lines busy.

From binomial distribution,

$$P(x) = {}^n C_x p^x q^{n-x}$$

$$\mathbf{P(x) = {}^{10} C_x (0.1)^x (0.9)^{10-x}}$$

i) Probability that no line is busy

$$P(x = 0) = {}^{10} C_0 (0.1)^0 (0.9)^{10-0} = (0.9)^{10} = \mathbf{0.3487}$$

ii) Probability that all lines are busy

$$P(x = 10) = {}^{10} C_{10} (0.1)^{10} (0.9)^0 = \mathbf{(0.1)^{10}}$$

iii) Probability that atleast one line is busy

$$P(x \geq 1) = 1 - P(x < 1)$$

$$P(x \geq 1) = 1 - P(x = 0)$$

$$P(x \geq 1) = 1 - {}^{10} C_0 (0.1)^0 (0.9)^{10}$$

$$P(x \geq 1) = 1 - 0.3487$$

$$\mathbf{P(x \geq 1) = 0.6513}$$

iv) Probability that atmost 2 lines are busy

$$P(x \leq 2) = P(x = 0) + P(x = 1) + P(x = 2)$$

$$P(x \leq 2) = (0.9)^{10} + 10c_1(0.1)^1(0.9)^2 + 10c_2(0.1)^2(0.9)^8$$

$$P(x \leq 2) = \mathbf{0.9298}$$

6. In a quiz contest of answering 'Yes' or 'No' what is the probability of guessing atleast 6 answers correctly out of 10 questions asked ? Also find the probability of the same if there are 4 options for a correct answer.

Sol: Given $n = 10$

Let x denote the correct answer .

$$\text{i) } p = \frac{1}{2} = \mathbf{0.5}$$

$$\therefore q = 1 - p = 1 - 0.5$$

$$\mathbf{q = 0.5}$$

From binomial distribution,

$$P(x) = {}^n C_x p^x q^{n-x}$$

$$P(x) = {}^{10} C_x (0.5)^x (0.5)^{10-x}$$

$$\mathbf{P(x) = {}^{10} C_x (0.5)^{10}}$$

Probability of guessing atleast 6 answers correctly out of 10 questions

$$P(x \geq 6) = P(x = 6) + P(x = 7) + P(x = 8) + P(x = 9) + P(x = 10)$$

$$P(x \geq 6) = {}^{10} C_6 (0.5)^{10} + {}^{10} C_7 (0.5)^{10} + {}^{10} C_8 (0.5)^{10} + {}^{10} C_9 (0.5)^{10} + {}^{10} C_{10} (0.5)^{10}$$

$$P(x \geq 6) = (0.5)^{10} [{}^{10} C_6 + {}^{10} C_7 + {}^{10} C_8 + {}^{10} C_9 + {}^{10} C_{10}]$$

$$P(x \geq 6) = (0.5)^{10} [210 + 120 + 45 + 10 + 1]$$

$$\mathbf{P(x \geq 6) = 0.3770}$$

$$\text{ii) } p = \frac{1}{4}$$

$$\therefore q = 1 - p = 1 - \frac{1}{4}$$

$$\mathbf{q = \frac{3}{4}}$$

From binomial distribution,

$$P(x) = {}^n C_x p^x q^{n-x}$$

$$P(x) = {}^{10} C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{10-x}$$

$$P(x) = 10c_x \frac{3^{10-x}}{4^{x+10-x}}$$

$$P(x) = 10c_x \frac{3^{10-x}}{4^{10}}$$

$$P(x) = \frac{1}{4^{10}} [3^{10-x} \cdot 10c_x]$$

Probability of guessing atleast 6 answers correctly out of 10 questions

$$P(x \geq 6) = P(x = 6) + P(x = 7) + P(x = 8) + P(x = 9) + P(x = 10)$$

$$P(x \geq 6) = \frac{1}{4^{10}} [3^4 \cdot 10c_6 + 3^3 \cdot 10c_7 + 3^2 \cdot 10c_8 + 3 \cdot 10c_9 + 10c_{10}]$$

$$P(x \geq 6) = \frac{1}{4^{10}} [81 \cdot 210 + 27 \cdot 120 + 9 \cdot 45 + 3 \cdot 10 + 1]$$

$$P(x \geq 6) = \mathbf{0.0197}$$

7. In sampling a large number of parts manufactured by a company, the mean number of defectives in samples of 20 is 2. Out of 1000 such samples how many would be expected to contain atleast 3 defective parts.

Sol: Given Mean, $\mu = 2$ and $n = 20$

$$np = 2$$

$$20p = 2$$

$$p = \frac{1}{10} = \mathbf{0.1}$$

$$q = 1 - p$$

$$q = 1 - \frac{1}{10} \quad q = \frac{9}{10} = \mathbf{0.9}$$

Let x denote the defective part.

From binomial distribution,

$$P(x) = {}^n c_x p^x q^{n-x}$$

$$P(x) = \mathbf{{}^{20} c_x (0.1)^x (0.9)^{20-x}}$$

Probability of atleast 3 defective parts

$$P(x \geq 3) = 1 - P(x < 3)$$

$$P(x \geq 3) = 1 - [P(0) + P(1) + P(2)]$$

$$P(x \geq 3) = 1 - [{}^{20} c_0 (0.1)^0 (0.9)^{20} + {}^{20} c_1 (0.1)^1 (0.9)^{19} + {}^{20} c_2 (0.1)^2 (0.9)^{18}]$$

$$P(x \geq 3) = 1 - 0.6769$$

$$P(x \geq 3) = \mathbf{0.3231}$$

Thus, the number of defectives in 1000 such samples = $1000 \cdot 0.3231 = 323.1 \cong \mathbf{323}$

8. If the mean and standard deviation of the number of correctly answered questions in a test given to 4096 students are 2.5 and $\sqrt{1.875}$. Find an estimate the number of candidates answering correctly i) 5 questions ii) 2 or less iii) 8 or more.

Sol: Given Mean = 2.5 $\sigma = \sqrt{1.875}$

Wkt, $\mu = np$ and $\sigma = \sqrt{V}$

Thus, $np = 2.5$ and $V = npq = 1.875$

Now, $\frac{npq}{np} = \frac{1.875}{2.5}$

We get $q = 0.75$, and $p = 1 - q = 0.25$

$n = 10$

Let x denote the number of correctly answered questions

$$P(x) = {}^nC_x p^x q^{n-x} = {}^{10}C_x (0.25)^x (0.75)^{10-x}$$

i) Probability that 5 questions are answered correctly

$$P(x = 5) = {}^{10}C_5 (0.25)^5 (0.75)^{10-5}$$

$P(x = 5) = 0.0584$

Estimation for 4096 students is $= 4096 * 0.0584 \cong 239$

Number of students correctly answering 5 questions is **239**

ii) Probability that 2 or less questions are answered correctly

$$P(x \leq 2) = P(x = 0) + P(x = 1) + P(x = 2)$$

$$= {}^{10}C_0 (0.25)^0 (0.75)^{10} + {}^{10}C_1 (0.25)^1 (0.75)^9 + {}^{10}C_2 (0.25)^2 (0.75)^8$$

$P(x \leq 2) = 0.5256$

Estimation for 4096 students is $= 4096 * 0.5256 \cong 2153$

Number of students correctly answering 2 or less than 2 questions is **2153**

iii) Probability that 8 or more questions are answered correctly

$$P(x \geq 8) = P(x = 8) + P(x = 9) + P(x = 10) \\ = {}^{10}C_8 (0.25)^8 (0.75)^2 + {}^{10}C_9 (0.25)^9 (0.75)^1 + {}^{10}C_{10} (0.25)^{10} (0.75)^0$$

$P(x \geq 8) = 0.0005$

Estimation for 4096 students is $= 4096 * 0.0005 \cong 2$

Number of students correctly answering 8 or more than 8 questions is **2**

. In 800 families with 5 children each how many families would be expected to have

i) 3 boys ii) 5 girls iii) either 2 or 3 boys iv) atmost 2 girls by assuming probabilities for boys and girls to be equal.

10. An airline knows that 5% of the people making reservations on a certain flight will not turn up. Consequently, their policy is to sell 52 tickets for a flight that can only hold 50 people. What is the probability that there will be a seat for every passenger who turns up?

11. 4 coins are tossed 100 times and the following results were obtained. Fit a binomial distribution for the data and calculate the theoretical frequencies.

Number of heads	0	1	2	3	4
Frequency	5	29	36	25	5

Sol: Let x denote the number of heads and f the corresponding frequency. Since the data is in the form of a frequency distribution we shall first calculate the mean.

$$\text{Mean, } \mu = \frac{\sum fx}{\sum f} = \frac{(0)(5) + (1)(29) + (2)(36) + (3)(25) + (4)(5)}{5 + 29 + 36 + 25 + 5} = \frac{196}{100} = 1.96$$

From binomial distribution, $\mu = np$. Here $n=4$

$$\text{Hence } 4p = 1.96$$

$$p = 0.49$$

$$q = 0.51$$

From binomial distribution,

$$P(x) = {}^n C_x p^x q^{n-x} \quad P(x) = {}^4 C_x (0.49)^x (0.51)^{4-x}$$

Since 4 coins were tossed, expected (theoretical) frequencies are obtained from

$$F(x) = 100P(x)$$

$$F(x) = 100 \cdot {}^4 C_x (0.49)^x (0.51)^{4-x} \quad \text{where } x = 0, 1, 2, 3, 4.$$

$$F(0) = 100 \cdot {}^4 C_0 (0.49)^0 (0.51)^4 = 6.765 \cong 7$$

$$F(1) = 100 \cdot {}^4 C_1 (0.49)^1 (0.51)^3 = 25.999 \cong 26$$

$$F(2) = 100 \cdot {}^4 C_2 (0.49)^2 (0.51)^2 = 37.47 \cong 37$$

$$F(3) = 100 \cdot {}^4 C_3 (0.49)^3 (0.51)^1 = 24.0004 \cong 24$$

$$F(4) = 100 \cdot {}^4 C_4 (0.49)^4 (0.51)^0 = 5.765 \cong 6$$

Thus the required theoretical frequencies are **7,26,37,24,6**.

Poisson Distribution

Poisson distribution is regarded as the limiting form of the binomial distribution when n is very large ($n \rightarrow \infty$) and p the probability of success is very small ($p \rightarrow 0$) so that np tends to a fixed finite constant say m .

Let m be a positive constant. Consider the infinite discrete probability distribution of a

random variable x is given by $P(x) = \frac{m^x}{x!} e^{-m}$

The above probability density function is called the **poisson probability** function.

Mean and Standard deviation of the Poisson Distribution

$$\text{Mean, } \mu = m$$

$$\text{Variance, } V = m$$

$$\text{Std. Deviation, } \sigma = \sqrt{m}$$

Problems

1. The probabilities of a poisson variate taking the values 3 and 4 are equal. Find the PDF. Also, calculate the probabilities of the variate taking the values 0 and 1.

Sol: Given $P(x = 3) = P(x = 4)$

$$\frac{m^3}{3!} e^{-m} = \frac{m^4}{4!} e^{-m}$$

$$\frac{4!}{3!} = \frac{m^4}{m^3}$$

$$\therefore m = 4.$$

From poisson distribution,

$$P(x) = \frac{m^x}{x!} e^{-m} = \frac{4^x}{x!} e^{-4}$$

$$P(x = 0) = \frac{4^0}{0!} e^{-4} = \mathbf{0.0183}$$

$$P(x = 1) = \frac{4^1}{1!} e^{-4} = \mathbf{0.0733}$$

2. In a certain factory turning out razor blades there is a small probability of $\frac{1}{500}$ for any blade to be defective. The blades are supplied in packets of 10. Use poisson distribution to calculate the approximate number of packets containing i) no defective ii) one defective iii) two defective blades in a consignment of 10,000 packets.

Sol: Probability of a defective blade = $p = \frac{1}{500} = 0.002$

In a packet of 10, the mean number of defective blades is

Mean, $m = np$

$$m = 10(0.002)$$

$$\mathbf{m = 0.02}$$

From poisson distribution,

$$P(x) = \frac{m^x}{x!} e^{-m}$$

$$P(x) = \frac{(0.02)^x}{x!} e^{-(0.02)}$$

$$P(x) = (0.9802) \frac{(0.02)^x}{x!} \quad (\text{Since } e^{-0.02} = 0.9802)$$

i) Probability of no defective

$$P(x = 0) = (0.9802) \frac{(0.02)^0}{0!} = 0.9802$$

$$\text{For consignment of 10,000 packets} = 10,000 * 0.9802 = 9802$$

ii) Probability of one defective

$$P(x = 1) = (0.9802) \frac{(0.02)^1}{1!} = 0.0196$$

$$\text{For consignment of 10,000 packets} = 10,000 * 0.0196 = 196$$

iii) Probability of one defective

$$P(x = 2) = (0.9802) \frac{(0.02)^2}{2!} = 0.0002$$

$$\text{For consignment of 10,000 packets} = 10,000 * 0.0002 = 2$$

3. The number of accidents in a year to taxi drivers in a city follows a Poisson distribution with mean 3. Out of 1000 taxi drivers find approximately the number of the drivers with
i) no accident in a year ii) more than 3 accidents in a year.

Sol: Given Mean=3

Mean, $\mu = m$

$$m = 3$$

From poisson distribution,

$$P(x) = \frac{m^x}{x!} e^{-m}$$

$$P(x) = \frac{(3)^x}{x!} e^{-3}$$

$$P(x) = (0.05) \frac{(3)^x}{x!} \quad (\text{Since } e^{-3} = 0.05)$$

i) Probability of number of drivers with no accident in a year

$$P(x = 0) = (0.05) \frac{(3)^0}{0!} = 0.05$$

$$\text{Number of drivers out of 1000 with no accident in a year} = 1,000 * 0.05 = 50$$

ii) Probability of more than 3 accidents in a year

$$P(x > 3) = 1 - P(x \leq 3)$$

$$P(x > 3) = 1 - [P(0) + P(1) + P(2) + P(3)]$$

$$P(x > 3) = 1 - \left[(0.05) \frac{(3)^0}{0!} + (0.05) \frac{(3)^1}{1!} + (0.05) \frac{(3)^2}{2!} + (0.05) \frac{(3)^3}{3!} \right]$$

$$P(x > 3) = 1 - (0.05)[1 + 3 + 4.5 + 4.5]$$

$$P(x > 3) = 0.35$$

Number of drivers out of 1000 with more than 3 accidents in a year = $1,000 * 0.35 = 350$

4. Given that 2% of the fuses manufactured by a firm are defective, find, by using poisson distribution, the probability that a box containing 200 fuses has i) no defective fuse ii) at least one defective fuse iii) exactly 3 defective fuses iv) 3 or more defective fuses.

Sol: The probability that a fuses is defective $p = 2/100 = 0.02$.

And $n = 200$

Mean, $\mu = np = 200 * 0.02$; **$m = 4$**

From poisson probability function $P(x) = \frac{m^x}{x!} e^{-m}$

$$P(x) = \frac{4^x}{x!} e^{-4}$$

Let x denotes the number of defective fuses in a box containing 200 fuses

i) Probability that the box contains no defective fuse

$$P(x = 0) = \frac{4^0}{0!} e^{-4} = \mathbf{0.0183}$$

ii) Probability that the box contains at least one defective fuse is

$$\begin{aligned} P(x \geq 1) &= 1 - P(x < 1) \\ &= 1 - P(x = 0) \\ &= 1 - 0.013 \end{aligned}$$

$$\mathbf{P(x \geq 1) = 0.9817}$$

iii) Probability that the box contains exactly 3 defective fuse

$$P(x = 3) = \frac{4^3}{3!} e^{-4}$$

$$\mathbf{P(x = 3) = 0.1952}$$

iv) Probability that the box contains 3 or more defective fuses

$$P(x \geq 3) = 1 - P(x < 3)$$

$$P(x \geq 3) = 1 - \left[\frac{4^0}{0!} e^{-4} + \frac{4^1}{1!} e^{-4} + \frac{4^2}{2!} e^{-4} \right]$$

$$\mathbf{P(x \geq 3) = 0.7621}$$

5. A certain screw making machine has a chance of producing 2 defectives out of 1000. The screws are packed in boxes of 100. Using poisson distribution , find the approximate number of boxes containing i) no defective screw ii) one defective screw iii) two defective screw, in a consignment of 5000 boxes.

Sol: The probability that a screw is defective is given as $p = 2/1000 = 0.002$.

Here $n=100$

Mean = $np=100 \times 0.002=0.2$; **$m = 0.2$**

From poisson probability function $P(x) = \frac{m^x}{x!} e^{-m}$

$$P(x) = \frac{(0.2)^x}{x!} e^{-0.2}$$

Let x denotes the number of defective fuses in a box containing 100 screws

i) Probability that the box contains no defective screw

$$P(x = 0) = \frac{0.2^0}{0!} e^{-0.2} = \mathbf{0.8187}$$

Number of boxes containing no defective screw in 5000 boxes = $5000 \times 0.8187 = 4094$

ii) Probability that the box contains one defective screw is

$$\begin{aligned} P(x = 1) &= \frac{(0.2)^1}{1!} e^{-0.2} \\ &= 0.2 \times 0.8187 \end{aligned}$$

$$P(x = 1) = \mathbf{0.1637}$$

Number of boxes containing one defective screw in 5000 boxes = $5000 \times 0.1637 = 819$

iii) Probability that the box contains two defective screw is

$$\begin{aligned} P(x = 2) &= \frac{(0.2)^2}{2!} e^{-0.2} \\ &= 0.02 \times 0.8187 \end{aligned}$$

$$P(x = 2) = \mathbf{0.0163}$$

Number of boxes containing two defective screw in 5000 boxes = $5000 \times 0.0163 = 82$

6. A communication channel receives independent pulses at the rate of 12 pulses per micro second. The probability of transmission error is 0.001 for each micro second. Compute the probabilities of i) no error during a micro second ii) one error per micro second iii) atleast one per micro second iv) two errors v) atmost two errors.

Sol: Given $n = 12$ and $p = 0.001$

Mean number of errors in one micro second

$$\mu = np$$

$$m = 12 \times 0.001 \quad (\text{Since } \mu = m, \text{ from poisson distribution})$$

$$\mathbf{m = 0.012}$$

From Poisson distribution

$$P(x) = \frac{m^x}{x!} \cdot e^{-m}$$

$$P(x) = \frac{(0.012)^x}{x!} \cdot e^{-(0.012)}$$

$$P(x) = (0.9880) \frac{(0.012)^x}{x!} \quad (\text{Since } e^{-(0.012)} = 0.9880)$$

i) Probability of no error during a micro second

$$P(x = 0) = (0.9880) \frac{(0.012)^0}{0!}$$

$$\mathbf{P(x = 0) = 0.9880}$$

ii) Probability of one error during a micro second

$$P(x = 1) = (0.9880) \frac{(0.012)^1}{1!}$$

$$\mathbf{P(x = 1) = 0.0119}$$

iii) Probability of atleast one error during a micro second

$$P(x \geq 1) = 1 - P(x < 1)$$

$$P(x \geq 1) = 1 - P(x = 0)$$

$$P(x \geq 1) = 1 - P(x = 0)$$

$$P(x \geq 1) = 1 - 0.9880$$

$$\mathbf{P(x \geq 1) = 0.012}$$

iv) Probability of two error during a micro second

$$P(x = 2) = (0.9880) \frac{(0.012)^2}{2!}$$

$$\mathbf{P(x = 2) = 0.00007}$$

v) Probability of atmost two error during a micro second

$$P(x \leq 2) = P(x = 0) + P(x = 1) + P(x = 2)$$

$$P(x \leq 2) = 0.9880 + 0.0119 + 0.00007$$

$$\mathbf{P(x \leq 2) = 0.99997}$$

Continuous Probability Distribution

If for every x belonging to the range of a continuous random variable X, we assign a real number f(x) satisfying the conditions,

i) $f(x) \geq 0$ ii) $\int_{-\infty}^{\infty} f(x)dx = 1$, then f(x) is called a Continuous probability function or probability density function (p.d.f)

Cumulative distribution function

If X is a continuous random variable with probability density function f(x) then the function F(x) defined by $F(x) = P(X \leq x) = \int_{-\infty}^x f(x)dx$ is called the Cumulative distribution function (c.d.f) of X.

NOTE: If r is any real number, then

$$1. P(x \geq r) = \int_r^{\infty} f(x) dx$$

$$P(x < r) = 1 - P(x \geq r) = 1 - \int_r^{\infty} f(x) dx$$

Mean and Variance

If X is a continuous random variable with probability density function $f(x)$ where $-\infty < x < \infty$, the mean (μ) or expectation $E(X)$ and the variance (σ^2) of X is defined as follows.

$$\text{Mean, } \mu = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

$$\text{Variance, } \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 \cdot f(x) dx \quad \text{or} \quad V = \int_{-\infty}^{\infty} x^2 \cdot f(x) dx - \mu^2$$

Problems

1. Find which of the following is a probability density function

$$a) f_1(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \quad b) f_2(x) = \begin{cases} |x|, & 0 < |x| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$c) f_3(x) = \begin{cases} 2x, & 0 < x \leq 1 \\ 4 - 4x, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Sol: Conditions for p.d.f are $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x) dx = 1$

a) Clearly $f_1(x) \geq 0$

$$\int_{-\infty}^{\infty} f_1(x) dx = \int_0^1 f_1(x) dx = \int_0^1 2x dx = 2 \cdot \left[\frac{x^2}{2} \right]_0^1 = [1 - 0] = 1$$

$\therefore f_1(x)$ is a p.d.f

b) Evidently $f_2(x) = |x| \geq 0$

$$\int_{-\infty}^{\infty} f_2(x) dx = \int_{-1}^1 f_2(x) dx = \int_{-1}^1 |x| dx$$

$$\text{But, } |x| = \begin{cases} -x, & -1 < x < 0 \\ +x, & 0 < x < 1 \end{cases}$$

$$\therefore \int_{-\infty}^{\infty} f_2(x) dx = \int_{-1}^0 (-x) dx + \int_0^1 x dx$$

$$\int_{-\infty}^{\infty} f_2(x) dx = - \left[\frac{x^2}{2} \right]_{-1}^0 + \left[\frac{x^2}{2} \right]_0^1$$

$$\int_{-\infty}^{\infty} f_2(x) dx = - \left[0 - \frac{1}{2} \right] + \left[\frac{1}{2} - 0 \right] = \frac{1}{2} + \frac{1}{2} = 1$$

$\therefore f_2(x)$ is a p.d.f

c) Given $f_3(x) = 2x > 0$ in $0 < x \leq 1$

But $f_3(x) = 4 - 4x$ is negative in $1 < x < 2$

The first condition is not satisfied.

$\therefore f_3(x)$ is not a p.d.f

2. Find the value of c such that $f(x) = \begin{cases} \frac{x}{6} + c, & 0 \leq x \leq 3 \\ 0, & \text{elsewhere} \end{cases}$ is a p.d.f. Also find $P(1 \leq x \leq 2)$.

Sol: Here $f(x) \geq 0$ if $c \geq 0$; Also we must have $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\text{i.e., } \int_0^3 \left(\frac{x}{6} + c \right) dx = 1$$

$$\text{i.e., } \left[\frac{x^2}{12} + cx \right]_0^3 = 1$$

$$\text{i.e., } \frac{1}{12}(9 - 0) + c(3 - 0) = 1$$

$$\text{i.e., } \frac{9}{12} + 3c = 1$$

$$\text{i.e., } 3c = 1 - \frac{9}{12}$$

$$3c = \frac{3}{12}$$

$$\therefore c = \frac{1}{12}$$

$$\text{Now, } P(1 \leq x \leq 2) = \int_1^2 f(x)dx$$

$$= \int_1^2 \left(\frac{x}{6} + \frac{1}{12} \right) dx$$

$$= \left[\frac{x^2}{12} + \frac{x}{12} \right]_1^2$$

$$= \frac{1}{12}(4 - 1) + \frac{1}{12}(2 - 1)$$

$$= \frac{3}{12} + \frac{1}{12} = \frac{4}{12}$$

$$P(1 \leq x \leq 2) = \frac{1}{3}$$

3. Find the constant k such that $f(x) = \begin{cases} kx^2, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$ is a p.d.f. Also compute

i) $P(1 < x < 2)$ ii) $P(x \leq 1)$ iii) $P(x > 1)$ iv) Mean v) Variance

Sol: Here $f(x) \geq 0$ if $k \geq 0$; Also we must have $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\text{i.e., } \int_0^3 kx^2 \, dx = 1$$

$$\text{i.e., } \left[\frac{kx^3}{3} \right]_0^3 = 1$$

$$\text{i.e., } \frac{k}{3}(27 - 0) = 1$$

$$\text{i.e., } 9k = 1$$

$$\text{i.e., } \mathbf{k} = \frac{1}{9}$$

$$\therefore f(x) = \begin{cases} \frac{x^2}{9}, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} \text{i) } P(1 < x < 2) &= \int_1^2 f(x) \, dx \\ &= \int_1^2 \frac{x^2}{9} \, dx \\ &= \left[\frac{x^3}{27} \right]_1^2 \\ &= \frac{1}{27}(8 - 1) \end{aligned}$$

$$\mathbf{P(1 < x < 2) = \frac{7}{27}}$$

$$\text{ii) } P(x \leq 1) = \int_0^1 f(x) \, dx$$

$$\begin{aligned} P(x \leq 1) &= \int_0^1 \frac{x^2}{9} \, dx \\ &= \left[\frac{x^3}{27} \right]_0^1 \\ &= \frac{1}{27}(1 - 0) \end{aligned}$$

$$\mathbf{P(x \leq 2) = \frac{1}{27}}$$

$$\text{iii) } P(x > 1) = \int_1^3 f(x) \, dx$$

$$= \int_1^3 \frac{x^2}{9} \, dx$$

$$\begin{aligned} P(x > 1) &= \left[\frac{x^3}{27} \right]_1^3 \\ &= \frac{1}{27}(27 - 1) \end{aligned}$$

$$\mathbf{P(x > 1) = \frac{26}{27}}$$

$$\text{iv) Mean, } \mu = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

$$= \int_0^3 x \cdot \frac{x^2}{9} dx$$

$$= \int_0^3 \frac{x^3}{9} dx$$

$$\mu = \left[\frac{x^4}{36} \right]_0^3$$

$$\mu = \frac{1}{36} (81 - 0)$$

$$\mu = \frac{81}{36}$$

$$\mu = \frac{9}{4}$$

$$\text{v) Variance, } V = \int_{-\infty}^{\infty} x^2 \cdot f(x) dx - \mu^2$$

$$= \int_0^3 x^2 \cdot \frac{x^2}{9} dx - \left(\frac{9}{4} \right)^2$$

$$= \int_0^3 \frac{x^4}{9} dx - \frac{81}{16}$$

$$= \left[\frac{x^5}{45} \right]_0^3 - \frac{81}{16}$$

$$V = \frac{1}{45} [3^5 - 0] - \frac{81}{16}$$

$$V = \frac{243}{45} - \frac{81}{16}$$

$$V = \frac{81}{240}$$

$$V = \frac{27}{80}$$

Exponential Distribution

The continuous probability distribution having the probability density function $f(x)$ given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & \text{for } x > 0 \\ 0, & \text{otherwise} \end{cases}, \text{ where } \alpha > 0 \text{ is known as the } \mathbf{\text{exponential distribution.}}$$

Evidently $f(x) \geq 0$ and we have $\int_{-\infty}^{\infty} f(x) dx = 1$.

$f(x)$ satisfy both the conditions required for a continuous probability function.

Mean and standard deviation of the Exponential Distribution

$$\mathbf{\text{Mean, } } \mu = \frac{1}{\alpha}$$

Std. deviation , $\sigma = \frac{1}{\alpha}$

Problems

1. If x is an exponential variate with mean 3 find i) $P(x > 1)$ ii) $P(x < 3)$

Sol: The p.d.f of the exponential distribution is given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & \text{for } 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

The mean of the distribution is given by $\frac{1}{\alpha}$.

By data, $\mu = \frac{1}{\alpha} = 3$

$$\therefore \alpha = \frac{1}{3}$$

Hence , $f(x) = \begin{cases} \frac{1}{3} e^{-\frac{x}{3}}, & \text{for } 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$

i) $P(x > 1) = 1 - P(x \leq 1)$

$$\begin{aligned} P(x > 1) &= 1 - P(x \leq 1) \\ &= 1 - \int_0^1 f(x) dx \\ &= 1 - \int_0^1 \frac{1}{3} e^{-\frac{x}{3}} dx \\ &= 1 - \frac{1}{3} \int_0^1 e^{-\frac{x}{3}} dx \\ &= 1 - \frac{1}{3} \left[\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}} \right]_0^1 \\ &= 1 + \left[e^{-\frac{1}{3}} - e^{-0} \right] \\ &= 1 + e^{-\frac{1}{3}} - 1 \\ &= e^{-\frac{1}{3}} \end{aligned}$$

$P(x > 1) = 0.7165$

$$\begin{aligned} \text{ii) } P(x < 3) &= \int_0^3 f(x) dx \\ &= \int_0^3 \frac{1}{3} e^{-\frac{x}{3}} dx \\ &= \frac{1}{3} \int_0^3 e^{-\frac{x}{3}} dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3} \left[\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}} \right]_0^3 \\
&= -[e^{-1} - e^{-0}] \\
&= -e^{-1} + 1 = 1 - \frac{1}{e}
\end{aligned}$$

$$\mathbf{P(x < 3) = 0.6321}$$

2. The length of telephone conversation in a booth has been an exponential distribution and found on an average to be 5 minutes. Find the probability that a random call made from this booth i) ends less than 5 minutes ii) between 5 and 10 minutes.

Sol: We have, $f(x) = \alpha e^{-\alpha x}$, $0 < x < \infty$

$$\text{By data, } \mu = \frac{1}{\alpha} = 5$$

$$\therefore \alpha = \frac{1}{5}$$

$$f(x) = \frac{1}{5} e^{-\frac{x}{5}}, \quad 0 < x < \infty$$

$$\begin{aligned}
\text{i) } P(x < 5) &= \int_0^5 f(x) dx \\
&= \int_0^5 \frac{1}{5} e^{-\frac{x}{5}} dx \\
&= \frac{1}{5} \int_0^5 e^{-\frac{x}{5}} dx
\end{aligned}$$

$$\begin{aligned}
P(x < 5) &= \frac{1}{5} \left[\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}} \right]_0^5 \\
&= -[e^{-1} - e^{-0}] \\
&= -e^{-1} + 1 = 1 - \frac{1}{e}
\end{aligned}$$

$$\mathbf{P(x < 5) = 0.6321}$$

$$\begin{aligned}
\text{ii) } P(5 < x < 10) &= \int_5^{10} f(x) dx \\
&= \int_5^{10} \frac{1}{5} e^{-\frac{x}{5}} dx \\
&= \frac{1}{5} \int_5^{10} e^{-\frac{x}{5}} dx
\end{aligned}$$

$$\begin{aligned}
 P(5 < x < 10) &= \frac{1}{5} \left[\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}} \right]_5^{10} \\
 &= -[e^{-2} - e^{-1}] \\
 &= -e^{-2} + e^{-1} = \frac{1}{e} - \frac{1}{e^2}
 \end{aligned}$$

$$P(5 < x < 10) = 0.2325$$

3. In a certain town the duration of a shower is exponentially distributed with mean 5 minutes. What is the probability that a shower will last for: (i) 10 minutes or more (ii) less than 10 minutes (iii) between 10 and 12 minutes.

Sol: We have, $f(x) = \alpha e^{-\alpha x}$, $0 < x < \infty$

$$\text{By data, } \mu = \frac{1}{\alpha} = 5$$

$$\therefore \alpha = \frac{1}{5}$$

$$f(x) = \frac{1}{5} e^{-\frac{x}{5}}, \quad 0 < x < \infty$$

$$\begin{aligned}
 \text{i) } P(x \geq 10) &= \int_{10}^{\infty} f(x) dx \\
 &= \int_{10}^{\infty} \frac{1}{5} e^{-\frac{x}{5}} dx \\
 &= \frac{1}{5} \int_{10}^{\infty} e^{-\frac{x}{5}} dx \\
 &= \frac{1}{5} \left[\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}} \right]_{10}^{\infty} \\
 &= -[e^{-\infty} - e^{-2}] \\
 &= -e^{-\infty} + e^{-2} = -0 + e^{-2}
 \end{aligned}$$

$$P(x \geq 10) = 0.1353$$

$$\text{ii) } P(x < 10) = 1 - P(x \geq 10)$$

$$P(x < 10) = 1 - 0.1353$$

$$P(x < 10) = 0.8647$$

$$\begin{aligned}
 \text{iii) } P(10 < x < 12) &= \int_{10}^{12} f(x) dx \\
 &= \int_{10}^{12} \frac{1}{5} e^{-\frac{x}{5}} dx
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{5} \int_{10}^{12} e^{-\frac{x}{5}} dx \\
&= \frac{1}{5} \left[\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}} \right]_{10}^{12} \\
&= -[e^{-2.4} - e^{-2}]
\end{aligned}$$

$$P(10 < x < 12) = e^{-2} - e^{-2.4}$$

$$P(10 < x < 12) = 0.0446$$

4. The life of a compressor manufactured by a company is known to be 200 months on an average following an exponential distribution. Find the probability that the life of a compressor of that company is i) less than 200 months ii) between 100 months and 25 years.

Sol: We have, $f(x) = \alpha e^{-\alpha x}$, $0 < x < \infty$

$$\text{By data, } \mu = \frac{1}{\alpha} = 200$$

$$\therefore \alpha = \frac{1}{200}$$

$$f(x) = \frac{1}{200} e^{-\frac{x}{200}}, \quad 0 < x < \infty$$

$$\begin{aligned}
\text{i) } P(x < 200) &= \int_0^{200} f(x) dx \\
&= \int_0^{200} \frac{1}{200} e^{-\frac{x}{200}} dx \\
&= \frac{1}{200} \int_0^{200} e^{-\frac{x}{200}} dx \\
&= \frac{1}{200} \left[\frac{e^{-\frac{x}{200}}}{-\frac{1}{200}} \right]_0^{200} \\
&= -[e^{-1} - e^{-0}] \\
&= -e^{-1} + 1 = 1 - \frac{1}{e}
\end{aligned}$$

$$P(x < 200) = 0.6321$$

Now, 25 years = 25 * 12 months = 300 months

$$\begin{aligned}
\text{ii) } P(100 < x < 300) &= \int_{100}^{300} f(x) dx \\
&= \int_{100}^{300} \frac{1}{200} e^{-\frac{x}{200}} dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{200} \int_{100}^{300} e^{-\frac{x}{200}} dx \\
&= \frac{1}{200} \left[\frac{e^{-\frac{x}{200}}}{-\frac{1}{200}} \right]_{100}^{300} \\
&= -[e^{-1.5} - e^{-2}]
\end{aligned}$$

$$\mathbf{P(100 < x < 300) = 0.3834}$$

5. Determine k if random variable x has the density function $f(x) = \frac{k}{1+x^2}$, $-\infty < x < \infty$
hence find i) $P(x \geq 0)$ ii) $P(0 < x < 1)$ iii) c. d. f

Sol: Given f(x) is a probability density function.

$$\therefore f(x) \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\text{i.e., } \int_{-\infty}^{\infty} \frac{k}{1+x^2} dx = 1$$

$$2k \int_0^{\infty} \frac{1}{1+x^2} dx = 1$$

$$2k[\tan^{-1}x]_0^{\infty} = 1 \quad ; \quad 2k[\tan^{-1}(\infty) - \tan^{-1}(0)] = 1$$

$$2k\left[\frac{\pi}{2} - 0\right] = 1$$

$$k \cdot \pi = 1 \quad k = \frac{1}{\pi}$$

$$f(x) = \frac{\frac{1}{\pi}}{1+x^2}$$

$$\text{i) } P(x \geq 0) = \int_0^{\infty} f(x) dx$$

$$= \int_0^{\infty} \frac{\frac{1}{\pi}}{1+x^2} dx$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{1}{1+x^2} dx$$

$$= \frac{1}{\pi} [\tan^{-1}x]_0^{\infty}$$

$$P(x \geq 0) = \frac{1}{\pi} [\tan^{-1}(\infty) - \tan^{-1}(0)]$$

$$P(x \geq 0) = \frac{1}{\pi} \left[\frac{\pi}{2} - 0 \right]$$

$$\mathbf{P(x \geq 0) = \frac{1}{2}}$$

$$\text{ii) } P(0 < x < 1) = \int_0^1 f(x) dx$$

$$= \int_0^1 \frac{\frac{1}{\pi}}{1+x^2} dx$$

$$= \frac{1}{\pi} \int_0^1 \frac{1}{1+x^2} dx$$

$$P(0 < x < 1) = \frac{1}{\pi} [\tan^{-1} x]_0^1$$

$$= \frac{1}{\pi} [\tan^{-1}(1) - \tan^{-1}(0)]$$

$$= \frac{1}{\pi} \left[\frac{\pi}{4} - 0 \right]$$

$$P(0 < x < 1) = \frac{1}{4}$$

$$\text{iii) c. d. f} = \int_{-\infty}^x f(x) dx$$

$$= \int_{-\infty}^x \frac{\frac{1}{\pi}}{1+x^2} dx$$

$$= \frac{1}{\pi} \int_{-\infty}^x \frac{1}{1+x^2} dx$$

$$\text{c. d. f} = \frac{1}{\pi} [\tan^{-1} x]_{-\infty}^x$$

$$= \frac{1}{\pi} [\tan^{-1}(x) - \tan^{-1}(-\infty)]$$

$$= \frac{1}{\pi} \left[\tan^{-1} x - \left(-\frac{\pi}{2} \right) \right]$$

$$\text{c. d. f} = \frac{1}{\pi} \left[\tan^{-1} x + \frac{\pi}{2} \right] \quad \text{if } -\infty < x < \infty.$$

6. The daily turn over in a medical shop is exponentially distributed with Rs.6000 as the average with a net profit of 8%. Find the probability that the net profit exceeds Rs.500 on a randomly chosen day.

Sol: Let 'x' denote the turn over per day.

∴ By exponential distribution

$$f(x) = \alpha e^{-\alpha x} ; x > 0.$$

Given, By data, $\mu = \frac{1}{\alpha} = 6000$

$$\therefore \alpha = \frac{1}{6000}$$

$$f(x) = \frac{1}{6000} e^{-\frac{x}{6000}}, \quad 0 < x < \infty$$

Let 'A' be the turn over for which the net profit is Rs.500.

Since the net profit is 8% of the turn over,

$$A * \frac{8}{100} = 500$$

$$\Rightarrow A = 6250$$

$\therefore$ The probability of profit exceeding Rs.500 is

$$P(x > 6250) = \int_{6250}^{\infty} f(x) dx$$

$$P(x > 6250) = \int_{6250}^{\infty} \frac{1}{6000} e^{-\frac{x}{6000}} dx$$

$$= \frac{1}{6000} \int_{6250}^{\infty} e^{-\frac{x}{6000}} dx$$

$$= \frac{1}{6000} \left[\frac{e^{-\frac{x}{6000}}}{-\frac{1}{6000}} \right]_{6250}^{\infty}$$

$$= - \left[e^{-\infty} - e^{-\frac{6250}{6000}} \right]$$

$$= e^{-\frac{6250}{6000}}$$

$$P(x > 6250) = 0.3529$$

Normal Distribution

The continuous probability distribution having the probability density function $f(x)$ given

by $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ where $-\infty < x < \infty$, $-\infty < \mu < \infty$ and $\sigma > 0$ is known as the normal distribution.

Evidently $f(x) \geq 0$ and we have $\int_{-\infty}^{\infty} f(x) dx = 1$.

$f(x)$ satisfy both the conditions required for a continuous probability function.

Mean and standard deviation of the Normal Distribution

$$\text{Mean} = \mu$$

$$\text{Variance} = \sigma^2$$

Geometrically we can write

$$\text{i) } \int_{-\infty}^{\infty} \phi(z) dz = 1 \quad \text{ii) } \int_{-\infty}^0 \phi(z) dz = \int_0^{\infty} \phi(z) dz = \frac{1}{2}$$

Note

$$1. P(-\infty \leq z \leq \infty) = 1$$

$$2. P(-\infty \leq z \leq 0) = \frac{1}{2}$$

$$3. P(0 \leq z \leq \infty) \text{ or } P(z \geq 0) = \frac{1}{2}$$

$$\text{Also, } P(-\infty < z < z_1) = P(-\infty < z \leq 0) + P(0 \leq z < z_1)$$

$$\text{i.e., } P(z < z_1) = 0.5 + \phi(z_1)$$

$$\text{Also, } P(z > z_2) = P(z \geq 0) - P(0 \leq z < z_2)$$

$$\text{i.e., } P(z > z_2) = 0.5 - \phi(z_2)$$

Problems

1. Evaluate the following probabilities with the help of normal probability tables

i) $P(z \geq 0.85)$ ii) $P(-1.64 \leq z \leq -0.88)$ iii) $P(z \leq -2.43)$ iv) $P(|z| \leq 1.94)$

$$\text{Sol: i) } P(z \geq 0.85) = P(z \geq 0) - P(0 \leq z \leq 0.85)$$

$$= 0.5 - \phi(0.85)$$

$$= 0.5 - 0.3023$$

$$\mathbf{P(z \geq 0.85) = 0.1977}$$

$$\text{ii) } P(-1.64 \leq z \leq -0.88)$$

$$\text{By Symmetry, } P(-1.64 \leq z \leq -0.88) = P(0.88 \leq z \leq 1.64)$$

$$= P(0 \leq z \leq 1.64) - P(0 \leq z \leq 0.88)$$

$$= \phi(1.64) - \phi(0.88)$$

$$= 0.4495 - 0.3106$$

$$\mathbf{P(-1.64 \leq z \leq -0.88) = 0.1389}$$

$$\text{iii) } P(z \leq -2.43) = P(z \geq 2.43)$$

$$P(z \leq -2.43) = P(z \geq 0) - P(0 \leq z \leq 2.43)$$

$$= 0.5 - \phi(2.43)$$

$$= 0.5 - 0.4925$$

$$\mathbf{P(z \geq -2.43) = 0.0075}$$

$$\text{iv) } P(|z| \leq 1.94) = P(-1.94 \leq z \leq 1.94)$$

$$= 2 P(0 \leq z \leq 1.94)$$

$$= 2. \phi(1.94)$$

$$= 2. (0.4738)$$

$$P(|z| \leq 1.94) = 0.9476$$

2. If x is a normal variate with mean 30 and standard deviation 5 find the probability that
i) $26 \leq x \leq 40$ ii) $x \geq 45$

Sol: We have standard normal variate, $z = \frac{x-\mu}{\sigma} = \frac{x-30}{5}$ [By data $\mu = 30, \sigma = 5$]

i) To find $P(26 \leq x \leq 40)$

$$\text{If } x = 26, z = \frac{26-30}{5} = -0.8 \quad ; \quad \text{If } x = 40, z = \frac{40-30}{5} = 2$$

Hence we need to find, $P(-0.8 \leq z \leq 2)$

$$\begin{aligned} P(-0.8 \leq z \leq 2) &= P(-0.8 \leq z \leq 0) + P(0 \leq z \leq 2) \\ &= P(0 \leq z \leq 0.8) + P(0 \leq z \leq 2) \\ &= \phi(0.8) + \phi(2) \\ &= 0.2881 + 0.4772 \end{aligned}$$

$$P(-0.8 \leq z \leq 2) = 0.7653$$

ii) To find $P(x \geq 45)$

$$\text{If } x = 45, z = \frac{45-30}{5} = 3$$

Hence we need to find, $P(z \geq 3)$

$$\begin{aligned} P(z \geq 3) &= P(z \geq 0) - P(0 \leq z \leq 3) \\ &= 0.5 - \phi(3) \\ &= 0.5 - 0.4987 \end{aligned}$$

$$P(z \geq 3) = 0.0013$$

3. The marks of 1000 students in an examination follows a normal distribution with mean 70 and standard deviation 5. Find the number of students whose marks will be
i) less than 65 ii) more than 75 iii) 65 to 75

Sol: Let x represent the marks of students

We have standard normal variate, $z = \frac{x-\mu}{\sigma} = \frac{x-70}{5}$ [By data $\mu = 70, \sigma = 5$]

i) To find $P(x < 65)$

$$\text{If } x = 65, z = \frac{65-70}{5} = -1$$

Hence we need to find, $P(z < -1)$

$$\begin{aligned} P(z < -1) &= P(z > 1) \\ &= P(z \geq 0) - P(0 \leq z \leq 1) \end{aligned}$$

$$= 0.5 - \phi(1)$$

$$= 0.5 - 0.3413$$

$$P(z < -1) = 0.1587$$

Number of students scoring less than 65 marks = $1000 * 0.1587 = 158.7 \cong 159$

ii) To find $P(x > 75)$

$$\text{If } x = 75, z = \frac{75-70}{5} = 1$$

Hence we need to find, $P(z > 1)$

$$P(z > 1) = P(z \geq 0) - P(0 \leq z \leq 1)$$

$$= 0.5 - \phi(1)$$

$$= 0.5 - 0.3413$$

$$P(z > 1) = 0.1587$$

Number of students scoring more than 75 marks = $1000 * 0.1587 = 158.7 \cong 159$

iii) To find $P(65 < x < 75)$

Hence we need to find, $P(-1 < z < 1)$

$$P(-1 < z < 1) = 2P(0 < z < 1)$$

$$= 2\phi(1)$$

$$= 2(0.3413)$$

$$P(-1 < z < 1) = 0.6826$$

Number of students scoring marks between 65 to 75 marks = $1000 * 0.6826 = 682.6 \cong 683$

4. In a test on electric bulbs, it was found that the life time of a particular brand was distributed normally with an average life of 2000 hours and S.D of 60 hours. If a firm purchases 2500 bulbs, find the number of bulbs that are likely to last for i) more than 2100 hours ii) less than 1950 hours iii) between 1900 to 2100 hours.

Sol: Given $\mu = 2000$, $\sigma = 60$

We have standard normal variate, $z = \frac{x-\mu}{\sigma} = \frac{x-2000}{60}$

i) To find $P(x > 2100)$

$$\text{If } x = 2100, z = \frac{2100-2000}{60} = 1.67$$

Hence we need to find, $P(z > 1.67)$

$$P(z > 1.67) = P(z \geq 0) - P(0 < z < 1.67)$$

$$= 0.5 - \phi(1.67)$$

$$= 0.5 - 0.4525$$

$$P(z > 1.67) = 0.0475$$

$$\begin{aligned}\therefore \text{Number of bulbs that are likely to last for more than 2100 hours} &= 2500 * 0.0475 \\ &= 118.75 \cong 119\end{aligned}$$

ii) To find $P(x < 1950)$

$$\text{If } x = 1950, z = \frac{1950-2000}{60} = -0.83$$

Hence we need to find, $P(z < -0.83)$

$$\begin{aligned}P(z < -0.83) &= P(z > 0.83) \\ &= P(z \geq 0) - P(0 < z < 0.83) \\ &= 0.5 - \phi(0.83) \\ &= 0.5 - 0.2967\end{aligned}$$

$$P(z < -0.83) = 0.2033$$

$$\begin{aligned}\therefore \text{Number of bulbs that are likely to last for more than 1950 hours} &= 2500 * 0.2033 \\ &= 508.25 \cong 508\end{aligned}$$

iii) To find $P(1900 < x < 2100)$

$$\text{If } x = 1900, z = \frac{1900-2000}{60} = -1.67 \quad ; \quad \text{If } x = 2100, z = \frac{2100-2000}{60} = 1.67$$

Hence we need to find, $P(-1.67 < z < 1.67)$

$$\begin{aligned}P(-1.67 < z < 1.67) &= 2P(0 < z < 1.67) \\ &= 2 * \phi(1.67) \\ &= 2 * 0.4525\end{aligned}$$

$$P(-1.67 < z < 1.67) = 0.905$$

$$\begin{aligned}\therefore \text{Number of bulbs that are likely to last between 1900 and 2100 hours} \\ &= 2500 * 0.905 \\ &= 2262.5 \cong 2263\end{aligned}$$

5. In a normal distribution 31% of the items are under 45 and 8% of the items are over 64.
Find the mean and S.D of the distribution.

Sol: Let μ and σ be the mean and S.D of the normal distribution.

By data, $P(x < 45) = 0.31$ and $P(x > 64) = 0.08$

We have standard normal variate, $z = \frac{x-\mu}{\sigma}$

When $x = 45$, $z = \frac{45-\mu}{\sigma} = z_1$ (Say)

$x = 64$, $z = \frac{64-\mu}{\sigma} = z_2$ (Say)

So we have,

$$P(z < z_1) = 0.31 \quad \text{and} \quad P(z > z_2) = 0.08$$

$$\text{i.e.,} \quad 0.5 + \phi(z_1) = 0.31 \quad \text{and} \quad 0.5 - \phi(z_2) = 0.08$$

$$\phi(z_1) = 0.31 - 0.5 \quad \text{and} \quad \phi(z_2) = 0.5 - 0.08$$

$$\phi(z_1) = -0.19 \quad \text{and} \quad \phi(z_2) = 0.42$$

Referring to the normal probability tables, we have

$$0.1915 (\cong 0.19) = \phi(0.5) \quad \text{and} \quad 0.4192 (\cong 0.42) = \phi(1.4)$$

$$\phi(z_1) = -\phi(0.5) \quad \text{and} \quad \phi(z_2) = \phi(1.4)$$

$$\Rightarrow \quad z_1 = -0.5 \quad \text{and} \quad z_2 = 1.4$$

$$\text{i.e.,} \quad \frac{45-\mu}{\sigma} = -0.5 \quad \text{and} \quad \frac{64-\mu}{\sigma} = 1.4$$

$$45 - \mu = -0.5\sigma \quad \text{and} \quad 64 - \mu = 1.4\sigma$$

$$\mu - 0.5\sigma = 45 \quad \text{and} \quad \mu + 1.4\sigma = 64$$

Solving above equations, we get

$$\mu = 50 \quad \text{and} \quad \sigma = 10$$

Thus, Mean=50 and S.D=10

6. The mean weight of 500 students during a medical examination was found to be 50 kgs and S.D weight 6 kgs. Assuming that the weights are normally distributed, find the number of student having weight i) between 40 and 50 kgs ii) more than 60 kgs. [$\phi(1.67) = 0.4525$]

Sol: Given $\mu = 50$, $\sigma = 6$

We have standard normal variate, $z = \frac{x-\mu}{\sigma} = \frac{x-50}{6}$

i) **To find $P(40 < x < 50)$**

$$\text{If } x = 40, z = \frac{40-50}{6} = -1.67 \quad ; \quad \text{If } x = 50, z = \frac{50-50}{6} = 0$$

Hence we need to find, **$P(-1.67 < z < 0)$**

$$P(-1.67 < z < 0) = P(0 < z < 1.67)$$

$$= \phi(1.67)$$

$$\mathbf{P(-1.67 < z < 0) = 0.4525}$$

$$\begin{aligned}\therefore \text{Number of students having weight between 40 and 50 kgs} &= 500 * 0.4525 \\ &= 226.25 \cong 226\end{aligned}$$

ii) To find $P(x > 60)$

$$\text{If } x = 60, z = \frac{60-50}{6} = 1.67$$

Hence we need to find, $P(z > 1.67)$

$$\begin{aligned}P(z > 1.67) &= P(z \geq 0) - P(0 < z < 1.67) \\ &= 0.5 - \phi(1.67) \\ &= 0.5 - 0.4525\end{aligned}$$

$$P(z > 1.67) = 0.0475$$

$$\begin{aligned}\therefore \text{Number of students having weight more than 60 kgs} &= 500 * 0.0475 \\ &= 23.75 \cong 24\end{aligned}$$

7. In an examination 7% of students score less than 35% marks and 89% of students score less than 60% of marks. Find the mean and S.D of the distribution if the marks are normally distributed. It is given that $P(1.2263)=0.39$ and $P(1.4757)=0.43$.

Sol: Let μ and σ be the mean and S.D of the normal distribution.

$$\text{By data, } P(x < 35) = 0.07 \quad \text{and} \quad P(x < 60) = 0.89$$

We have standard normal variate, $z = \frac{x-\mu}{\sigma}$

$$\text{When } x = 35, z = \frac{35-\mu}{\sigma} = z_1 \text{ (Say)}$$

$$x = 60, z = \frac{60-\mu}{\sigma} = z_2 \text{ (Say)}$$

So we have ,

$$P(z < z_1) = 0.07 \quad \text{and} \quad P(z < z_2) = 0.89$$

$$\text{i.e., } 0.5 + \phi(z_1) = 0.07 \quad \text{and} \quad 0.5 + \phi(z_2) = 0.89$$

$$\phi(z_1) = 0.07 - 0.5 \quad \text{and} \quad \phi(z_2) = 0.89 - 0.5$$

$$\phi(z_1) = -0.43 \quad \text{and} \quad \phi(z_2) = 0.39$$

Using the given data in the RHS of these, we have

$$\phi(z_1) = -\phi(1.4757) \quad \text{and} \quad \phi(z_2) = \phi(1.2263)$$

$$\Rightarrow \quad z_1 = -1.4757 \quad \text{and} \quad z_2 = 1.2263$$

$$\text{i.e., } \frac{35-\mu}{\sigma} = -1.4757 \quad \text{and} \quad \frac{60-\mu}{\sigma} = 1.2263$$

$$\begin{array}{ll}
35 - \mu = -1.4757\sigma & \text{and} & 60 - \mu = 1.2263\sigma \\
\mu - 1.4757\sigma = 35 & \text{and} & \mu + 1.2263\sigma = 60
\end{array}$$

Solving above equations, we get

$$\mu = 48.65 \quad \text{and} \quad \sigma = 9.25$$

Thus, Mean = 48.65 and S.D = 9.25

Sampling Theory

Random Sampling

A large collection individuals or attributes or numerical data can be understood as a *population* or *universe*.

A finite subset of the universe is called a *sample*. The number of individuals in a sample is called a *sample size*(n).

If $n \leq 30$ then the sample is said to be *small* otherwise it is a *large sample*.

The selection of an individual or item from the population in such a way that each has the same chance of being selected is called as *random sampling*.

Sampling where a member of the population may be selected more than once is called as *sampling with replacement*.

If a member cannot be chosen more than once is called as *sampling without replacement*.

Parameter and Statistics

The statistical constants of population is mean(μ), standard deviation (σ), correlation (ρ) are called the *parameter*.

The statistical constants drawn from the given population is mean($\bar{x}$), standard deviation (s), correlation (r) are called *statistic*.

Testing of Hypothesis

To reach the decisions about the populations and the basis of sample information, we make certain assumptions about the population involved. Such assumptions which may or may not be true are called *hypothesis*. **OR** Quantitative statement about the population. Explanation on the basis of limited evidences.

Null Hypothesis

The hypothesis formulated for the purpose of its rejection under the assumption that it is true is called the *Null Hypothesis* denoted by H_0 .

Any hypothesis which is complimentary to the null hypothesis is called *Alternative Hypothesis* denoted by H_1 .

Type-I and Type-II Errors

Type-I Error: Wrong decision to reject the null hypothesis when it is actually true.

Type-II Error: Wrong decision to accept the null hypothesis when it is actually false.

	Accepting the hypothesis	Rejecting the hypothesis
Hypothesis True	Correct decision	Wrong decision (Type-I Error)
Hypothesis False	Wrong decision (Type-II Error)	Correct decision

Significance level

The probability level, below which leads to the rejection of the hypothesis is known as the *significance level*. The probability levels are fixed at 0.05 or 0.01 being 5% or 1%. These are called *Significance level*.

Test of Significance

The process which enables us to decide about the acceptance or rejection of the hypothesis is called the *test of significance*.

Confidence Intervals

Let us suppose that we have a normal population with mean μ and S.D σ . If $\bar{x}$ is the sample

mean of a random sample size n the quantity z defined by $z = \frac{\bar{x}-\mu}{\left(\frac{\sigma}{\sqrt{n}}\right)}$ is called the *Standard normal variate*.

From the normal distribution table we find the 95% of the area lies between $z = -1.96$ and $z = +1.96$ i.e., 95% confidence interval lies between -1.96 and $+1.96$.

Further 5% level of significance is denoted by $z_{0.05}$. Mathematically we can write

$$-1.96 \leq \frac{\bar{x} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} \leq 1.96$$

$$-\frac{\sigma}{\sqrt{n}}(1.96) \leq \bar{x} - \mu \leq (1.96) \frac{\sigma}{\sqrt{n}}$$

$$\mu \leq \bar{x} + \frac{\sigma}{\sqrt{n}}(1.96) \quad \text{and} \quad \bar{x} - \frac{\sigma}{\sqrt{n}}(1.96) \leq \mu$$

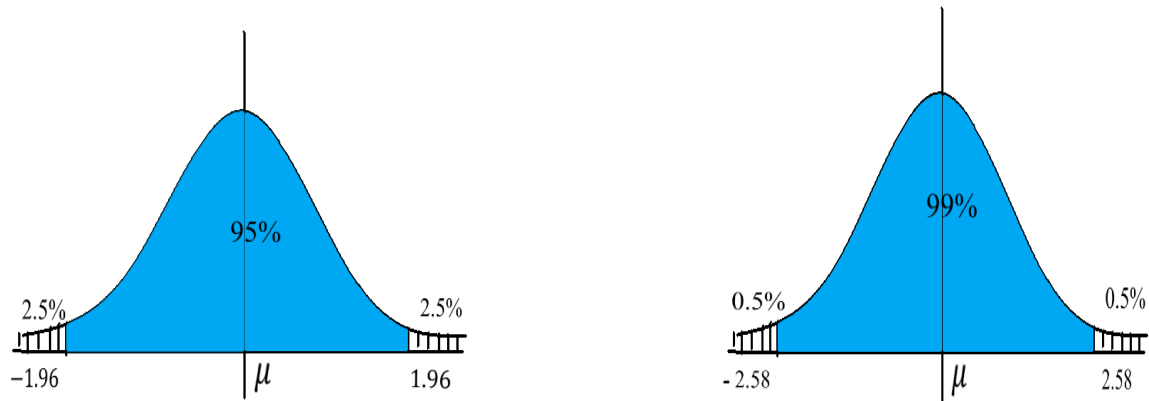
Also we can write combining the two results in the form,

$$\bar{x} - 1.96 \left(\frac{\sigma}{\sqrt{n}} \right) \leq \mu \leq \bar{x} + 1.96 \left(\frac{\sigma}{\sqrt{n}} \right) \quad \text{--- (1)}$$

Similarly, from the table of normal areas 99% of the area lies between -2.58 and 2.58. This is equivalent to the form,

$$\bar{x} - 2.58 \left(\frac{\sigma}{\sqrt{n}} \right) \leq \mu \leq \bar{x} + 2.58 \left(\frac{\sigma}{\sqrt{n}} \right) \quad \text{--- (2)}$$

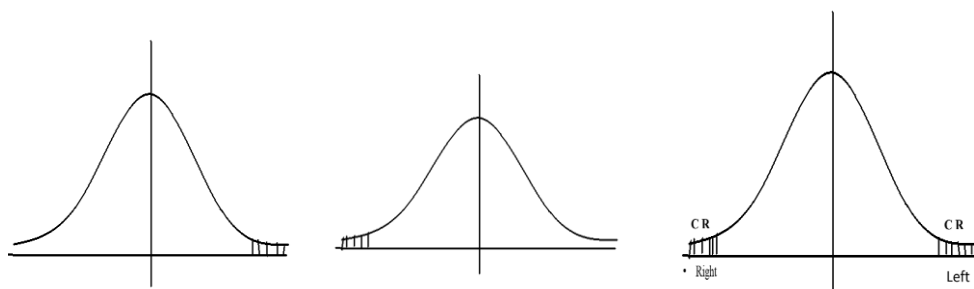
We can say that (1) is the 95% confidence interval and (2) is the 99% confidence interval.



One tailed and two tailed tests

In our test of acceptance or non acceptance of a hypothesis, we concentrated on the value of z on both sides of the mean. This can be categorically stated that, the focus of the attention lies in the two “tails” of the distribution and hence such a test is called a *two tailed test*.

Sometimes we will be interested in the extreme values to only one side of the mean in which the region of significance will be a region to one side of the distribution. Obviously the area of such a region will be equal to the level of significance itself. Such a tail is called a *one tailed test*.



One tailed test

Two tailed test

The following table will be useful for working problems

Test	Critical Value of z	
	5% level	1% level

One-tailed test	-1.645 or 1.645	-2.33 or 2.33
Two-tailed test	-1.96 or 1.96	-2.58 or 2.58

Tests of significance of Proportions

1. Let x be the observed number of successes in a sample size of n and $\mu = np$ be the expected number of successes let the associated standard normal variate Z be defined by $z = \frac{x-\mu}{\sigma} = \frac{x-np}{\sqrt{npq}}$

2. If $|Z| > 2.58$, we conclude that the differences is highly significant and reject the hypothesis .Probable limits $= p \pm 2.58 \sqrt{\frac{pq}{n}}$

3. If $Z < 1.96$, difference between the observed and expected number of successes is not significant.

4. If $Z > 1.96$, difference is significant at 5% level of significance.

Test of hypothesis for means

Let μ_1 and μ_2 be the mean of two populations. Let $(\bar{x}_1, \sigma_1)$; $(\bar{x}_2, \sigma_2)$ be the mean and standard deviation of two large samples of size n_1 and n_2 respectively. To test the null hypothesis H_0 that there is no difference between the population means. i.e., $H_0: \mu_1 = \mu_2$

The static for this is given by $Z = \frac{(\bar{x}_1 - \bar{x}_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$

Test of significance of proportions

1.A coin is tossed 1000 times and head turns up 540 times. Decide on the hypothesis that the coin is unbiased.

Sol: Let us suppose that the coin is unbiased

p = Probability of getting a head in one toss $= \frac{1}{2}$

Since $p + q = 1$, $q = \frac{1}{2}$

Expected number of heads in 1000 tosses $= np$

$$= 1000 \times \frac{1}{2} = 500$$

Actual Number of heads $= 540 = x$

Now $x - np = 540 - 500 = 40$

Consider $z = \frac{x-\mu}{\sqrt{npq}} = \frac{x-np}{\sqrt{npq}} = \frac{540-500}{\sqrt{1000 \times 0.5 \times 0.5}}$

$$z = 2.53 < 2.58$$

Thus, we can say that the coin is unbiased.

2. A sample of 900 days was taken in a coastal town and it was found that on 100 days the weather was very hot. Obtain the probable limits of the percentage of very hot weather.

Sol: Probability of very hot weather, $p = \frac{100}{900} = \frac{1}{9}$

$$\therefore q = \frac{8}{9}, n = 900$$

$$\begin{aligned} \text{Probable limits} &= p \pm (2.58) \sqrt{\frac{pq}{n}} \\ &= \frac{1}{9} \pm (2.58) \sqrt{\frac{1}{9} \cdot \frac{8}{9} \cdot \frac{1}{900}} \\ &= 0.111 \pm 0.027 \\ &= 0.084 \text{ and } 0.138 \end{aligned}$$

Probability limits of very hot weather is 8.4% to 13.8%

3. In a sample of 500 men it was found that 60% of them had over weight. What can we infer about the proportion of people having over weight in the population?

Sol: Probability of persons having over weight is, $p = \frac{60}{100} = 0.6$

$$\therefore q = 1 - p = 0.4, n = 500$$

$$\begin{aligned} \text{Probable limits} &= p \pm (2.58) \sqrt{\frac{pq}{n}} \\ &= 0.6 \pm (2.58) \sqrt{\frac{(0.6)(0.4)}{500}} \\ &= 0.6 \pm 0.0565 \end{aligned}$$

Probable limits = 0.5435 and 0.6565

Thus, the probable limits of people having over weight is 54.35% and 65.65%

4. A survey was conducted in a slum locality of 2000 families by selecting a sample size 800. It was revealed that 180 families were illiterates. Find the probable limits of the illiterate families in the population of 2000.

Sol: Probability of illiterate families is, $p = \frac{180}{800} = 0.225$

$$\therefore q = 1 - p = 0.775, n = 800$$

$$\text{Probable limits} = p \pm (2.58) \sqrt{\frac{pq}{n}}$$

$$= 0.225 \pm (2.58) \sqrt{\frac{(0.225)(0.775)}{800}}$$

$$= 0.225 \pm 0.038$$

Probable limits = 0. 187 and 0.263

The probable limits of the illiterate families in the population of 2000

$$= 0. 187 * 2000 \text{ and } 0.263 * 2000$$

$$= 374 \text{ and } 526$$

Thus, 374 to 526 are probably illiterate families.

5. In 324 throws of a six faced 'die', an odd number turned up 181 times. Is it reasonable to think that the 'die' is an unbiased one?

Sol: Probability of the turn up of an odd number , $p = 3/6 = \frac{1}{2}$

Since $p + q = 1$, $q = \frac{1}{2}$

Expected number of successes = np

$$= 324 \times \frac{1}{2} = 162$$

Actual Number of heads = 181 = x

$$\text{Now } x - np = 181 - 162 = 19$$

$$\text{Consider } z = \frac{x - \mu}{\sqrt{npq}} = \frac{x - np}{\sqrt{npq}} = \frac{19}{\sqrt{324 \times 0.5 \times 0.5}} = \frac{19}{9}$$

$$z = 2.11 < 2.58$$

Thus, we can say that the die is unbiased.

6. A sample of 100 days is taken from meteorological records of a certain district and 10 of them are found to be foggy. What are the probable limits of the percentage of foggy days in the district.

Sol: Proportion of foggy days in a sample of 100 days is , $p = \frac{10}{100} = 0.1$

$$\therefore q = 1 - p = 0.9 \quad , n = 100$$

$$\text{Probable limits} = p \pm (2.58) \sqrt{\frac{pq}{n}}$$

$$= 0.1 \pm (2.58) \sqrt{\frac{(0.1)(0.9)}{100}}$$

$$= 0.1 \pm 0.0774$$

Probable limits = 0. 0226 and 0.1774

Thus, the percentage of foggy days lies between 2.26% and 17.74%

7. A random sample of 500 apples was taken from a large consignment and 65 were found to be bad. Estimate the proportion of bad apples in the consignment as well as the standard error of the estimate. Also find the percentage of bad apples in the consignment.

Sol: Proportion of bad apples in the sample is given by , $p = \frac{65}{500} = 0.13$

$$\therefore q = 1 - p = 0.87 \quad , n = 500$$

$$\text{S.E proportion of bad apples} = \sqrt{\frac{pq}{n}} = \sqrt{\frac{(0.13)(0.87)}{500}} = 0.015$$

$$\begin{aligned}\text{Probable limits} &= p \pm (2.58) \sqrt{\frac{pq}{n}} \\ &= 0.13 \pm (2.58)(0.015) \\ &= 0.13 \pm 0.0387\end{aligned}$$

$$\begin{aligned}\text{Probable limits} &= 0.0913 \text{ and } 0.1687 \\ &= 9.13\% \text{ and } 16.87\%\end{aligned}$$

Thus, the required percentage of bad apples in the consignment lies between 9.13 and 16.87.

8. The mean and standard deviation of the maximum loads supported by 60 cables are 11.09 tonnes and 0.73 tonnes respectively. Find a) 95% b) 99% confidence limits for mean of the maximum loads of all cables produced by the company.

Sol: By data $\bar{x} = 11.09$, $\sigma = 0.73$, $n = 60$

a) 95% confidence limits for mean of the maximum loads are given by

$$\begin{aligned}\bar{x} \pm 1.96 \left(\frac{\sigma}{\sqrt{n}} \right) &= 11.09 \pm (1.96) \left(\frac{0.73}{\sqrt{60}} \right) \\ &= 11.09 \pm 0.18 \\ &= 10.91 \text{ and } 11.27\end{aligned}$$

Thus, 10.91 tonnes to 11.27 tonnes are the 95% confidence limits for the mean of the maximum loads.

b) 99% confidence limits for mean of the maximum loads are given by

$$\begin{aligned}\bar{x} \pm 2.58 \left(\frac{\sigma}{\sqrt{n}} \right) &= 11.09 \pm (2.58) \left(\frac{0.73}{\sqrt{60}} \right) \\ &= 11.09 \pm 0.24 \\ &= 10.85 \text{ and } 11.33\end{aligned}$$

Thus, 10.85 tonnes to 11.33 tonnes are the 95% confidence limits for the mean of the maximum loads.

9. An unbiased coin is thrown n times. It is desired that the relative frequency of the appearance of heads should lie between 0.49 and 0.51. Find the smallest value of n that will ensure this result with a) 95% confidence b) 99% confidence

Sol: Probability of getting a head, $p = \frac{1}{2}$

$$q = \frac{1}{2}$$

$$\text{S.E proportion of heads} = \sqrt{\frac{pq}{n}}$$

$$\text{S.E proportion of heads} = \sqrt{\frac{\frac{1}{2} \cdot \frac{1}{2}}{n}} = \frac{1}{2\sqrt{n}}$$

a) Probable limits for 95% confidence level is given by : $p \pm (1.96)\sqrt{\frac{pq}{n}}$, which should lie

between 0.51 and 0.49

$$p \pm (1.96)\sqrt{\frac{pq}{n}} = 0.51 \text{ or } 0.49$$

$$0.5 \pm (1.96)\frac{1}{2\sqrt{n}} = 0.51 \text{ or } 0.49$$

$$0.5 \pm \frac{1.96}{2\sqrt{n}} = 0.51 \text{ or } 0.49$$

$$0.5 + \frac{1.96}{2\sqrt{n}} = 0.51$$

$$0.5 - \frac{1.96}{2\sqrt{n}} = 0.49$$

$$\frac{1.96}{2\sqrt{n}} = 0.51 - 0.5$$

$$\frac{1.96}{2\sqrt{n}} = 0.5 - 0.49$$

$$\frac{1.96}{2\sqrt{n}} = 0.01$$

$$\frac{1.96}{2\sqrt{n}} = 0.01$$

$$\sqrt{n} = \frac{1.96}{0.02}$$

$$\sqrt{n} = 98$$

Thus, **n = 9604**

b) Taking the confidence coefficient equal to 1.645 for 90% confidence level we have as before

$$\frac{1.645}{2\sqrt{n}} = 0.01$$

$$\sqrt{n} = \frac{1.645}{0.02}$$

$$\sqrt{n} = 82.25$$

Thus, **n = 6765**

Test of significance of a sample mean

1. A manufacturer claimed that atleast 95% of the equipment which he supplied to a factory conformed to specifications. An examination of a sample of 200 pieces of equipment revealed that 18 of them were faulty. Test his claim at a significance level of 1% and 5%.

Sol: Let p be the probability of success which being the probability of the equipment supplied to the factory conformal to the specifications.

$\therefore$ By data $p = 0.95$ and hence $q = 0.05$

$H_0: p = 0.95$ and the claim is correct.

$H_1: p < 0.95$ and the claim is false.

We choose the one tailed test to determine whether the supply is conformal to the specification.

$$\mu = np = 200 * 0.95 = 190$$

$$\sigma = \sqrt{npq} = \sqrt{200 * 0.95 * 0.05} = 3.082$$

Expected number of equipment's according to the specification =190

Actual number =182 , since 18 out of 200 were faulty.

$$\text{Now, } Z = \frac{x - np}{\sigma}$$

$$Z = \frac{190 - 182}{3.082}$$

$$\mathbf{Z = 2.6}$$

The value of Z is greater than the critical value 1.645 at 5% level and 2.33 at 1% level of significance.

The claim of the manufacturer (null hypothesis that claim is correct) is rejected at 5% as well as 1% level of significance in accordance with the one tailed test.

2. It has been found from experience that the mean breaking strength of a particular brand of thread is 275.6gms with standard deviation of 39.7gms. Recently a sample of 36 pieces of thread showed a mean breaking strength of 253.2gms. Can one conclude at a significance level of a) 0.05 b) 0.01 that the thread has become inferior ?

Sol: We have to decide between the two hypothesis,

$H_0: \mu = 275.6 \text{ gms}$, mean breaking strength.

$H_1: \mu < 275.6 \text{ gms}$, inferior in breaking strength.

We choose the one tailed test.

Mean breaking strength of a sample of 36 pieces=253.2

$\therefore \text{Difference} = 275.6 - 253.2 = 22.4 ; n = 36$

$$\text{Now, } Z = \frac{\text{Difference}}{\left(\frac{\sigma}{\sqrt{n}}\right)}$$

$$Z = \frac{22.4}{\left(\frac{39.7}{\sqrt{36}}\right)}$$

$$Z = 3.38$$

The value of Z is greater than the critical value of $Z=1.645$ at 5% level and 2.33 at 1% level of significance.

Under the hypothesis H_1 that the thread has become inferior is accepted at both 0.05 and 0.01 levels in accordance with one tailed test.

Test of significance of difference between means

1. In an elementary school examination the mean grade of 32 boys was 72 with a standard deviation of 8, while the mean grade of 36 girls was 75 with a standard deviation of 6.

Test the hypothesis that the performance of girls is better than boys.

Sol: We have , $\bar{x}_B = 72$, $\sigma_B = 8$, $n_B = 32$ [Boys]

$$\bar{x}_G = 75 , \sigma_G = 6 , n_G = 36 \quad \text{[Girls]}$$

$$\text{Consider, } Z = \frac{\bar{x}_G - \bar{x}_B}{\sqrt{\frac{\sigma_G^2}{n_G} + \frac{\sigma_B^2}{n_B}}}$$

$$Z = \frac{75 - 72}{\sqrt{\frac{36}{36} + \frac{64}{32}}}$$

$$Z = \sqrt{3}$$

$$Z = 1.73$$

$$Z = 1.73 \begin{cases} > Z_{0.05} = 1.645 \text{ (one tailed test)} \\ < Z_{0.01} = 2.33 \text{ (one tailed test)} \end{cases}$$

The difference in the performance of girls and boys in the examination is significant at 5% level but not at 1% level.

2. A sample of 100 bulbs produced by a company A showed a mean life of 1190 hours and a standard deviation of 90 hours. Also a sample of 75 bulbs produced by a company B showed a mean life of 1230 hours and a standard deviation of 120 hours. Is there a difference between the mean life time of the bulbs produced by the two companies at
a) 5% level of significance b) 1% level of significance.

Sol: By data $\bar{x}_A = 1190$, $\sigma_A = 90$, $n_A = 100$ [Company A]

$$\bar{x}_B = 1230, \sigma_B = 120, n_B = 75 \quad [\text{Company B}]$$

$$\text{Consider, } Z = \frac{\bar{x}_B - \bar{x}_A}{\sqrt{\frac{\sigma_B^2}{n_B} + \frac{\sigma_A^2}{n_A}}}$$

$$Z = \frac{1230 - 1190}{\sqrt{\frac{(120)^2}{75} + \frac{(90)^2}{100}}}$$

$$Z = 2.42$$

$$Z = 2.42 \begin{cases} > Z_{0.05} = 1.96 \text{ (Two tailed test)} \\ < Z_{0.01} = 2.58 \text{ (Two tailed test)} \end{cases}$$

The null hypothesis that there is no difference between the mean life time of bulbs is rejected at 5% level but not at 1% level of significance.

The null hypothesis that there is no difference between the mean life time of bulbs is rejected at both levels of significance in a one tailed test as the respective critical values are 1.645 and 2.33.

3. The mean life of two large samples of 1000 and 2000 members are 168.75 cms and 170 cms respectively. Can this be regarded as drawn from the sample population of standard deviation 6.25 cms ?

Sol: By data $\bar{x}_1 = 168.75, n_1 = 1000$

$$\bar{x}_2 = 170, n_2 = 2000, \sigma = 6.25$$

$$\text{Consider, } Z = \frac{\bar{x}_2 - \bar{x}_1}{\sigma \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$Z = \frac{170 - 168.75}{6.25 \sqrt{\frac{1}{1000} + \frac{1}{2000}}}$$

$$Z = 5.16$$

$Z = 5.16$ is very much greater than $Z_{0.05} = 1.96$ and $Z_{0.01} = 2.58$. Thus we say that the difference between the sample means is significant and we conclude that the samples cannot be regarded as drawn from the same population.

4. A random sample for 1000 workers in company has mean wage of Rs.50 per day and S.D of Rs.15. Another sample of 1500 workers from another company has mean wage of Rs.45 per day and S.D of Rs.20. Does the mean rate of wages varies between the two companies? Find the 95% confidence limits for the difference of the wages of the population of the two companies?

Sol: By data $\bar{x}_1 = 50, \sigma_1 = 15, n_1 = 1000$ [Company 1]

$$\bar{x}_2 = 45, \sigma_2 = 20, n_2 = 1500 \quad [\text{Company 2}]$$

Consider, $Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$

$$Z = \frac{50 - 45}{\sqrt{\frac{15^2}{1000} + \frac{20^2}{1500}}}$$

$$Z = 7.1307$$

The value of $Z = 7.1307$ is greater than $Z_{0.05} = 1.96$ and $Z_{0.01} = 2.58$. Hence we can say that the difference between the mean wages is significant both at 5% and 1% level of significance.

Also 95% confidence limits for the difference of mean wages is given by

$$\begin{aligned} (\bar{x}_1 - \bar{x}_2) \pm 1.96 \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} &= 5 \pm 1.96(0.7012) \\ &= 5 \pm 1.374 \\ &= 3.626 \text{ and } 6.374 \end{aligned}$$

Thus we can say with 95% confidence that the difference of population mean of wages between the two companies lies between Rs.3.63 and Rs.6.37.

Student's t distribution/test

We need to test the hypothesis, whether the sample mean $\bar{x}$ differs significantly from the population mean or hypothetical value μ .

We compute, $t = \frac{\bar{x} - \mu}{s} \sqrt{n}$ and consider $|t|$.

Here, $\bar{x} = \frac{\sum x}{n}$ and $s^2 = \frac{1}{(n-1)} \sum_{i=1}^n (x_i - \bar{x})^2$

$v = n - 1$, denote the number of degrees of freedom 't'.

If $|t| > t_{0.05}$, the difference between $\bar{x}$ and μ is said to be significant at 5% level of significance.

If $|t| > t_{0.01}$, the difference between $\bar{x}$ and μ is said to be significant at 1% level of significance.

If $|t|$ is less than the table value at a certain level of significance, the data is said to be conformal/consistent with the hypothesis that μ is the mean of the population.

We have 95% confidence limits for μ is given by $\bar{x} \pm \frac{s}{\sqrt{n}} t_{0.05}$

Similarly 99% confidence limits for μ is given by $\bar{x} \pm \frac{s}{\sqrt{n}} t_{0.01}$

Note: 1. Confidence limits are also called **Fiducial limits**.

2. Test of significance for difference between sample means is

$$t = \frac{\bar{x} - \bar{y}}{s \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$\text{Where, } s^2 = \frac{1}{n_1 + n_2 - 2} \left\{ \sum_{i=1}^{n_1} (x_i - \bar{x})^2 + \sum_{j=1}^{n_2} (y_j - \bar{y})^2 \right\}$$

And degrees of freedom, $v = n_1 + n_2 - 2$

Problems

1. Find the student's t for the following variable values in a sample of eight:

-4, -2, -2, 0, 2, 2, 3, 3 taking the mean of the universe to be zero.

$$\text{Sol: Wkt } t = \frac{\bar{x} - \mu}{s} \sqrt{n}$$

By data $\mu = 0$ and we have $n = 8$

$$\bar{x} = \frac{\sum x}{n}$$

$$\bar{x} = \frac{1}{8} (-4 - 2 - 2 + 0 + 2 + 2 + 3 + 3)$$

$$\bar{x} = \frac{1}{4} = 0.25$$

$$s^2 = \frac{1}{(n-1)} \sum_{i=1}^n (x_i - \bar{x})^2$$

$$s^2 = \frac{1}{(8-1)} \{ (-4.25)^2 + (-2.25)^2 + (-2.25)^2 + (-0.25)^2 + (1.75)^2 + (1.75)^2 + (2.75)^2 + (2.75)^2 \}$$

$$s^2 = \frac{1}{(7)} (49.5)$$

$$s^2 = 7.07$$

$$\therefore s = 2.66$$

$$\text{Thus, } t = \frac{0.25 - 0}{2.66} \sqrt{8}$$

$$t = 0.266$$

2. A machine is expected to produce nails of length 3 inches. A random sample of 25 nails gave an average length of 3.1 inch with standard deviation 0.3. Can it be said that the machine is producing nails as per specification ? ($t_{0.05}$ for 24 d.f is 2.064)

Sol: By data, we have $\mu = 3$, $\bar{x} = 3.1$, $n = 25$, $s = 0.3$

$$t = \frac{\bar{x} - \mu}{s} \sqrt{n}$$

$$t = \frac{3.1 - 3}{0.3} \sqrt{25}$$

$$t = 1.67 < 2.064$$

Thus, the hypothesis that the machine is producing nails as per specification is accepted at 5% level of significance.

3. Ten individuals are chosen at random from a population and their heights in inches are found to be 63, 63, 66, 67, 68, 69, 70, 70, 71, 71. Test the hypothesis that the mean height of the universe is 66 inches. ($t_{0.05} = 2.262$ for 9 d.f.).

Sol: We have $\mu = 66$, $n = 10$

$$\bar{x} = \frac{\sum x}{n}$$

$$\bar{x} = \frac{678}{10}$$

$$\bar{x} = 67.8$$

$$s^2 = \frac{1}{(n-1)} \sum_{i=1}^n (x_i - \bar{x})^2$$

$$s^2 = \frac{1}{9} [(63 - 67.8)^2 + (63 - 67.8)^2 + \dots + (71 - 67.8)^2]$$

$$s^2 = 9.067$$

$$s = 3.011$$

$$\text{Wkt } t = \frac{\bar{x} - \mu}{s} \sqrt{n}$$

$$t = \frac{(67.8 - 66)}{3.011} \sqrt{10}$$

$$t = 1.89 < 2.262$$

Thus the hypothesis is accepted at 5% level of significance.

4. A sample of 10 measurements of the diameter of a sphere gave a mean of 12 cm and a standard deviation 0.15 cm. Find the 95% confidence limits for the actual diameter.

Sol: By data, $n = 10$, $\bar{x} = 12$, $s = 0.15$

Also $t_{0.05} = 2.262$ for 9 d.f

Confidence limits for the actual diameter is given by $\bar{x} \pm \left[\frac{s}{\sqrt{n}} \right] t_{0.05}$

$$\bar{x} \pm \left[\frac{s}{\sqrt{n}} \right] t_{0.05} = 12 + \left[\frac{0.15}{\sqrt{10}} \right] (2.262) = 12 \pm 0.1073$$

Thus 11.893 cm to 12.107 cm is the confidence limits for the actual diameter.

5. A certain stimulus administered to each of the 12 patients resulted in the following change in blood pressure 5, 2, 8, -1, 3, 0, 6, -2, 1, 5, 0, 4. Can it be concluded that the stimulus will increase the blood pressure ? ($t_{0.05}$ for 11 d. f is 2.201)

Sol: $\bar{x} = \frac{\sum x}{n}$

$$\bar{x} = \frac{31}{12}$$

$$\bar{x} = 2.5833$$

$$s^2 = \frac{1}{(n-1)} \sum_{i=1}^n (x_i - \bar{x})^2$$

$$s^2 = \frac{1}{11} [(5 - 2.58)^2 + (2 - 2.58)^2 + \dots + (4 - 2.58)^2]$$

$$s^2 = 9.538$$

$$s = 3.088$$

Wkt $t = \frac{\bar{x} - \mu}{s} \sqrt{n}$

Let us suppose that the stimulus administration is not accompanied with increase in blood pressure, we can take $\mu = 0$.

$$t = \frac{(2.5833 - 0)}{3.088} \sqrt{12}$$

$$t = 2.8979 > 2.201$$

Hence the hypothesis is rejected at 5% level of significance. We conclude with 95% confidence that the stimulus in general is accompanied with increase in blood pressure.

6. A group of boys and girls were given an intelligence test. The mean score, S.D score and numbers in each group are as follows.

	Boys	Girls
Mean	74	70
S D	8	10
n	12	10

Is the difference between the means of the two groups significant at 5% level of significance ($t_{0.05}$ for 20 d. f is 2.086).

Sol: By data , $\bar{x} = 74$, $s_1 = 8$, $n_1 = 12$ [Boys]

$\bar{y} = 70$, $s_2 = 10$, $n_2 = 10$ [Girls]

We have $t = \frac{\bar{x} - \bar{y}}{s \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$

Where, $s^2 = \frac{1}{n_1 + n_2 - 2} \left\{ \sum_{i=1}^{n_1} (x_i - \bar{x})^2 + \sum_{j=1}^{n_2} (y_j - \bar{y})^2 \right\}$

Or $s^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$

Now $s^2 = \frac{12(64) + 10(100)}{12 + 10 - 2}$

$$s^2 = \frac{1768}{20}$$

$$s^2 = 88.4$$

$$s = 9.402$$

Hence $t = \frac{74 - 70}{(9.402) \sqrt{\frac{1}{12} + \frac{1}{10}}}$

$$t = 0.9936 < 2.086$$

Thus the hypothesis that there is a difference between the means of the two groups is accepted at 5% level of significance.

7. A sample of 11 rats from a central population had an average blood viscosity of 3.92 with standard deviation of 0.61. On the basis of this sample, establish 95% fiducial limits for μ , the mean blood viscosity of the central population. ($t_{0.05} = 2.228$ for 10 d. f).

Sol: By data $\bar{x} = 3.92$, $s = 0.61$, $n_1 = 11$

95% fiducial limits for $\mu = \bar{x} \pm \left[\frac{s}{\sqrt{n}} \right] t_{0.05}$

$$\bar{x} \pm \left[\frac{s}{\sqrt{n}} \right] t_{0.05} = 3.92 \pm \left[\frac{0.61}{\sqrt{11}} \right] (2.228)$$

$$= 3.92 \pm 0.41$$

$$= 3.51 \text{ and } 4.33$$

Thus 95% confidence limits for μ are 3.51 and 4.33.

8. Two types of batteries are tested for their length of life and the following results were obtained.

Battery A : $n_1 = 10$, $\bar{x}_1 = 500$ hrs, $s_1^2 = 100$

Battery B : $n_2 = 10$, $\bar{x}_2 = 500$ hrs, $s_1^2 = 121$

Compute student's t and test whether there is a significant difference in the two means.

Sol: By data Battery A : $n_1 = 10$, $\bar{x}_1 = 500$ hrs, $s_1^2 = 100$

Battery B : $n_2 = 10$, $\bar{x}_2 = 560$ hrs, $s_1^2 = 121$

$$\text{Now, } s^2 = \frac{n_1 s_1^2 + n_2 s_1^2}{n_1 + n_2 - 2}$$

$$s^2 = \frac{(10 \cdot 100) + (10 \cdot 121)}{10 + 10 - 2}$$

$$s^2 = 122.78$$

$$s = 11.0806$$

$$\text{We have } t = \frac{\bar{x}_2 - \bar{x}_1}{s \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$t = \frac{60}{(11.0806) \sqrt{\frac{1}{10} + \frac{1}{10}}}$$

$$t = 12.1081 > 2.101 \quad (t_{0.05} = 2.101 \text{ for } 18 \text{ d.f.})$$

This value of t is greater than the table value of t for 18 d.f at all levels of significance.

The null hypothesis that there is no significant difference in the two means rejected at all significance levels.

9. A group of 10 boys fed on a diet A and another group of 8 boys fed on a different diet B for a period of 6 months recorded the following increase in weight (lbs.)

Diet A : 5 6 8 1 12 4 3 9 6 10

Diet B : 2 3 6 8 10 1 2 8

Test whether diets A and B differ significantly regarding their effect on increase in weight.

Sol: Let the variable x correspond to the diet A and y to the diet B.

$$\bar{x} = \frac{\sum x}{n_1}$$

$$\bar{y} = \frac{\sum y}{n_2}$$

$$\bar{x} = \frac{64}{10}$$

$$\bar{y} = \frac{40}{8}$$

$$\bar{x} = 6.4$$

$$\bar{y} = 5$$

$$\sum_{i=1}^{n_1} (x_i - \bar{x})^2 = 102.4 \quad ; \quad \sum_{j=1}^{n_2} (y_j - \bar{y})^2 = 82$$

$$s^2 = \frac{1}{n_1 + n_2 - 2} \left\{ \sum_{i=1}^{n_1} (x_i - \bar{x})^2 + \sum_{j=1}^{n_2} (y_j - \bar{y})^2 \right\}$$

$$s^2 = \frac{1}{16} \{102.4 + 82\}$$

$$s^2 = \frac{184.4}{16} = 11.525$$

$$s = 3.395$$

$$\text{We have } t = \frac{\bar{x} - \bar{y}}{s \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$t = \frac{1.4}{(3.395) \sqrt{\frac{1}{10} + \frac{1}{8}}}$$

$$t = 0.869 < 2.12 \text{ for 16 d.f}$$

Thus we conclude that the two diets do not differ significantly regarding their effect on increase in weight.

10. Two horses A and B were tested according to the time (in seconds) to run a particular race with the following results.

Horse A : 28 30 32 33 33 29 34

Horse B : 29 30 30 24 27 29

Test whether you can discriminate between the two horses.

Sol: Let the variables x and y respectively correspond to horse A and horse B.

$$\bar{x} = \frac{\sum x}{n_1} = \frac{219}{7} = 31.3$$

$$\bar{y} = \frac{\sum y}{n_2} = \frac{169}{6} = 28.2$$

$$\sum_{i=1}^{n_1} (x_i - \bar{x})^2 = 31.43$$

$$\sum_{j=1}^{n_2} (y_j - \bar{y})^2 = 26.84$$

$$s^2 = \frac{1}{n_1 + n_2 - 2} \left\{ \sum_{i=1}^{n_1} (x_i - \bar{x})^2 + \sum_{j=1}^{n_2} (y_j - \bar{y})^2 \right\}$$

$$s^2 = \frac{1}{11} (31.43 + 26.84)$$

$$s^2 = 5.2973$$

$$\therefore s = 2.3016$$

$$\text{We have } t = \frac{\bar{x} - \bar{y}}{s \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$t = \frac{31.3 - 28.2}{(2.3016) \sqrt{\frac{1}{7} + \frac{1}{6}}}$$

$$t = 2.42 = \begin{cases} > t_{0.05} = 2.2 \\ < t_{0.02} = 2.72 \end{cases}$$

The discrimination between the horses is significant at 5% level but not at 2% level of significance.

Chi-Square distribution

Chi-square distribution provides a measure of correspondence between the theoretical frequencies and observed frequencies.

If $O_i (i = 1, 2, \dots, n)$ and $E_i (i = 1, 2, \dots, n)$ respectively denotes a set of observed and estimated frequencies, the quantity **chi-square** denoted by χ^2 is defined as follows.

$$\chi^2 = \sum_{i=1}^n \frac{(O_i - E_i)^2}{E_i} ; \text{ degrees of freedom} = n - 1$$

Note: If the expected frequencies are less than 10, we group them suitably for computing the value of chi. Square.

Chi-square test as a test of goodness of fit

It is possible to test the hypothesis about the association of two attributes. We already discussed the fitting of Binomial distribution, Normal distribution, Poisson distribution to a given data. It is easily possible to find the theoretical frequencies from the distribution of fit.

Chi-square test helps us to test the goodness of fit of these distributions.

If the calculated value of χ^2 is less than the value of χ^2 at a specified level of significance, the hypothesis is accepted. Otherwise the hypothesis is rejected.

Problems

1. A dice is thrown 264 times and the number appearing on the face (x) follows the following frequency distribution.

x	1	2	3	4	5	6
f	40	32	28	58	54	60

Calculate the value of χ^2 .

Sol: The frequencies in the given data are the observed frequencies. Assuming that the dice is unbiased, the expected number of frequencies for the numbers 1, 2, 3, 4, 5, 6 to appear on the face is $\frac{264}{6} = 44$ each.

Now the data is as follows:

No. on the dice	1	2	3	4	5	6
Observed frequency (O_i)	40	32	28	58	54	60
Expected frequency (E_i)	44	44	44	44	44	44

$$\chi^2 = \sum_{i=1}^n \frac{(O_i - E_i)^2}{E_i}$$

$$\chi^2 = \frac{(40 - 44)^2}{44} + \frac{(32 - 44)^2}{44} + \dots + \frac{(60 - 44)^2}{44}$$

$$\chi^2 = \frac{1}{44} [16 + 144 + 256 + 196 + 100 + 256]$$

$$\chi^2 = \frac{968}{44}$$

Thus, $\chi^2 = 22$

2. Five dice were thrown 96 times and the numbers 1, 2 or 3 appearing on the face of the dice follows the frequency distribution as below.

No. of dice showing 1,2 or 3	5	4	3	2	1	0
Frequency	7	19	35	24	8	3

Test the hypothesis that the data follows a binomial distribution. ($\chi_{0.05}^2 = 11.07$ for 5 d. f)

Sol: The data gives the observed frequencies and we need to calculate the expected frequencies.

Probability of a single dice throwing 1, 2 or 3 is $p = \frac{3}{6} = \frac{1}{2}$

$$\therefore q = 1 - p = \frac{1}{2}$$

From the binomial distribution, $P(x) = {}^nC_x p^x q^{(n-x)}$

The theoretical frequencies of getting 5, 4, 3, 2, 1, 0 successes with 5 dice are respectively the successive terms of the binomial expansion.

They are respectively $96 * {}^5C_0 * \frac{1}{2^5}$, $96 * {}^5C_1 * \frac{1}{2^5}$, $96 * {}^5C_2 * \frac{1}{2^5}$, $96 * {}^5C_3 * \frac{1}{2^5}$, $96 * {}^5C_4 * \frac{1}{2^5}$, $96 * {}^5C_5 * \frac{1}{2^5}$ and they are 3,15,30,30,15,3.

We have the table of observed and expected frequencies.

O_i	7	19	35	24	8	3
E_i	3	15	30	30	15	3

$$\chi^2 = \sum_{i=1}^n \frac{(O_i - E_i)^2}{E_i}$$

$$\chi^2 = \frac{16}{3} + \frac{16}{15} + \frac{25}{30} + \frac{36}{30} + \frac{49}{15} + \frac{0}{3}$$

$$\chi^2 = 11.7 > \chi_{0.05}^2 = 11.07$$

Thus the hypothesis that the data follows a binomial distribution is rejected.

3. A sample analysis of examination results of 500 students was made. It was found that 220 students had failed, 170 had secured third class 90 had secured second class and 20 had secured first class. Do these figures support the general examination result which is in the ratio 4:3:2:1 for the respective categories ($\chi_{0.05}^2 = 7.81$ for 3 d.f).

Sol: Let us take the hypothesis that these figures support to the general result in the ratio

4:3:2:1

The expected frequencies in the respective category are

$$\frac{4}{10} * 500, \frac{3}{10} * 500, \frac{2}{10} * 500, \frac{1}{10} * 500 \text{ or } 200, 150, 100, 50$$

We have the following table.

O_i	220	170	90	20
E_i	200	150	100	50

$$\chi^2 = \sum_{i=1}^n \frac{(O_i - E_i)^2}{E_i}$$

$$\chi^2 = \frac{400}{200} + \frac{400}{150} + \frac{100}{100} + \frac{900}{50}$$

$$\chi^2 = 23.67 > \chi_{0.05}^2 = 7.81$$

Thus the hypothesis is rejected.

4. 4 coins are tossed 100 times and the following results were obtained. Fit a binomial distribution for the data and calculate the theoretical frequencies ($\chi_{0.05}^2 = 9.49$ for 4 d.f).

Number of heads	0	1	2	3	4
Frequency	5	29	36	25	5

Sol: Let x denote the number of heads and f the corresponding frequency. Since the data is in the form of a frequency distribution we shall first calculate the mean.

$$\text{Mean, } \mu = \frac{\sum fx}{\sum f} = \frac{(0)(5)+(1)(29)+(2)(36)+(3)(25)+(4)(5)}{5+29+36+25+5} = \frac{196}{100} = 1.96$$

From binomial distribution, $\mu = np$. Here $n=4$

$$\text{Hence } 4p = 1.96$$

$$p = 0.49$$

$$q = 0.51$$

From binomial distribution,

$$P(x) = {}^n C_x p^x q^{n-x}$$

$$P(x) = {}^4 C_x (0.49)^x (0.51)^{4-x}$$

Since 4 coins were tossed, expected(theoretical) frequencies are obtained from

$$F(x) = 100P(x)$$

$$F(x) = 100.4c_x (0.49)^x (0.51)^{4-x} \text{ where } x = 0,1,2,3,4.$$

$$F(0) = 100.4c_0 (0.49)^0 (0.51)^4 = 6.765 \cong 7$$

$$F(1) = 100.4c_1 (0.49)^1 (0.51)^3 = 25.999 \cong 26$$

$$F(2) = 100.4c_2 (0.49)^2 (0.51)^2 = 37.47 \cong 37$$

$$F(3) = 100.4c_3 (0.49)^3 (0.51)^1 = 24.0004 \cong 24$$

$$F(4) = 100.4c_4 (0.49)^4 (0.51)^0 = 5.765 \cong 6$$

Thus the required theoretical frequencies are **7,26,37,24,6**.

We have the following table

O_i	5	29	36	25	5
E_i	7	26	37	24	6

$$\chi^2 = \sum_{i=1}^n \frac{(O_i - E_i)^2}{E_i}$$

$$\chi^2 = \frac{4}{7} + \frac{9}{26} + \frac{1}{37} + \frac{1}{24} + \frac{1}{6}$$

$$\chi^2 = 1.15 < \chi_{0.05}^2 = 9.49$$

Thus the hypothesis that the fitness is good can be accepted.

5. Fit a poisson distribution for the following data and test the goodness of fit given that $\chi^2_{0.05} = 7.815$ for 3 d.f

x	0	1	2	3	4
f	122	60	15	2	1

Sol: Let x denote the number of heads and f the corresponding frequency. Since the data is in the form of a frequency distribution we shall first calculate the mean.

$$\text{Mean, } \mu = \frac{\sum fx}{\sum f} = \frac{(0)(122)+(1)(60)+(2)(15)+(3)(2)+(4)(1)}{122+60+15+2+1} = \frac{100}{200} = 0.2$$

From poisson distribution, $\mu = m$.

$$\therefore m = 0.2$$

$$\text{Wkt, } P(x) = \frac{m^x e^{-m}}{x!}$$

$$\text{Let } F(x) = 200 * P(x)$$

$$F(x) = 200 * \frac{m^x e^{-m}}{x!}$$

$$F(x) = 200 * \frac{(0.5)^x e^{-(0.5)}}{x!} \quad \text{But } e^{-(0.5)} = 0.6065$$

$$F(x) = 121.3 \frac{(0.5)^x}{x!}$$

$$F(0) = 121, F(1) = 61, F(2) = 15, F(3) = 3, F(4) = 0$$

Thus the required theoretical frequencies are 121,61,15,3,0.

Since the last of the expected frequency is 0 we shall club it with the previous one.

We have the following table

O_i	122	60	15	2+1=3
E_i	121	61	15	3+0=3

$$\chi^2 = \sum_{i=1}^n \frac{(O_i - E_i)^2}{E_i}$$

$$\chi^2 = \frac{1}{121} + \frac{1}{61} + 0 + 0$$

$$\chi^2 = 0.025 < \chi^2_{0.05} = 7.815$$

Thus the hypothesis that the fitness is good can be accepted.

6. The number of accidents per day (x) as recorded in a textile industry over a period of 400

days is given below. Test the goodness of fit in respect of Poisson distribution of fit to the given data ($\chi^2_{0.05} = 9.49$ for 4 d. f.).

x	0	1	2	3	4	5
f	173	168	37	18	3	1

Sol: Let x denote the number of heads and f the corresponding frequency. Since the data is in the form of a frequency distribution we shall first calculate the mean.

$$\text{Mean, } \mu = \frac{\sum fx}{\sum f} = \frac{(0)(173) + (1)(168) + (2)(37) + (3)(18) + (4)(3) + (5)(1)}{173 + 168 + 37 + 18 + 3 + 1} = \frac{313}{400} = 0.7825$$

From poisson distribution, $\mu = m$.

$$\therefore m = 0.7825$$

$$\text{Wkt, } P(x) = \frac{m^x e^{-m}}{x!}$$

$$\text{Let } F(x) = 400 * P(x)$$

$$F(x) = 400 * \frac{m^x e^{-m}}{x!}$$

$$F(x) = 400 * \frac{(0.7825)^x e^{-(0.7825)}}{x!} \quad \text{But } e^{-(0.7825)} = 0.4573$$

$$F(x) = 182.92 \frac{(0.7825)^x}{x!}$$

$$F(0) = 183, F(1) = 143, F(2) = 56, F(3) = 15, F(4) = 3, F(5) = 0$$

Thus the required theoretical frequencies are 183, 143, 56, 15, 3, 0.

Since the last of the expected frequency is 0 we shall club it with the previous one.

We have the following table

O_i	173	168	37	18	3+1=4
E_i	183	143	56	15	3+0=3

$$\chi^2 = \sum_{i=1}^n \frac{(O_i - E_i)^2}{E_i}$$

$$\chi^2 = \frac{100}{183} + \frac{625}{143} + \frac{361}{56} + \frac{9}{15} + \frac{1}{3}$$

$$\chi^2 = 12.297 > \chi^2_{0.05} = 9.49$$

Thus the hypothesis that the fitness is good can be rejected.

Module-5

Linear Programming Problems

Linear programming is a decision making technique under the given constraints on the condition that the relationship among the variables involved is linear. A general relationship among the variables involved is called **objective function**. The variables involved are called **decision variables**.

Canonical and Standard forms of LPP

After the formulation of LPP, the next step is to obtain its solution. But before any method is used to find its solution, the problem must be presented in a suitable form. Two forms are dealt with here, the canonical form and the standard form.

The canonical form : The general linear programming problem can always be expressed in the following form :

$$\text{Maximize } Z = c_1 x_1 + c_2 x_2 + c_3 x_3 + \dots + c_n x_n$$

subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$$

.....

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$$

and the non-negativity restrictions

$$x_1, x_2, x_3, \dots, x_n \geq 0.$$

This form of LPP is called **the canonical form of the LPP**.

In matrix notation the canonical form of LPP can be expressed as :

$$\text{Maximize } Z = CX \text{ (objective function)}$$

$$\text{Subject to } AX \leq b \text{ (constraints)}$$

$$\text{and } X \geq 0 \text{ (non-negativity restrictions)}$$

$$\text{where } C = (c_1 \ c_2 \ \dots c_n),$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

The Standard Form :

The general linear programming problem in the form

Maximize or Minimize

$$Z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

Subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

.....

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

and $x_1, x_2, \dots, x_n \geq 0$ is known as **standard form**.

In matrix notation the standard form of LPP can be expressed as :

Maximize or Minimize $Z = CX$ (objective function)

Subject to constraints $AX = b$ and $X \geq 0$

Where, $C = (c_1, c_2, \dots, c_n)$,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

The set of real values $X = (x_1, x_2, \dots, x_n)$ which satisfies the constraint $AX \leq (\geq) B$ is called a **basic solution**.

A set of real values x_i which satisfies the constraints and also satisfy non negativity constraints $x_i \geq 0$ is called **feasible solution**.

A set of real values x_i which satisfies the constraints along with non negativity restrictions and optimizes the objective function is called **optimal solution**.

Simplex Method

It is an iterative method to solve a Linear programming Problem.

Slack Variable: It is a variable introduced to convert " $\leq$ " (less than or equal to) to " $=$ " (equal to).

Surplus Variable: It is a variable introduced to convert " $\geq$ " (greater than or equal to) to " $=$ " (equal to).

Problems

1. Use Simplex method to maximize $z = x + 1.5y$ subject to the constraints $x + 2y \leq 160$ and $3x + 2y \leq 240$ where $x \geq 0$ and $y \geq 0$.

Sol: Introduce two slack variables s_1 and s_2 to convert linear inequality into equalities

$$x + 2y + 1.s_1 + 0.s_2 = 160 \quad \text{--- (1)}$$

$$3x + 2y + 0.s_1 + 1.s_2 = 240 \quad \text{--- (2)}$$

$x + 1.5y + 0.s_1 + 0.s_2 = z$ is the objective function.

The first two rows in the above table contains the various co-efficients of equations (1) and (2). The last row contains the co-efficients of the variables in the objective function with their signs reversed and are called as indicators.

NZV	x	y	s_1	s_2	Qty	Ratio
s_1	1	2	1	0	160	$\frac{160}{2} = 80$
s_2	3	2	0	1	240	$\frac{240}{2} = 120$
$\Delta(\text{Indicator})$	-1	-1.5	0	0	0	-

Pivotal Row

↑
*Pivotal
column*

- i) Identify the least negative indicator, it is -1.5. Therefore, the second column is known as the key or the Pivotal Column.
- ii) In each row examine the ratio of the entry in the quantity (Qty), column corresponding entry in the pivotal column.
- iii) Find the least non-negative ratio (ignore negative ratios and -0 also).
 - The row in which the least ratio is present is known as the pivotal row.

- The entry which is common to both pivotal column and pivotal row is termed as the pivotal element. The variable corresponding to the pivotal row is known as outgoing variable or departing variable and the variable corresponding to pivotal column is known as incoming variable or arriving variable.
- Make the pivotal element unity (one) (Gauss elimination method) by dividing the pivotal row by the pivotal element.

$$R'_1 = \frac{R_1}{2} \quad \text{Outgoing Variable - } S_1 \text{ Incoming Variable - } y$$

NZV	x	y	s_1	s_2	Ratio
s_1	$\frac{1}{2}$	1	$\frac{1}{2}$	0	80
s_2	3	2	0	1	240
$\Delta(\text{Indicator})$	-1	-1.5	0	0	-

Pivotal Row

↑
*Pivotal
column*

$$R'_2 = R_2 - 2R_1$$

$$R'_3 = R_3 + 1.5R_1$$

NZV	x	y	s_1	s_2	Qty	Ratio
y	$\frac{1}{2}$	1	$\frac{1}{2}$	0	80	$\frac{80}{\frac{1}{2}} = 160$
s_2	2	0	-1	1	80	$\frac{80}{2} = 40$
$\Delta(\text{Indicator})$	-0.25	0	0.75	0	120	-

Pivotal Row

↑
*Pivotal
column*

Again, continue from step (i) as shown previously.

$$\text{Outgoing Variable - } S_2 \text{ Incoming Variable - } x ; R'_2 = \frac{R_2}{2}$$

NZV	x	y	s_1	s_2	Ratio
y	$\frac{1}{2}$	1	$\frac{1}{2}$	0	80
x	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	40
$\Delta(\text{Indicator})$	-1	-1.5	0	0	120

$$R'_1 = R_1 - \frac{1}{2}R_2$$

$$R'_3 = R_3 + 0.25R_1$$

NZV	x	y	s_1	s_2	Ratio
y	0	1	$\frac{3}{4}$	$-\frac{1}{4}$	60
x	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	40
$\Delta(\text{Indicator})$	0	0	0.625	0.125	130

As all the indicators in the above Simplex table are non-negative, the iterative procedure is complete.

$$z_{\max} = 130 \text{ at } y = 60 ; x = 40$$

2. Use Simplex method to maximize $z = 3x + 4y$ subject to the constraints $2x + y \leq 40$ and $2x + 5y \leq 180$ where $x \geq 0$ and $y \geq 0$.

Sol: Introduce two slack variables s_1 and s_2 to convert linear inequality into equalities

$$2x + y + 1.s_1 + 0.s_2 = 40 \quad \text{--- (1)}$$

$$2x + 5y + 0.s_1 + 1.s_2 = 180 \quad \text{--- (2)}$$

$3x + 4y + 0.s_1 + 0.s_2 = z$ is the objective function.

NZV	x	y	s_1	s_2	Qty	Ratio
s_1	2	1	1	0	40	$\frac{40}{1} = 40$
s_2	2	5	0	1	180	$\frac{180}{5} = 36$
$\Delta(\text{Indicator})$	-3	-4	0	0	0	—

↑
*Pivotal
column*

Outgoing Variable - s_2 Incoming Variable - y ; $R'_2 = \frac{R_2}{5}$

NZV	x	y	s_1	s_2	Qty
s_1	2	1	1	0	40
y	$\frac{2}{5}$	1	0	$\frac{1}{5}$	36
$\Delta(\text{Indicator})$	-3	-4	0	0	0

$$R'_1 = R_1 - R_2$$

$$R'_3 = R_3 + 4R_2$$

NZV	x	y	s_1	s_2	Ratio
s_1	$\frac{8}{5}$	0	1	$-\frac{1}{5}$	4
y	$\frac{2}{5}$	1	0	$\frac{1}{5}$	36
$\Delta(\text{Indicator})$	$-\frac{7}{5}$	0	0	$\frac{4}{5}$	144

2nd iteration

NZV	x	y	s_1	s_2	Qty	Ratio	
s_1	$\frac{8}{5}$	0	1	$-\frac{1}{5}$	4	$\frac{4}{\frac{8}{5}} = \frac{5}{2}$	<i>Pivotal Row</i>
y	$\frac{2}{5}$	1	0	$\frac{1}{5}$	36	$\frac{36}{\frac{2}{5}} = 90$	
$\Delta(\text{Indicator})$	$-\frac{7}{5}$	0	0	$\frac{4}{5}$	144	—	

↑
Pivotal column

Outgoing Variable - s_1 Incoming Variable - x ; $R'_1 = \frac{5}{8}R_1$

NZV	x	y	s_1	s_2	Qty
x	1	0	$\frac{5}{8}$	$-\frac{1}{8}$	$\frac{5}{2}$
y	$\frac{2}{5}$	1	0	$\frac{1}{5}$	36
$\Delta(\text{Indicator})$	$-\frac{7}{5}$	0	0	$\frac{4}{5}$	144

$$R'_2 = R_2 - \frac{2}{5}R_1$$

$$R'_3 = R_3 + \frac{7}{5}R_1$$

NZV	x	y	s_1	s_2	Qty
x	1	0	$\frac{5}{8}$	$-\frac{1}{8}$	$\frac{5}{2}$
y	0	1	$-\frac{1}{4}$	$\frac{1}{4}$	35
$\Delta(\text{Indicator})$	0	0	$\frac{7}{8}$	$\frac{5}{8}$	147.5

As all the indicators in the above simplex table are non-negative, the iterative procedure is complete.

$$z_{\max} = 147.5 \text{ at } y = 35 ; x = \frac{5}{2}$$

3. Maximize $P = 6x + 10y + 2z$ subject to the constraints $2x + 4y + 3z \leq 40$, $x + y \leq 10$, $2y + z \leq 12$, $x, y, z \geq 0$ using simplex method.

Sol: $2x + 4y + 3z + 1.s_1 + 0.s_2 + 0.s_3 = 40$ --- (1)

$x + y + 0.z + 0.s_1 + 1.s_2 + 0.s_3 = 10$ --- (2)

$0.x + 2y + 1.z + 0.s_1 + 0.s_2 + 1.s_3 = 12$ --- (3)

$6x + 10y + 2z + 0.s_1 + 0.s_2 + 0.s_3 = P$ is the objective function.

NZV	x	y	z	s_1	s_2	s_3	Qty	Ratio
s_1	2	4	3	1	0	0	40	$\frac{40}{4} = 10$
s_2	1	1	0	0	1	0	10	$\frac{10}{1} = 10$
s_3	0	2	1	0	0	1	12	$\frac{12}{2} = 6$
$\Delta(\text{Indicator})$	-6	-10	-2	0	0	0	0	-

*Pivotal
Row*

↑
*Pivotal
column*

Outgoing Variable - s_3 Incoming Variable - y ; $R'_3 = \frac{R_3}{2}$

NZV	x	y	z	s_1	s_2	s_3	Qty
s_1	2	4	3	1	0	0	40
s_2	1	1	0	0	1	0	10
y	0	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	6
$\Delta(\text{Indicator})$	-6	-10	-2	0	0	0	0

$R'_1 = R_1 - 4R_3$

$R'_2 = R_2 - R_3$

$R'_4 = R_4 + 10R_3$

NZV	x	y	z	s_1	s_2	s_3	Qty
s_1	2	0	1	1	0	-2	16
s_2	1	0	$-\frac{1}{2}$	0	1	$-\frac{1}{2}$	4
y	0	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	6
$\Delta(\text{Indicator})$	-6	0	3	0	0	5	60

*Pivotal
Row*

↑
*Pivotal
column*

$\frac{16}{2} = 8$ $\frac{4}{1} = 4$ (4 is least, 1 is pivot)

Outgoing Variable - s_2 Incoming Variable - x ; $R'_2 = \frac{R_2}{1}$

NZV	x	y	z	s_1	s_2	s_3	Qty
s_1	2	0	1	1	0	-2	16
x	1	0	$-\frac{1}{2}$	0	1	$-\frac{1}{2}$	4
y	0	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	6
$\Delta(\text{Indicator})$	-6	0	3	0	0	5	60

$$R'_1 = R_1 - 2R_2$$

$$R'_4 = R_4 + 6R_2$$

NZV	x	y	z	s_1	s_2	s_3	Qty
s_1	0	0	2	0	-2	-1	8
x	1	0	$-\frac{1}{2}$	0	1	$-\frac{1}{2}$	4
y	0	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	6
$\Delta(\text{Indicator})$	0	0	0	0	6	2	84

As all the indicators in the above simplex table are non-negative, the iterative procedure is complete.

$P_{max} = 84$ at $x = 4$; $y = 6$; $z = 0$.

4. Maximize $P = 4x_1 - 2x_2 - x_3$ subject to the constraints $x_1 + x_2 + x_3 \leq 3$,

$2x_1 + 2x_2 + x_3 \leq 4$, $x_1 - x_2 \leq 0$ $x_1, x_2, x_3 \geq 0$ using simplex method.

Sol: $x_1 + x_2 + x_3 + 1.s_1 + 0.s_2 + 0.s_3 = 3$ --- (1)

$2x_1 + 2x_2 + x_3 + 0.s_1 + 1.s_2 + 0.s_3 = 4$ --- (2)

$x_1 - x_2 + 0.x_3 + 0.s_1 + 0.s_2 + 1.s_3 = 0$ --- (3)

$4x_1 - 2x_2 - x_3 + 0.s_1 + 0.s_2 + 0.s_3 = P$ is the objective function.

NZV	x_1	x_2	x_3	s_1	s_2	s_3	Qty	
s_1	1	1	1	1	0	0	3	
s_2	2	2	1	0	1	0	4	
s_3	1	-1	0	0	0	1	0	Pivotal Row
$\Delta(\text{Indicator})$	-4	2	1	0	0	0	0	

↑
Pivotal
column

$$\frac{3}{1} = 3 \quad \frac{4}{2} = 2 \quad \frac{0}{1} = 0 \quad (0 \text{ is least, } 1 \text{ is pivot})$$

Outgoing Variable - S_3 Incoming Variable - x_1 ; $R'_3 = \frac{R_3}{1}$

NZV	x_1	x_2	x_3	s_1	s_2	s_3	Qty
s_1	1	1	1	1	0	0	3
s_2	2	2	1	0	1	0	4
x_1	1	-1	0	0	0	1	0
$\Delta(\text{Indicator})$	-4	2	1	0	0	0	0

$$R'_1 = R_1 - R_3$$

$$R'_2 = R_2 - 2R_3$$

$$R'_4 = R_4 + 4R_3$$

NZV	x_1	x_2	x_3	s_1	s_2	s_3	Qty
s_1	0	2	1	1	0	-1	3
s_2	0	4	1	0	1	-2	4
x_1	1	-1	0	0	0	1	0
$\Delta(\text{Indicator})$	0	-2	1	0	0	4	0

*Pivotal
Row*

↑
*Pivotal
column*

$$\frac{3}{2} = 1.5 \quad \frac{4}{4} = 1 \quad \frac{0}{-1} = \text{neglected} \quad (1 \text{ is least, } 4 \text{ is pivot})$$

Outgoing Variable - S_2 Incoming Variable - x_2 ; $R'_2 = \frac{R_2}{4}$

NZV	x_1	x_2	x_3	s_1	s_2	s_3	Qty
s_1	0	2	1	1	0	-1	3
x_2	0	1	$\frac{1}{4}$	0	$\frac{1}{4}$	$-\frac{1}{2}$	1
x_1	1	-1	0	0	0	1	0
$\Delta(\text{Indicator})$	0	-2	1	0	0	4	0

$$R'_1 = R_1 - 2R_2$$

$$R'_3 = R_3 + R_2$$

$$R'_4 = R_4 + 2R_2$$

NZV	x_1	x_2	x_3	s_1	s_2	s_3	Qty
s_1	0	0	0.5	1	-0.5	0	1
x_2	0	1	$\frac{1}{4}$	0	$\frac{1}{4}$	$-\frac{1}{2}$	1
x_1	1	0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1
$\Delta(\text{Indicator})$	0	0	$\frac{3}{2}$	0	$\frac{1}{2}$	3	2

As all the indicators in the above simplex table are non-negative, the iterative procedure is complete.

$$P_{max} = 2 \text{ at } x_1 = 1; x_2 = 1; x_3 = 0.$$

5. Use Simplex method to minimize $P = x - 3y + 2z$ subject to the constraints

$$3x - y + 2z \leq 7 \quad -2x + 4y \leq 12 \quad \text{and} \quad -4x + 3y + 8z \leq 10 \quad \text{where } x, y, z \geq 0.$$

Sol: Minimization of P is equivalent to maximisation of $-P$.

$$P' = -P$$

$$\text{Let } P' = -x + 3y - 2z$$

$$3x - y + 2z + 1.s_1 + 0.s_2 + 0.s_3 = 7 \quad \text{--- (1)}$$

$$-2x + 4y + 0.z + 0.s_1 + 1.s_2 + 0.s_3 = 12 \quad \text{--- (2)}$$

$$-4x + 3y + 8z + 0.s_1 + 0.s_2 + 1.s_3 = 10 \quad \text{--- (3)}$$

$$-x + 3y - 2z + 0.s_1 + 0.s_2 + 0.s_3 = P' \quad \text{is the objective function.}$$

NZV	x	y	z	s_1	s_2	s_3	Qty
s_1	3	-1	2	1	0	0	7
s_2	-2	4	0	0	1	0	12
s_3	-4	3	8	0	0	1	10
$\Delta(\text{Indicator})$	1	-3	2	0	0	0	0

↑
*Pivotal
column*

$$\frac{7}{-1} = -7 \quad \frac{12}{4} = 3 \quad \frac{10}{3} = 3.333 \quad (3 \text{ is least, } 4 \text{ is pivot})$$

$$\text{Outgoing Variable - } s_2 \quad \text{Incoming Variable - } y; \quad R'_2 = \frac{R_2}{4}$$

NZV	x	y	z	s_1	s_2	s_3	Qty
s_1	3	-1	2	1	0	0	7
y	$-\frac{1}{2}$	1	0	0	$\frac{1}{4}$	0	3
s_3	-4	3	8	0	0	1	10
$\Delta(\text{Indicator})$	1	-3	2	0	0	0	0

$$R'_1 = R_1 + R_2$$

$$R'_3 = R_3 - 3R_2$$

$$R'_4 = R_4 + 3R_2$$

NZV	x	y	z	s_1	s_2	s_3	Qty
s_1	$\frac{5}{2}$	0	2	1	$\frac{1}{4}$	0	10
y	$-\frac{1}{2}$	1	0	0	$\frac{1}{4}$	0	3
s_3	$-\frac{5}{2}$	0	8	0	$-\frac{3}{4}$	1	1
$\Delta(Indicator)$	$-\frac{1}{2}$	0	2	0	$\frac{3}{4}$	0	9

*Pivotal
Row*

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*Pivotal
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$$\frac{10}{\frac{5}{2}} = 4 \quad \frac{3}{-\frac{1}{2}} = -6 \quad \frac{1}{-\frac{5}{2}} = -0.4 \quad (4 \text{ is least, } \frac{5}{2} \text{ is pivot})$$

Outgoing Variable - s_1 Incoming Variable - x ; $R'_1 = \frac{R_1}{\frac{5}{2}}$

NZV	x	y	z	s_1	s_2	s_3	Qty
x	1	0	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	0	4
y	$-\frac{1}{2}$	1	0	0	$\frac{1}{4}$	0	3
s_3	$-\frac{5}{2}$	0	8	0	$-\frac{3}{4}$	1	1
$\Delta(Indicator)$	$-\frac{1}{2}$	0	2	0	$\frac{3}{4}$	0	9

$$R'_2 = R_2 + \frac{1}{2}R_1 \quad R'_3 = R_3 + \frac{5}{2}R_1 \quad R'_4 = R_4 + \frac{1}{2}R_1$$

NZV	x	y	z	s_1	s_2	s_3	Qty
x	1	0	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	0	4
y	0	1	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	0	5
s_3	0	0	10	1	$-\frac{1}{2}$	1	11
$\Delta(Indicator)$	0	0	$\frac{12}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	0	11

As all the indicators in the above simplex table are non-negative, the iterative procedure is complete.

$P'_{max} = 11$ at $x = 4$; $y = 5$.

$P_{min} = -11$ at $x = 4$ and $y = 5$.

L.P.P. and Simplex Method

Syllabus:

General linear programming problems, canonical and standard forms of L.P.P. Basic solution, Basic feasible solution, Optimal solution.

Simplex Method-Problems, Artificial variables, Big-M method, Two-Phase Method-Problems.

General Linear Programming Problem

Basic terminology:

Linear equation: Straight line relationship among variables under considerations.

Programming: Process of determining a particular course of action.

Decision variables: The variables whose values determine the solution of the problem.

Optimization: Maximization or minimization.

Objective function: A function of decision variables.

Constraints: Restrictions expressed as a set of linear equations or inequalities.

Definition:

The method of optimizing a linear function of variables (Objective function) subject to a set of linear equations and/or inequalities (constraints) is called linear programming.

Requirements:

- ❖ The relationship among the decision variables should be linear.
- ❖ There must be a well-defined objective function which may be either to maximize or minimize.
- ❖ There must be limitations on resources (Constraints) like production, capacity, time, man power, machine, etc. which are to be allocated among various computing activities.
- ❖ There must be alternative courses of action to choose from men, machines, materials, markets, etc.
- ❖ All decision variables must be non-negative.
- ❖ Both the objective function and constraints must be expressed in terms of linear equations or inequalities.

General form of an L.P.P		Parts of an L.P.P
Max (or) min $z = c_1x_1 + c_2x_2 + c_3x_3$	}	<i>Objective function</i> (1)
Subject to $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \leq (or) \geq (or) = d_1$ $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \leq (or) \geq (or) = d_2$ $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \leq (or) \geq (or) = d_3$	}	<i>constraints</i> (2)
and $x_1, x_2, x_3 \geq 0$	}	<i>non – negativity restrictions</i> (3)

- ❖ The set of values x_1, x_2, x_3 that satisfies (2) is called solution.
- ❖ The set of values x_1, x_2, x_3 that satisfies (2) and (3) is called feasible solution.
- ❖ The set of values x_1, x_2, x_3 that satisfies (1), (2) and (3) is called optimum solution.

Characteristics:

- ❖ The objective function is of max (or) min type.
- ❖ Constraints are $\leq (or) \geq (or) =$ type.
- ❖ Decision variables may be non-negative or unrestricted.

Canonical form of an L.P.P	
Max	$z = c_1x_1 + c_2x_2 + c_3x_3$
Subject to	$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \leq b_1$ $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \leq b_2$ $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \leq b_3$
and	$x_1, x_2, x_3 \geq 0.$

Characteristics:

- ❖ The objective function is of max type.
- ❖ All the constraints are $\leq$ type.
- ❖ All the decision variables are non-negative.

Standard form of an L.P.P	
Max	$z = c_1x_1 + c_2x_2 + c_3x_3$
Subject to	$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$ $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$ $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$
and	$x_1, x_2, x_3 \geq 0.$

Characteristics:

- ❖ The objective function is of max type.
- ❖ All the constraints are expressed as equations.
- ❖ Right side of each constraint (b_i) is non-negative.
- ❖ All the decision variables are non-negative.

Hints to convert General form of L.P.P to Standard form of L.P.P

Min $z = c_1x_1 + c_2x_2 + c_3x_3$	Max $z^* = -c_1x_1 - c_2x_2 - c_3x_3$
$ax_1 + bx_2 + cx_3 \leq d$	$ax_1 + bx_2 + cx_3 + s_1 = d$, where s_1 –Slack variable
$ax_1 + bx_2 + cx_3 \geq -d$	$-ax_1 - bx_2 - cx_3 \leq d$, $-ax_1 - bx_2 - cx_3 + s_1 = d$, where s_1 –Slack variable.
$ax_1 + bx_2 + cx_3 \geq d$	$ax_1 + bx_2 + cx_3 - s_1 + A_1 = d$ Where s_1 – Surplus variable A_1 – Artificial variable
x_1 is unrestricted in sign	$x_1 = x_1' - x_1''$, where $x_1', x_1'' \geq 0$

Basic feasible solution

- ❖ Consider a system of m linear equations with n variables ($m < n$).
- ❖ **Basic solution:** The values of m variables by keeping $n - m$ variables as zero.
- ❖ **Basic feasible solution:** Basic solution satisfying non-negativity restrictions.

Problems:

1. Find the initial basic feasible solution of the L.P.P Max $z = 3x_1 + 2x_2$
subject to $x_1 + x_2 \leq 4$, $x_1 - x_2 \leq 2$ and $x_1, x_2 \geq 0$.

Standard form:

$$\text{Max } z = 3x_1 + 2x_2 + 0s_1 + 0s_2$$

$$\text{Subject to } x_1 + x_2 + s_1 = 4, x_1 - x_2 + s_2 = 2$$

$$\text{And } x_1, x_2, s_1, s_2 \geq 0$$

Initial basic feasible solution:

$$\text{Number of equations} = m = 2,$$

$$\text{Number of variables} = n = 4, n > m.$$

$$\text{Put } x_1 = 0, x_2 = 0.$$

$$\text{Initial basic feasible solution is } s_1 = 4, s_2 = 2.$$

2. Find the initial basic feasible solution of the L.P.P Min $z = x_2 - 3x_3 + 2x_5$
subject to $3x_2 - x_3 + 2x_5 \leq 7$, $-2x_2 + 4x_3 \leq 12$, $-4x_2 + 3x_3 + 8x_5 \leq 10$
and $x_2, x_3, x_5 \geq 0$.

Standard form:

$$\text{Max } z^* = -x_2 + 3x_3 - 2x_5 + 0s_1 + 0s_2 + 0s_3$$

$$\text{Subject to}$$

$$3x_2 - x_3 + 2x_5 + s_1 = 7, -2x_2 + 4x_3 + s_2 = 12, -4x_2 + 3x_3 + 8x_5 + s_3 = 10$$

$$\text{and } x_2, x_3, x_5, s_1, s_2, s_3 \geq 0$$

Initial basic feasible solution:

Number of equations = $m = 3$

Number of variables = $n = 6, n > m$.

Put $x_2 = 0, x_3 = 0, x_5 = 0$

Initial basic feasible solution is $s_1 = 7, s_2 = 12, s_3 = 10$.

3. Find the initial basic feasible solution of the L.P.P $\text{Max } z = 3x_1 + 2x_2$
subject to $2x_1 + x_2 \leq 2, 3x_1 + 4x_2 \geq 12$ and $x_1, x_2 \geq 0$.

Standard form:

$\text{Max } z = 3x_1 + 2x_2 + 0s_1 + 0s_2 + 0A_1$

Subject to $2x_1 + x_2 + s_1 = 2, 3x_1 + 4x_2 - s_2 + A_1 = 12$

And $x_1, x_2, s_1, s_2, A_1 \geq 0$

Initial basic feasible solution:

Number of equations = $m = 2$,

Number of variables = $n = 5, n > m$.

Put $x_1 = 0, x_2 = 0, s_2 = 0$.

Initial basic feasible solution is $s_1 = 2, A_1 = 12$.

4. Find the initial basic feasible solution of the L.P.P $\text{Min } z = 5x + 3y$
subject to $2x + 4y \leq 12, 2x + 2y = 10, 5x + 2y \geq 10$ and $x, y \geq 0$.

Standard form:

$\text{Max } z^* = -5x - 3y + 0s_1 + 0s_2 + 0A_1 + 0A_2$

Subject to $2x + 4y + s_1 = 12, 2x + 2y + A_1 = 10, 5x + 2y - s_2 + A_2 = 10$

And $x, y, s_1, s_2, A_1, A_2 \geq 0$

Initial basic feasible solution:

Number of equations = $m = 3$,

Number of variables = $n = 6, n > m$.

Put $x = 0, y = 0, s_2 = 0$

Initial basic feasible solution is $s_1 = 12, A_1 = 10, A_2 = 10$.

5. Express the following L.P.P in the canonical form: $\text{Max } z = 2x_1 + 3x_2 + x_3$
subject to $4x_1 - 3x_2 + x_3 \leq 6, x_1 + 5x_2 - 7x_3 \geq -4$ and $x_1, x_3 \geq 0, x_2$ is
unrestricted in sign.

$x_2 = x'_2 - x_2''$, where $x'_2, x_2'' \geq 0$

Canonical form:

$\text{Max } z = 2x_1 + 3(x'_2 - x_2'') + x_3$

subject to $4x_1 - 3(x'_2 - x_2'') + x_3 \leq 6, -x_1 - 5(x'_2 - x_2'') + 7x_3 \leq 4$

and $x_1, x'_2, x_2'', x_3 \geq 0$

6. Express the following L.P.P in the standard form: $\text{Max } z = 4x_1 + 2x_2 + 6x_3$
 subject to $2x_1 + 3x_2 + 2x_3 \geq 6$, $3x_1 - 4x_2 = 8$, $6x_1 - 4x_2 + x_3 \leq 10$ and
 $x_1, x_2, x_3 \geq 0$.

Standard form:

$$\text{Max } z = 4x_1 + 2x_2 + 6x_3 + 0s_1 + 0s_2 + 0A_1 + 0A_2$$

$$\text{subject to } 2x_1 + 3x_2 + 2x_3 - s_1 + A_1 = 6$$

$$3x_1 - 4x_2 + A_2 = 8$$

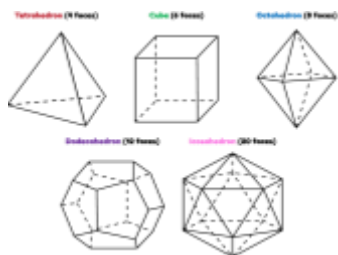
$$6x_1 - 4x_2 + x_3 + 0s_2 = 10$$

$$\text{and } x_1, x_2, x_3, s_1, s_2, A_1, A_2 \geq 0$$

4.2 Simplex Method

Simplex shape: All polyhedrons are simplex in shape.

Simplex:



Not simplex:



L.P.P with two variables can be solved using graphical method. Simplex method is useful to solve L.P.P with 2 or more variables.

1. Solve the following L.P.P using simplex method: $\text{Max } z = 4x_1 + 10x_2$ subject to $2x_1 - 1 + x_2 \leq 50$, $2x_1 + 5x_2 \leq 100$, $2x_1 + 3x_2 \leq 90$ and $x_1, x_2 \geq 0$.

Standard form:

Max $z = 4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3$

subject to $2x_1 - 1 + x_2 + s_1 = 50$

$2x_1 + 5x_2 + s_2 = 100$

$2x_1 + 3x_2 + s_3 = 90$

And $x_1, x_2, s_1, s_2, s_3 \geq 0$.

Simplex table: 1

	C_j	4	10	0	0	0		
C_B	B	x_1	x_2	s_1	s_2	s_3	x_B	x_B/x_2
0	s_1	2	1	1	0	0	50	$50/1 = 50$
0	s_2	2	5	0	1	0	100	$100/5 = 20$
0	s_3	2	3	0	0	1	90	$90/3 = 30$
	z_j	0	0	0	0	0	0	
	$z_j - C_j$	-4	-10	0	0	0		



Entering variable = x_2 , leaving variable = s_2 , Pivotal element = 5.

Column with pivotal element (Entering column) is called pivotal column.

Modify entering column to $(0 \ 1 \ 0)^T$ using elementary row operation.

$R_2 \rightarrow R_2/5$	$R_1 \rightarrow R_1 - R_2$	$R_3 \rightarrow R_3 - 3R_2$
2/5	$2 - 2/5 = 8/5$	$2 - 3(2/5) = 4/5$
1	$1 - 1 = 0$	$3 - 3(1) = 0$
0	$1 - 0 = 1$	$0 - 3(0) = 0$
1/5	$0 - 1/5 = -1/5$	$0 - 3(1/5) = -3/5$
0	$0 - 0 = 0$	$1 - 3(0) = 1$
$100/5 = 20$	$50 - 20 = 30$	$90 - 3(20) = 30$

Simplex table:2

		C_j	4	10	0	0	0		
C_B	B_v	x_1	x_2	s_1	s_2	s_3	x_B		
0	s_1	8/5	0	1	-1/5	0	30		
10	x_2	2/5	1	0	1/5	0	20		
0	s_3	4/5	0	0	-3/5	1	30		
		z_j	4	10	0	2	0		
		$z_j - c_j$	0	0	0	2	0		

Since all $z_j - c_j \geq 0$, current basic feasible solution is optimal.

Therefore, optimal feasible solution is $\text{Max } z_j = 200, x_1 = 0, x_2 = 20$.

2. Solve the following L.P.P using simplex method: $\text{Min } z = 8x_1 - 2x_2$ subject to $-4x_1 + 2x_2 \leq 1, 5x_1 - 4x_2 \leq 3$ and $x_1, x_2 \geq 0$.

Standard form:

$$\text{Max } z^* = -8x_1 + 2x_2 + 0s_1 + 0s_2$$

$$\text{subject to } -4x_1 + 2x_2 + s_1 = 1$$

$$5x_1 - 4x_2 + s_2 = 3$$

$$\text{and } x_1, x_2, s_1, s_2 \geq 0.$$

Simplex table: 1

		C_j	-8	2	0	0		
C_B	B_v	x_1	x_2	s_1	s_2	x_B	x_B/x_2	
0	s_1	-4	2	1	0	1	1/2	→
0	s_2	5	-4	0	1	3		
	z_j^*	0	0	0	0	0		
	$z_j^* - c_j$	8	-2	0	0			

Entering variable = x_2 , leaving variable = s_1 , Pivotal element = 2.

Column with pivotal element (Entering column) is called pivotal column.

Modify entering column to $(0 \ 1 \ 0)^T$ using elementary row operation.

$R_1 \rightarrow R_1/2$	$R_2 \rightarrow R_2 + 4R_1'$
-2	$5 + 4(-2) = -3$
1	$-4 + 4(1) = 0$
1/2	$0 + 4(1/2) = 2$
0	$1 + 4(0) = 1$
1/2	$3 + 4(1/2) = 5$

Simplex table: 2

		C_j	-8	2	0	0		
C_B	B_v	x_1	x_2	s_1	s_2	x_B		
2	x_2	-2	1	1/2	0	1/2		
0	s_2	-3	0	2	1	5		
	z_j^*	-4	2	1	0	1		
	$z_j^* - c_j$	4	0	1	0			

Since all $z_j - c_j \geq 0$, current basic feasible solution is optimal.

Therefore, optimal feasible solution is $x_1 = 0, x_2 = \frac{1}{2}, \text{Max } z_j^* = 1$.

Otherwise, $x_1 = 0, x_2 = \frac{1}{2}, \text{min } z_j = -1$.

3. Solve the following L.P.P using simplex method: $\text{Max } z = 3x_1 + 2x_2 + 5x_3$ subject to $x_1 + 4x_2 \leq 420$, $3x_1 + 2x_3 \leq 460$, $x_1 + 2x_2 + x_3 \leq 430$ and $x_1, x_2, x_3 \geq 0$.

Standard form:

$$\text{Max } z = 3x_1 + 2x_2 + 5x_3 + 0s_1 + 0s_2 + 0s_3$$

$$\text{subject to } x_1 + 4x_2 + s_1 = 420$$

$$3x_1 + 2x_3 + s_2 = 460$$

$$2x_2 + x_3 + s_3 = 430 \text{ and } x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$$

Simplex table: 1

	C_j	3	2	5	0	0	0	
C_B	B_v	x_1	x_2	x_3	s_1	s_2	s_3	x_B
0	s_1	1	4	0	1	0	0	420
0	s_2	3	0	2	0	1	0	460
0	s_3	1	2	1	0	0	1	430
	z_j	0	0	0	0	0	0	0
	$z_j^* - c_j$	-3	-2	-5	0	0	0	-

x_B/x_2
—
230
430 →

Entering variable = x_3 , leaving variable = s_2 , Pivotal element = 2.

Modify entering column as $(0 \ 1 \ 0)^T$.

$R_2 \rightarrow R_2/2$	$R_1 \rightarrow R_1$	$R_3 \rightarrow R_3 - R_2$
3/2	1	$1 - 3/2 = -1/2$
0	4	$2 - 0 = 2$
1	0	$1 - 1 = 0$
0	1	$0 - 0 = 0$
1/2	0	$0 - 1/2 = -1/2$
0	0	$1 - 0 = 1$
230	420	$430 - 230 = 200$

Simplex table: 2

	C_j	3	2	5	0	0	0	
C_B	B_v	x_1	x_2	x_3	s_1	s_2	s_3	x_B
0	s_1	1	4	0	1	0	0	420
5	x_3	3/2	0	1	0	1/2	0	230
0	s_3	-1/2	2	0	0	-1/2	1	200
	z_j	0	0	0	0	0	0	1150
	$z_j^* - c_j$	9/2	-2	0	0	5/2	0	-

x_B/x_2
105
—
100 →

Entering variable = x_2 , leaving variable = s_3 , Pivotal element = 2.

Modify entering column as $(0 \ 0 \ 1)^T$.

$R_3' \rightarrow R_3'/2$	$R_1' \rightarrow R_1' - 4R_3'$	$R_2' \rightarrow R_2'$
-1/4	$1 - 4(-1/4) = 2$	3/2
1	$4 - 4(1) = 0$	0
0	$0 - 4(0) = 0$	1
0	$1 - 4(0) = 1$	0
-1/4	$0 - 4(-1/4) = 1$	1/2
1/2	$0 - 4(1/2) = -2$	0
100	$420 - 4(100) = 20$	230

Simplex table: 3

C_j		3	2	5	0	0	0	
C_B	B_v	x_1	x_2	x_3	s_1	s_2	s_3	x_B
0	s_1	2	0	0	1	1	-2	20
5	x_3	3/2	0	1	0	1/2	0	230
2	x_2	-1/4	1	0	0	-1/4	1/2	100
	z_j	7	2	5	0	2	1	1150
	$z_j - c_j$	4	0	0	0	2	1	-

Since all $z_j - c_j \geq 0$, current basic feasible solution is optimal.

Therefore, optimal feasible solution is $x_1 = 0, x_2 = 100, x_3 = 230$, $Max \ z_j = 1350$.

4. Solve the following L.P.P using simplex method: $Min \ z = x_2 - 3x_3 + 2x_5$
subject to $3x_2 - x_3 + 2x_5 \leq 7, -2x_2 + 4x_3 \leq 12, -4x_2 + 3x_3 + 8x_5 \leq 10$
and $x_2, x_3, x_5 \geq 0$.

Standard form:

Max $z^* = -x_2 + 3x_3 - 2x_5 + 0s_1 + 0s_2 + 0s_3$

subject to $3x_2 - x_3 + 2x_5 + s_1 = 7$

$-2x_2 + 4x_3 + s_2 = 12$

$-4x_2 + 3x_3 + 8x_5 + s_3 = 10$ and $x_2, x_3, x_5, s_1, s_2, s_3 \geq 0$.

Simplex table: 1

C_j		-1	3	-2	0	0	0	
C_B	B_v	x_2	x_3	x_5	s_1	s_2	s_3	x_B
0	s_1	3	-1	2	1	0	0	7
0	s_2	-2	4	0	0	1	0	12
0	s_3	-4	3	8	0	0	1	10
	z_j^*	0	0	0	0	0	0	0
	$z_j^* - c_j$	1	-3	2	0	0	0	-

x_B/x_2
-
3
10/3



Entering variable = x_3 , leaving variable = s_2 , Pivotal element = 4.

Modify entering column as $(0 \ 1 \ 0)^T$.

$R_2 \rightarrow R_2/4$	$R_1 \rightarrow R_1 + R_2'$	$R_3 \rightarrow R_3 - 3R_2'$
-1/2	$3 - 1/2 = 5/2$	$-4 - 3(-1/2) = -5/2$
1	$-1 + 1 = 0$	$3 - 3(1) = 0$
0	$2 + 0 = 2$	$8 - 3(0) = 8$
0	$1 + 0 = 1$	$0 - 3(0) = 0$
1/4	$0 + 1/4 = 1/4$	$0 - 3(1/4) = -3/4$
0	$0 + 0 = 0$	$1 - 3(0) = 1$
3	$7 + 3 = 10$	$10 - 3(3) = 1$

Simplex table: 2

C_j		-1	3	-2	0	0	0	
C_B	B_v	x_2	x_3	x_5	s_1	s_2	s_3	x_B
0	s_1	5/2	0	2	1	1/4	0	10
3	x_3	-1/2	1	0	0	1/4	0	3
0	s_3	-5/2	0	8	0	-3/4	1	1
	z_j^*	-3/2	3	0	0	3/4	0	9
	$z_j^* - c_j$	-1/2	0	2	0	3/4	0	-

x_B/x_2
4 $\rightarrow$
-
-



Entering variable = x_2 , leaving variable = s_1 , Pivotal element = 5/2.

Modify elementary column as $(0 \ 1 \ 0)^T$.

$R_1'' \rightarrow \frac{2}{5}R_1'$	$R_2'' \rightarrow R_2' + 1/2R_1''$	$R_3 \rightarrow R_3' - 3R_1''$
1	$-\frac{1}{2} + \frac{1}{2}(1) = 0$	$-\frac{5}{2} + \frac{5}{2}(1) = 0$
0	$1 + \frac{1}{2}(0) = 1$	$0 + \frac{5}{2}(0) = 0$
$\frac{4}{5}$	$0 + \frac{1}{2}(\frac{4}{5}) = \frac{2}{5}$	$8 + \frac{5}{2}(\frac{4}{5}) = 10$
$\frac{2}{5}$	$0 + \frac{1}{2}(\frac{2}{5}) = \frac{1}{5}$	$0 + \frac{5}{2}(\frac{2}{5}) = 1$
$\frac{1}{10}$	$\frac{1}{4} + \frac{1}{2}(\frac{1}{10}) = \frac{3}{10}$	$-\frac{3}{4} + \frac{5}{2}(\frac{1}{10}) = -\frac{1}{2}$
0	$0 + \frac{1}{2}(0) = 0$	$1 + \frac{5}{2}(0) = 1$
4	$3 + \frac{1}{2}(4) = 5$	$1 + \frac{5}{2}(4) = 11$

Simplex table: 3

	C_j		-1	3	-2	0	0	0	
C_B	B_v		x_2	x_3	x_5	s_1	s_2	s_3	x_B
-1	x_2		1	0	4/5	2/5	1/10	0	4
3	x_3		0	1	2/5	1/5	3/10	0	5
0	s_3		0	0	10	1	-1/2	1	11
	z_j^*		-1	3	2/5	1/5	8/10	0	11
	$z_j^* - c_j$		0	0	12/5	1/5	4/5	0	-

Since all $z_j^* - c_j \geq 0$, current basic feasible solution is optimal.

Therefore, optimal feasible solution is $x_2 = 4, x_3 = 5, x_5 = 0, \text{Max } z^* = 11$.

Otherwise, $x_2 = 4, x_3 = 5, x_5 = 0, \text{Min } z = -11$.

5. Solve the following L.P.P using Big M method:

Max $z = 3x_1 + 2x_2$ subject to $2x_1 + x_2 \leq 2, 3x_1 + 4x_2 \geq 12, x_1, x_2 \geq 0$.

Standard form:

Max $z = 3x_1 + 2x_2 + 0s_1 + 0s_2 - MA_1$

subject to $2x_1 + x_2 + s_1 = 2$

$3x_1 + 4x_2 - s_2 + A_1 = 12$

$x_1 + 2x_2 + s_2 = 4$

and $x_1, x_2, s_1, s_2, A_1 \geq 0$.

Simplex table: 1

	C_j		3	2	0	0	-M	
C_B	B_v		x_1	x_2	s_1	s_2	s_3	x_B
0	s_1		2	1	1	0	0	2
-M	A_1		3	4	0	-1	1	12
	z_j		-3M	-4M	0	M	-M	-12M
	$z_j - c_j$		-3M - 3	-4M - 2	0	M	0	

x_B/x_2
 $2/1 = 2 \rightarrow$
 $12/4 = 3$



Entering variable = x_2 , leaving variable = s_1 , Pivotal element = 1.

Column with pivotal element (Entering column) is called pivotal column.

Modify entering column to $(1 \ 0)^T$ using elementary row operation.

$R_1' \rightarrow R_1$	$R_2' \rightarrow R_2 - 4R_1'$
2	$3 - 4(2) = -5$
1	$4 - 4(1) = 0$
1	$0 - 4(1) = -4$
0	$-1 - 4(0) = -1$
0	$1 - 4(0) = 1$
2	$12 - 4(2) = 4$

Simplex table: 2

	C_j	3	2	0	0	$-M$	
C_B	B_v	x_1	x_2	s_1	s_2	s_3	x_B
2	x_2	2	1	1	0	0	2
$-M$	A_1	-5	0	-4	-1	1	4
	z_j	$4 + 5M$	2	$2 + 4M$	M	$-M$	$4 - 4M$
	$z_j - c_j$	$5M + 1$	0	$2 + 4M$	M	0	

All $z_j - c_j \geq 0$ and artificial variable appears in the basis at positive level.

Therefore, the given L.P.P does not have any feasible solution.

6. Solve the following L.P.P using Big M method:

Max $z = -2x_1 - x_2$ subject to $3x_1 + x_2 = 3$, $4x_1 + 3x_2 \geq 6$,
 $x_1 + 2x_2 \leq 4$, $x_1, x_2 \geq 0$.

Standard form:

Max $z = -2x_1 - x_2 + 0s_1 + 0s_2 - MA_1 - MA_2$

subject to $3x_1 + x_2 + A_1 = 3$

$4x_1 + 3x_2 - s_1 + A_2 = 6$

$x_1 + 2x_2 + s_2 = 4$ and $x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$.

Simplex table: 1

	C_j	-2	-1	0	0	$-M$	$-M$	
C_B	B_v	x_1	x_2	s_1	s_2	A_1	A_2	x_B
$-M$	A_1	3	1	0	0	1	0	3
$-M$	A_2	4	3	-1	0	0	1	6
0	s_2	1	2	0	1	0	0	4
	z_j	$-7M$	$-4M$	M	0	$-M$	$-M$	$-9M$
	$z_j - c_j$	$2 - 7M$	$1 - 4M$	M	0	0	0	-

x_B/x_1
 $3/3 \rightarrow$
 $6/4$
 $4/1$

Entering variable $\uparrow = x_1$, leaving variable = A_1 , Pivotal element = 3.

Ignore the leaving artificial variable column.

Modify Pivotal column as $(1 \ 0 \ 0)^T$.

$R_1' \rightarrow R_1/3$	$R_1' \rightarrow R_2 - 4R_1'$	$R_3' \rightarrow R_3 - R_1'$
1	$4 - 4(1) = 0$	$1 - 1 = 0$
$\frac{1}{3}$	$3 - 4(\frac{1}{3}) = \frac{5}{3}$	$2 - \frac{1}{3} = \frac{5}{3}$
0	$-1 - 4(0) = -1$	$0 - 0 = 0$
0	$0 - 4(0) = 0$	$1 - 0 = 1$
0	$1 - 4(0) = 1$	$0 - 0 = 0$
1	$6 - 4(1) = 2$	$4 - 1 = 3$

Simplex table: 2

C_j		-1	3	-2	0	0	x_B / x_2
C_B	B_v	x_1	x_2	s_1	s_2	A_2	
-1	x_1	1	1/3	0	0	0	1
-M	A_2	0	5/3	-1	0	1	2
0	s_2	0	5/3	0	1	0	3
	z_j^*	-1	$\frac{-1-5M}{3}$	M	0	-M	-1-6M
	$z_j^*-c_j$	0	$\frac{8-5M}{3}$	M+2	0	0	-



Entering variable = x_2 , leaving variable = A_2 , Pivotal element = $5/3$.

Ignore the leaving artificial variable column.

Modify entering column as $(0 \ 1 \ 0)^T$

$R_2'' \rightarrow \frac{3}{5} R_2'$	$R_1'' \rightarrow R_1' - \frac{1}{3} R_2''$	$R_3'' \rightarrow R_3' - \frac{5}{3} R_2''$
0	$1 - \frac{1}{3}(0) = 1$	$0 - \frac{5}{3}(0) = 0$
1	$\frac{1}{3} - \frac{1}{3}(1) = 0$	$\frac{5}{3} - \frac{5}{3}(1) = 0$
$-\frac{3}{5}$	$0 - \frac{1}{3}(-\frac{3}{5}) = \frac{1}{5}$	$0 - \frac{5}{3}(-\frac{3}{5}) = 1$
0	$0 - \frac{1}{3}(0) = 0$	$1 - \frac{5}{3}(0) = 1$
$\frac{6}{5}$	$1 - \frac{1}{3}(\frac{6}{5}) = \frac{3}{5}$	$3 - \frac{5}{3}(\frac{6}{5}) = 1$

Simplex table: 3

C_j		-1	3	-2	0	
C_B	B_v	x_1	x_2	s_1	s_2	x_B
-1	x_1	1	0	1/5	0	3/5
3	x_2	0	1	-3/5	0	6/5
0	s_2	0	0	1	1	1
	z_j^*	-1	3	-2	0	-12/5
	$z_j^*-c_j$	0	0	0	0	-

Since all $z_j^* - c_j \geq 0$, current basic feasible solution is optimal.

Therefore, optimal feasible solution is $x_1 = \frac{3}{5}, x_2 = \frac{6}{5}, x_3 = 0, \text{Max } z^* = -12/5$.