

# FOURIER SERIES

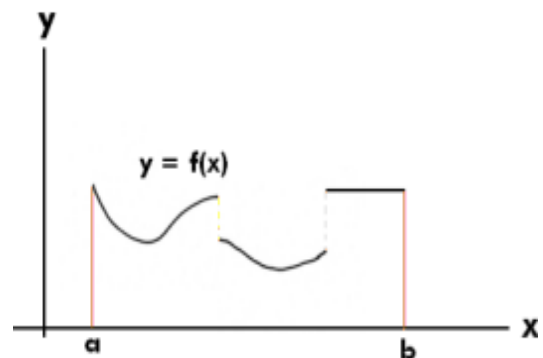
## 1.1 Introduction

Fourier series introduced by a French physicist Joseph Fourier (1768-1830), is a mathematical tool that converts some specific periodic signals into everlasting sinusoidal waveforms, which is of utmost importance in scientific and engineering applications.

The Fourier series allows us to model any arbitrary periodic signal or function  $f(x)$  in the form  $\frac{a_0}{2} + (a_1 \cos x + a_2 \cos 2x + \dots) + (b_1 \sin x + b_2 \sin 2x + \dots)$  the interval  $[C, C + 2l]$  under some conditions called **Dirichlet's conditions** as given below:

- (i)  $f(x)$  is periodic with a period  $2l$
- (ii)  $f(x)$  and its integrals are finite and single valued in  $[C, C + 2l]$
- (iii)  $f(x)$  is piecewise continuous\* in the interval  $[C, C + 2l]$
- (iv)  $f(x)$  has a finite no of maxima & minima in  $[C, C + 2l]$

\* A function  $f(x)$  is said to be **piecewise continuous** in an interval  $[a, b]$ , if the interval can be subdivided into a finite number of intervals in each of which the function is continuous and has finite left and right hand limits i.e. it is bounded. In other words, a piecewise continuous function is a function that has a finite number of discontinuities and doesn't blow up to infinity anywhere in the given interval.



Thus any function  $f(x)$  defined in  $[C_1, C_2]$  and satisfying Dirichlet's conditions can be expressed in Fourier series given by  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$  where  $a_0, a_n, b_n$  are constants called the Fourier coefficients of  $f(x)$ , required to determine any function into Fourier series.

## Periodic Functions

A function  $f(x)$  is said to be periodic if there exists a positive number  $T$  such that  $f(x + T) = f(x) \forall x \in R$ .

Here  $T$  is the smallest positive real number such that  $f(x + T) = f(x) \forall x \in R$  and is called the fundamental period of  $f(x)$ .

We know that  $\sin x, \cos x, \sec x, \operatorname{cosec} x$  are periodic functions with period  $2\pi$  whereas  $\tan x$  and  $\cot x$  are periodic with a period  $\pi$ . The functions  $\sin nx$  and  $\cos nx$  are periodic with period  $\frac{2\pi}{n}$ , while fundamental period of  $\tan nx$  is  $\frac{\pi}{n}$ .

**Example1** Determine the period of the following functions:

i.  $\sin(5x + 3)$       ii.  $\sin^2 x$       iii.  $|\cos x|$       iv.  $\sin x + \cos x$

v.  $K$ , where  $K$  is a constant      vi.  $\cos x + \frac{1}{3} \cos 2x + \frac{1}{2} \cos \frac{x}{3}$

**Solution:** i.  $f(x) = \sin(5x + 3)$

$\sin 5x$  is a periodic function with a period  $\frac{2\pi}{5}$

$\therefore f(x)$  is periodic with a period  $\frac{2\pi}{5}$

ii.  $f(x) = \sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2} - \frac{\cos 2x}{2}$

$\cos 2x$  is a periodic function with a period  $\frac{2\pi}{2} = \pi$

$\therefore f(x)$  is periodic with a period  $\pi$

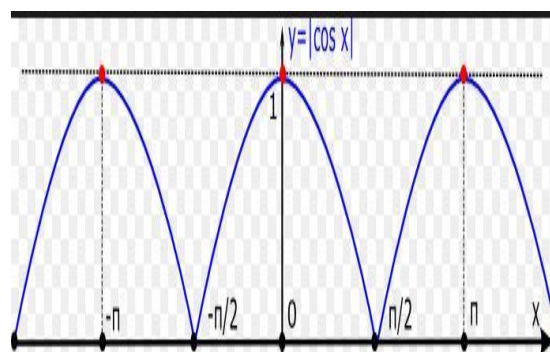
iii.  $f(x) = |\cos x| = \sqrt{\cos^2 x} \therefore |x| = \sqrt{x^2}$   

$$= \sqrt{\frac{1 + \cos 2x}{2}}$$

Now  $\cos 2x$  is a periodic function with a period  $\frac{2\pi}{2} = \pi$

$\therefore f(x)$  is periodic with a period  $\pi$

Note: From the graph of  $|\cos x|$ , period is  $\pi$



iv.  $f(x) = (\sin x + \cos x) = \sqrt{2} \left( \frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x \right)$   

$$= \sqrt{2} \left( \sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4} \right)$$
  

$$= \sqrt{2} \sin \left( x + \frac{\pi}{4} \right)$$

Now  $\sin x$  is a periodic function with a period  $2\pi$

$\therefore f(x)$  is periodic with a period  $2\pi$

Or

Periods of  $\sin x$  is  $2\pi$  and period of  $\cos x$  is also  $2\pi$

$\therefore f(x)$  is periodic with a period which is lowest common multiple (l.c.m) of  $(2\pi, 2\pi) = 2\pi$

v.  $f(x) = K$ , where  $K$  is a constant

$$\Rightarrow f(x + T) = K = f(x) \forall x \in R$$

$\therefore f(x)$  is periodic function, but fundamental period of  $f(x)$  i.e.  $T$  can not be defined since it is the smallest positive real number.

vi.  $f(x) = \cos x + \frac{1}{3} \cos 2x + \frac{1}{2} \cos \frac{x}{3}$

Periods of  $\cos x$ ,  $\cos 2x$  and  $\cos \frac{x}{3}$  are  $2\pi$ ,  $\pi$  and  $6\pi$  respectively.

$\therefore f(x)$  is periodic with a period which is lowest common multiple i.e.

$$(\text{l.c.m}) \text{ of } (2\pi, \pi, 6\pi) = 6\pi$$

**Example 2** Determine a sinusoidal periodic function whose period is given by:

i.  $\frac{\pi}{k}$

ii.  $\frac{k}{2}$

iii.  $2 - k$

**Solution:** The function  $\sin nx$  is periodic with period  $\frac{2\pi}{n}$

i.  $\frac{\pi}{k} = \frac{2\pi}{n} \Rightarrow n = 2k$

$\therefore$  The required periodic function is  $\sin 2kx$

ii.  $\frac{k}{2} = \frac{2\pi}{n} \Rightarrow n = \frac{4\pi}{k}$

$\therefore$  The required periodic function is  $\sin \frac{4\pi x}{k}$

iii.  $2 - k = \frac{2\pi}{n} \Rightarrow n = \frac{2\pi}{2-k}$

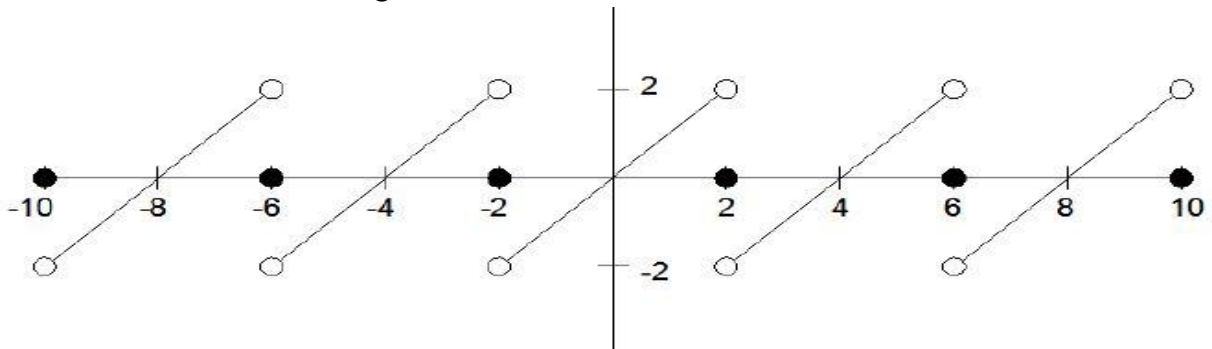
$\therefore$  The required periodic function is  $\sin \frac{2\pi x}{2-k}$

**Example 3** Draw the graph of the following functions which are assumed to be periodic with a period '4', such that  $f(x + 4) = f(x)$

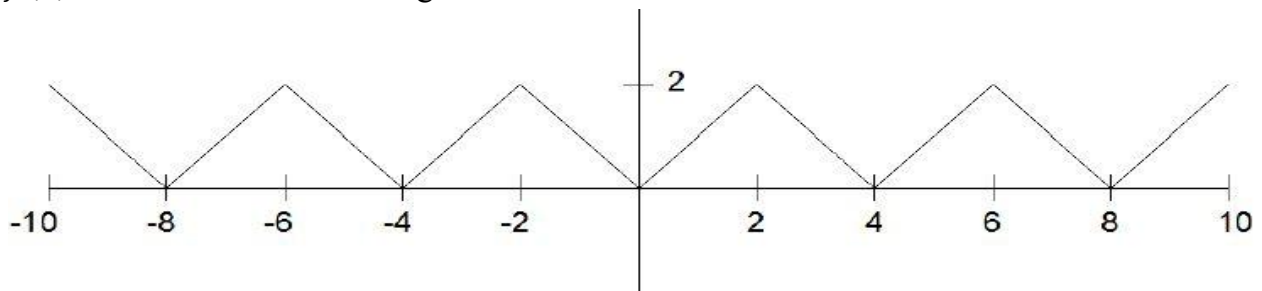
i.  $f(x) = x$ ,  $-2 < x < 2$

ii.  $f(x) = |x|$ ,  $-2 < x < 2$

**Solution:** i. Plotting graph of  $f(x) = x$  in the interval  $(-2, 2)$  and repeating periodically in the intervals  $(-6, -2)$ ,  $(-10, -6)$  on the left and  $(2, 6)$ ,  $(6, 10)$  on the right, since period of  $f(x)$  is given to be '4' units



ii. Plotting graph of  $f(x) = |x|$  in the interval  $(-2, 2)$  and repeating periodically in the intervals  $(-6, -2)$ ,  $(-10, -6)$  on the left and  $(2, 6)$ ,  $(6, 10)$  on the right, since period of  $f(x)$  is given to be '4' units



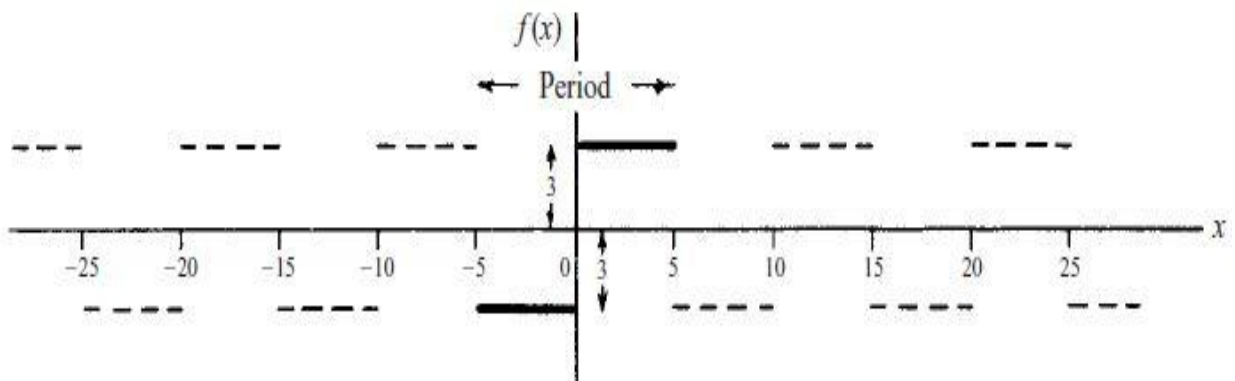
**Example 4** Draw the graph of the following functions which are assumed to be periodic with a period 'T', such that  $f(x + T) = f(x)$

i.  $f(x) = \begin{cases} -3, & -5 < x < 0 \\ 3, & 0 < x < 5 \end{cases} \quad f(x + 10) = f(x)$

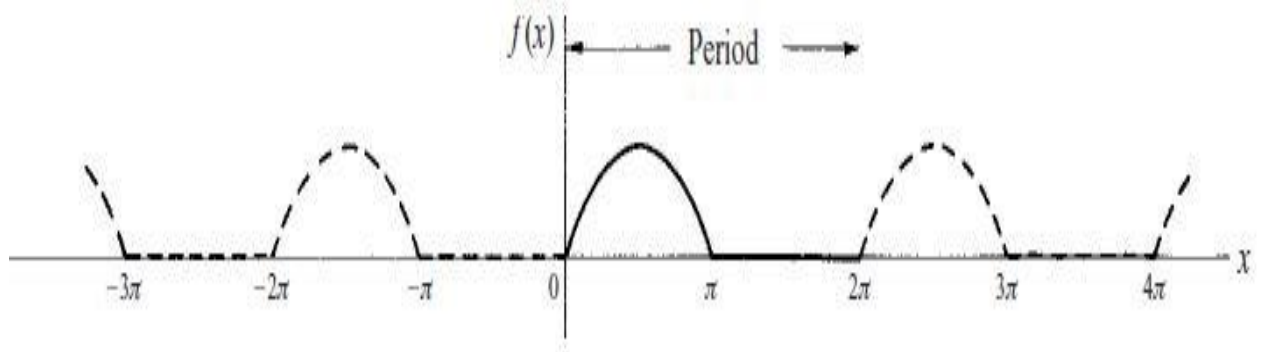
ii.  $f(x) = \begin{cases} \sin x, & 0 \leq x \leq \pi \\ 0, & \pi \leq x \leq 2\pi \end{cases} \quad f(x + 2\pi) = f(x)$

iii.  $f(x) = \begin{cases} 0, & 0 \leq x < 2 \\ 1, & 2 \leq x < 4 \\ 0, & 4 \leq x < 6 \end{cases} \quad f(x + 6) = f(x)$

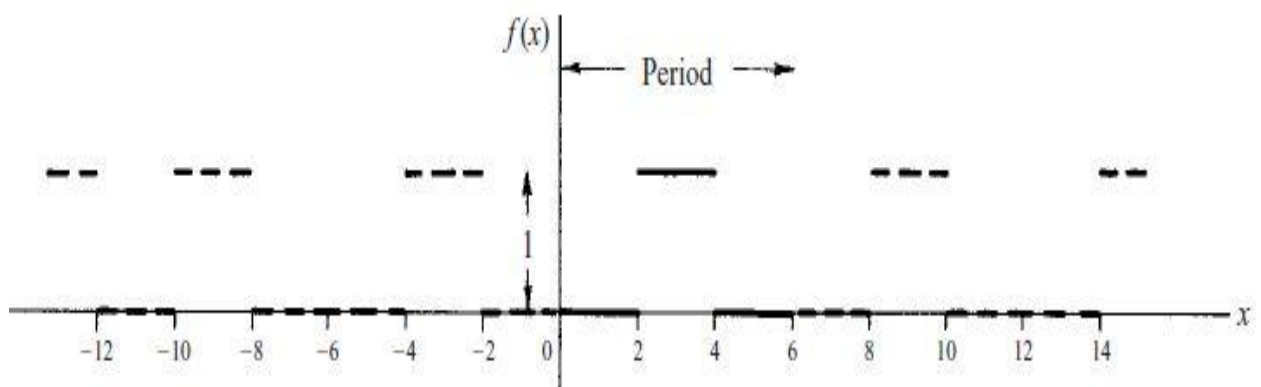
**Solution:** i. Plotting graph of  $f(x)$  in the interval  $(-5, 5)$  and repeating periodically taking period of 10 units



ii. Plotting graph of  $f(x)$  in the interval  $(0, 2\pi)$  and repeating periodically taking period of  $2\pi$  units



iii. Plotting graph of  $f(x)$  in the interval  $(0, 6)$  and repeating periodically taking period of 6 units



### Some useful results in computation of the Fourier series:

If  $m, n$  are non - zero integers then:

$$(i) \quad \int_c^{c+2\pi} \sin nx \, dx = - \left[ \frac{\cos nx}{n} \right]_c^{c+2\pi} = 0$$

$$(ii) \quad \int_c^{c+2\pi} \cos nx \, dx = 0, n \neq 0$$

$$(iii) \quad \int_c^{c+2\pi} \sin mx \cdot \sin nx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

$$(iv) \quad \int_c^{c+2\pi} \cos mx \cdot \cos nx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

$$(v) \quad \int_c^{c+2\pi} \sin mx \cdot \cos nx \, dx = 0$$

$$(vi) \quad \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$$

$$(vii) \quad \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$$

$$(viii) \quad \sin n\pi = 0$$

$$(ix) \quad \cos n\pi = (-1)^n$$

(x) Integration by parts when first function vanishes after a finite number of differentiations:

If  $u$  and  $v$  are functions of  $x$

$$\int u \cdot v \, dx = uv_1 - u^{(1)}v_2 + u^{(2)}v_3 - u^{(3)}v_4 + \dots$$

Here  $u^{(n)}$  is derivative of  $u^{(n-1)}$  and  $v_n$  is integral of  $v_{n-1}$

For example

$$\begin{aligned} \int x^2 \cdot \sin nx \, dx &= (x^2) \left( -\frac{\cos nx}{n} \right) - (2x) \left( -\frac{\sin nx}{n^2} \right) + (2) \left( \frac{\cos nx}{n^3} \right) \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x \\ &= -\frac{x^2 \cos nx}{n} + \frac{2x \sin nx}{n^2} + \frac{2 \cos nx}{n^3} \end{aligned}$$

## 1.2 Euler's Formulae (To find Fourier coefficients $a_0, a_n, b_n$ when interval length is $2\pi$ )

If a function  $f(x)$  satisfies Dirichlet's conditions, it can be expanded into Fourier series given by  $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \dots \dots \textcircled{1}$

**To find  $a_0$**

Integrating both sides of  $\textcircled{1}$  with respect to  $x$  within the limits  $C$  to  $C+2\pi$

$$\begin{aligned} \int_C^{C+2\pi} f(x) dx &= \frac{a_0}{2} \int_C^{C+2\pi} dx + \int_C^{C+2\pi} (a_1 \cos x + a_2 \cos 2x + \dots) dx + \\ &\quad \int_C^{C+2\pi} (b_1 \sin x + b_2 \sin 2x + \dots) dx \end{aligned}$$

$$\Rightarrow \int_C^{C+2\pi} f(x) dx = \frac{a_0}{2} [x]_C^{C+2\pi} + 0 + 0 \quad \text{using results (i) and (ii)}$$

$$= \frac{a_0}{2} [C + 2\pi - C] = a_0\pi$$

$$\Rightarrow a_0 = \frac{1}{\pi} \int_C^{C+2\pi} f(x) dx$$

**To find  $a_n$**

Multiplying both sides of ① by  $\cos nx$  and integrating respect to  $x$  within the limits  $C$  to  $C+2\pi$

$$\begin{aligned} \int_C^{C+2\pi} f(x) \cos nx \, dx &= \frac{a_0}{2} \int_C^{C+2\pi} \cos nx \, dx + \\ &\quad \int_C^{C+2\pi} (a_1 \cos x \cos nx + \dots a_n \cos nx \cos nx + \dots) dx + \\ &\quad \int_C^{C+2\pi} (b_1 \sin x \cos nx + \dots + b_n \sin nx \cos nx + \dots) dx \\ \Rightarrow \int_C^{C+2\pi} f(x) \cos nx \, dx &= 0 + a_n \int_C^{C+2\pi} \cos^2 nx \, dx + 0 \quad \text{using results (i),(ii),(iv), (v)} \\ &= \frac{a_n}{2} \int_C^{C+2\pi} (1 + \cos 2nx) \, dx \\ &= \frac{a_n}{2} \left[ x + \frac{\sin 2nx}{2n} \right]_C^{C+2\pi} \\ &= \frac{a_n}{2} [C + 2\pi - C] = a_n\pi \end{aligned}$$

$$\Rightarrow a_n = \frac{1}{\pi} \int_C^{C+2\pi} f(x) \cos nx \, dx$$

**To find  $b_n$**

Multiplying both sides of ① by  $\sin nx$  and integrating respect to  $x$  within the limits  $C$  to  $C+2\pi$

$$\begin{aligned} \int_C^{C+2\pi} f(x) \sin nx \, dx &= \frac{a_0}{2} \int_C^{C+2\pi} \sin nx \, dx + \\ &\quad \int_C^{C+2\pi} (a_1 \cos x \sin nx + \dots a_n \cos nx \sin nx + \dots) dx + \\ &\quad \int_C^{C+2\pi} (b_1 \sin x \sin nx + \dots + b_n \sin nx \sin nx + \dots) dx \\ \Rightarrow \int_C^{C+2\pi} f(x) \sin nx \, dx &= 0 + 0 + b_n \int_C^{C+2\pi} \sin^2 nx \, dx \quad \text{using results (i),(ii),(iii), (v)} \\ &= \frac{b_n}{2} \int_C^{C+2\pi} (1 - \cos 2nx) \, dx \\ &= \frac{b_n}{2} \left[ x - \frac{\sin 2nx}{2n} \right]_C^{C+2\pi} \\ &= \frac{b_n}{2} [C + 2\pi - C] = b_n\pi \end{aligned}$$

$$\Rightarrow b_n = \frac{1}{\pi} \int_C^{C+2\pi} f(x) \sin nx \, dx$$

**Example 5** State giving reasons whether the following functions can be expanded into Fourier series in the interval  $[-\pi, \pi]$

i.  $\sin \frac{1}{x}$

ii.  $\operatorname{cosec} x$

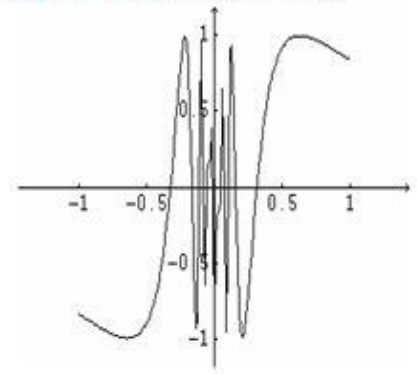
iii.  $\frac{1}{2-x}$

**Solution:** i.  $f(x) = \sin \frac{1}{x}$  is not single valued at  $x = 0$ .

Also graph of  $\sin \frac{1}{x}$  oscillates infinite number of times in vicinity of  $x = 0$  as shown in adjoining figure.  $\therefore f(x)$  is having infinite number of maxima and minima in  $[-\pi, \pi]$ .

Hence  $f(x) = \sin \frac{1}{x}$  violates Dirichlet's conditions and cannot be expanded into Fourier series.

Graph of Function:  $f(x) = \sin(1/x)$



ii.  $f(x) = \operatorname{cosec} x$  is not piecewise continuous in the interval  $[-\pi, \pi]$  as

$$\lim_{x \rightarrow 0^-} f(x) = -\infty \text{ and } \lim_{x \rightarrow 0^+} f(x) = +\infty$$

Hence  $f(x) = \operatorname{cosec} x$  violates Dirichlet's conditions and cannot be expanded into Fourier series.

iii.  $f(x) = \frac{1}{2-x}$  is not piecewise continuous in the interval  $[-\pi, \pi]$  as

$$\lim_{x \rightarrow 2^-} f(x) = +\infty \text{ and } \lim_{x \rightarrow 2^+} f(x) = -\infty$$

Hence  $f(x) = \frac{1}{2-x}$  violates Dirichlet's conditions and cannot be expanded into Fourier series.

**Example 6** If  $f(x + 2\pi) = f(x)$ , find the Fourier expansion  $f(x) = x$  in the interval  $[0, 2\pi]$

$$\text{Hence or otherwise prove that } \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

**Solution:**  $f(x) = x$  is integrable and piecewise continuous in the interval  $[0, 2\pi]$ .

$\therefore f(x)$  can be expanded into Fourier series given by:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \dots \dots \textcircled{1}$$

$$a_0 = \frac{1}{\pi} \int_C^{C+2\pi} f(x) dx = \frac{1}{\pi} \int_0^{2\pi} x dx = \frac{1}{\pi} \left[ \frac{x^2}{2} \right]_0^{2\pi} = 2\pi$$

$$a_n = \frac{1}{\pi} \int_C^{C+2\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{2\pi} x \cos nx dx$$

$$= \frac{1}{\pi} \left[ (x) \left( \frac{\sin nx}{n} \right) - (1) \left( \frac{-\cos nx}{n^2} \right) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[ x \frac{\sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[ \frac{\cos nx}{n^2} \right]_0^{2\pi} \quad \because \sin nx = 0 \text{ when } x = 0 \text{ or } x = 2\pi$$

$$= \frac{1}{\pi} \left[ \frac{1}{n^2} - \frac{1}{n^2} \right] = 0 \quad \because \cos 2n\pi = 1$$

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_C^{C+2\pi} f(x) \sin nx \, dx \\
&= \frac{1}{\pi} \int_0^{2\pi} x \sin nx \, dx \\
&= \frac{1}{\pi} \left[ (x) \left( \frac{-\cos nx}{n} \right) - (1) \left( \frac{-\sin nx}{n^2} \right) \right]_0^{2\pi} \\
&= \frac{1}{\pi} \left[ -\frac{x \cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{2\pi} \\
&= -\frac{1}{\pi} \left[ \frac{2\pi}{n} \right] = -\frac{2}{n} \quad \because \sin nx = 0 \text{ when } x = 0 \text{ or } x = 2\pi \text{ and } \cos 2n\pi = 1
\end{aligned}$$

Substituting values of  $a_0, a_n, b_n$  in ①

$$f(x) \approx \pi - 2 \left[ \frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots \right]$$

Putting  $x = \frac{\pi}{2}$  on both sides

$$\begin{aligned}
\frac{\pi}{2} &= \pi - 2 \left[ \frac{1}{1} + 0 - \frac{1}{3} + 0 + \frac{1}{5} + \dots \right] \\
\Rightarrow \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots &= \frac{\pi}{4}
\end{aligned}$$

**Example 7** If  $f(x + 2\pi) = f(x)$ , find the Fourier series expansion of

$$f(x) = \begin{cases} 0, & -\pi \leq x \leq 0 \\ x, & 0 \leq x \leq \pi \end{cases}$$

Hence or otherwise prove that  $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$

**Solution:**  $f(x)$  is integrable and piecewise continuous in the interval  $[-\pi, \pi]$ .

$\therefore f(x)$  can be expanded into Fourier series given by:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \dots \dots \text{①}$$

$$a_0 = \frac{1}{\pi} \int_C^{C+2\pi} f(x) \, dx = \frac{1}{\pi} \left[ \int_{-\pi}^0 0 \, dx + \int_0^{\pi} x \, dx \right] = \frac{1}{\pi} \left[ \frac{x^2}{2} \right]_0^{\pi} = \frac{\pi}{2}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_C^{C+2\pi} f(x) \cos nx \, dx \\
&= \frac{1}{\pi} \left[ \int_{-\pi}^0 0 \cos nx \, dx + \int_0^{\pi} x \cos nx \, dx \right] \\
&= \frac{1}{\pi} \left[ (x) \left( \frac{\sin nx}{n} \right) - (1) \left( \frac{-\cos nx}{n^2} \right) \right]_0^{\pi} \\
&= \frac{1}{\pi} \left[ \frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi} \\
&= \frac{1}{\pi} \left[ \frac{(-1)^n}{n^2} - \frac{1}{n^2} \right] \quad \because \sin nx = 0 \text{ when } x = 0 \text{ or } x = \pi \\
&= \frac{1}{\pi n^2} [(-1)^n - 1] = \begin{cases} \frac{-2}{\pi n^2} & , n \text{ is odd} \\ 0 & , n \text{ is even} \end{cases}
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_C^{C+2\pi} f(x) \sin nx \, dx \\
&= \frac{1}{\pi} \left[ \int_{-\pi}^0 0 \sin nx \, dx + \int_0^\pi x \sin nx \, dx \right] \\
&= \frac{1}{\pi} \left[ (x) \left( \frac{-\cos nx}{n} \right) - (1) \left( \frac{-\sin nx}{n^2} \right) \right]_0^\pi \\
&= \frac{1}{\pi} \left[ -\frac{x \cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^\pi \\
&= -\frac{1}{\pi} \left[ \frac{\pi(-1)^n}{n} \right] \quad \because \frac{\sin nx}{n^2} = 0 \text{ when } x = 0 \text{ or } x = \pi \\
&= -\frac{1}{n} [(-1)^n] = \frac{(-1)^{n+1}}{n} = \begin{cases} \frac{1}{n}, & n \text{ is odd} \\ -\frac{1}{n}, & n \text{ is even} \end{cases}
\end{aligned}$$

Substituting values of  $a_0, a_n, b_n$  in ①

$$f(x) \approx \frac{\pi}{4} - \frac{2}{\pi} \left[ \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right] + \left[ \frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \frac{\sin 5x}{5} - \dots \right]$$

Putting  $x = \frac{\pi}{2}$  on both sides

$$\begin{aligned}
\frac{\pi}{2} &= \frac{\pi}{4} - 0 + \left[ \frac{1}{1} - 0 - \frac{1}{3} - 0 + \frac{1}{5} - \dots \right] \\
\Rightarrow \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots &= \frac{\pi}{4}
\end{aligned}$$

**Example 8** If  $f(x + 2\pi) = f(x)$ , find the Fourier series expansion of

$$f(x) = \begin{cases} \cos x, & -\pi \leq x \leq 0 \\ \sin x, & 0 \leq x \leq \pi \end{cases}$$

**Solution:**  $f(x)$  is integrable and piecewise continuous in the interval  $[-\pi, \pi]$ .

$\therefore f(x)$  can be expanded into Fourier series given by:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \dots\dots \textcircled{1}$$

$$a_0 = \frac{1}{\pi} \int_C^{C+2\pi} f(x) \, dx$$

$$= \frac{1}{\pi} \left[ \int_{-\pi}^0 \cos x \, dx + \int_0^\pi \sin x \, dx \right]$$

$$= \frac{1}{\pi} [\sin x]_{-\pi}^0 - \frac{1}{\pi} [\cos x]_0^\pi$$

$$= 0 - \frac{1}{\pi} [\cos \pi - \cos 0] = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi} \int_C^{C+2\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \left[ \int_{-\pi}^0 \cos x \cos nx \, dx + \int_0^\pi \sin x \cos nx \, dx \right]$$

$$= \frac{1}{2\pi} \left[ \int_{-\pi}^0 [\cos(n+1)x + \cos(n-1)x] \, dx + \int_0^\pi [\sin(n+1)x - \sin(n-1)x] \, dx \right]$$

$$\begin{aligned}
&= \frac{1}{2\pi} \left[ \frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right]_{-\pi}^0 + \frac{1}{2\pi} \left[ -\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right]_0^{\pi}, n \neq 1 \\
&= 0 + \frac{1}{2\pi} \left[ -\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right] \\
&= \frac{1}{2\pi} \left[ -\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right] \\
&= \begin{cases} \frac{1}{2\pi} \left[ -\frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right], & n \text{ is odd } (n \neq 1) \\ \frac{1}{2\pi} \left[ \frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right], & n \text{ is even} \end{cases} = \begin{cases} 0, & n \text{ is odd } (n \neq 1) \\ \frac{-2}{\pi(n^2-1)}, & n \text{ is even} \end{cases}
\end{aligned}$$

$$a_1 = \frac{1}{\pi} \left[ \int_{-\pi}^0 \cos x \cos x \, dx + \int_0^{\pi} \sin x \cos x \, dx \right]$$

$$= \frac{1}{\pi} \left[ \int_{-\pi}^0 \cos^2 x \, dx + \frac{1}{2} \int_0^{\pi} \sin 2x \, dx \right]$$

$$= \frac{1}{\pi} \left[ \frac{1}{2} \int_{-\pi}^0 (1 + \cos 2x) \, dx + \frac{1}{2} \int_0^{\pi} \sin 2x \, dx \right]$$

$$= \frac{1}{2\pi} \left[ x + \frac{\sin 2x}{2} \right]_{-\pi}^0 + \frac{1}{2\pi} \left[ -\frac{\cos 2x}{2} \right]_0^{\pi}$$

$$= \frac{1}{2\pi} [\pi] - \frac{1}{4\pi} [1 - 1] = \frac{1}{2}$$

$$b_n = \frac{1}{\pi} \int_C^{C+2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \left[ \int_{-\pi}^0 \cos x \sin nx \, dx + \int_0^{\pi} \sin x \sin nx \, dx \right]$$

$$= \frac{1}{2\pi} \left[ \int_{-\pi}^0 [\sin(n+1)x + \sin(n-1)x] \, dx + \int_0^{\pi} [\cos(n-1)x - \cos(n+1)x] \, dx \right]$$

$$= -\frac{1}{2\pi} \left[ \frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right]_{-\pi}^0 + \frac{1}{2\pi} \left[ \frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1} \right]_0^{\pi}, n \neq 1$$

$$= -\frac{1}{2\pi} \left[ \frac{1}{n+1} + \frac{1}{n-1} - \frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} \right] + 0 \because \sin nx = 0 \text{ when } x = 0 \text{ or } x = \pi$$

$$= -\frac{1}{2\pi} \left[ \frac{1}{n+1} + \frac{1}{n-1} - \frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1} \right]$$

$$= \begin{cases} -\frac{1}{2\pi} \left[ \frac{1}{n+1} + \frac{1}{n-1} - \frac{1}{n+1} - \frac{1}{n-1} \right], & n \text{ is odd } (n \neq 1) \\ -\frac{1}{2\pi} \left[ \frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} + \frac{1}{n-1} \right], & n \text{ is even} \end{cases} = \begin{cases} 0, & n \text{ is odd } (n \neq 1) \\ \frac{-2n}{\pi(n^2-1)}, & n \text{ is even} \end{cases}$$

$$b_1 = \frac{1}{\pi} \left[ \int_{-\pi}^0 \cos x \sin x \, dx + \int_0^{\pi} \sin x \sin x \, dx \right]$$

$$= \frac{1}{\pi} \left[ \frac{1}{2} \int_{-\pi}^0 \sin 2x \, dx + \int_0^{\pi} \sin^2 x \, dx \right]$$

$$= \frac{1}{\pi} \left[ \frac{1}{2} \int_{-\pi}^0 \sin 2x \, dx + \frac{1}{2} \int_0^{\pi} (1 - \cos 2x) \, dx \right]$$

$$= -\frac{1}{2\pi} \left[ \frac{\cos 2x}{2} \right]_{-\pi}^0 + \frac{1}{2\pi} \left[ x - \frac{\sin 2x}{2} \right]_0^{\pi}$$

$$= -\frac{1}{4\pi} [1 - 1] + \frac{1}{2\pi} [\pi - 0 - 0 + 0] = \frac{1}{2}$$

Substituting values of  $a_0, a_n, b_n$  in ①

$$f(x) \approx \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi} \left[ \frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \dots \right] + \frac{1}{2} - \frac{2}{\pi} \left[ \frac{2\sin 2x}{2^2-1} + \frac{4\sin 4x}{4^2-1} + \dots \right]$$

$$\Rightarrow f(x) \approx \frac{1}{\pi} + 1 - \frac{2}{\pi} \left[ \frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \dots \right] - \frac{2}{\pi} \left[ \frac{2\sin 2x}{2^2-1} + \frac{4\sin 4x}{4^2-1} + \dots \right]$$

**Example 9** If  $f(x+2\pi) = f(x)$ , find the Fourier expansion  $f(x) = e^{ax}$  in the interval  $[-\pi, \pi]$

**Solution:**  $f(x) = e^{ax}$  is integrable and piecewise continuous in the interval  $[-\pi, \pi]$ .

$\therefore f(x)$  can be expanded into Fourier series given by:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \dots\dots \textcircled{1}$$

$$a_0 = \frac{1}{\pi} \int_C^{C+2\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} dx$$

$$= \frac{1}{a\pi} [e^{ax}]_{-\pi}^{\pi} = \frac{1}{a\pi} [e^{a\pi} - e^{-a\pi}] = \frac{2}{a\pi} \sinh a\pi \quad \because \frac{e^x - e^{-x}}{2} = \sinh x$$

$$a_n = \frac{1}{\pi} \int_C^{C+2\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \cos nx dx$$

$$= \frac{1}{\pi(a^2+n^2)} [e^{ax} (a \cos nx + n \sin nx)]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(a^2+n^2)} [e^{a\pi} (a \cos n\pi + n \sin n\pi) - e^{-a\pi} (a \cos n\pi - n \sin n\pi)]$$

$$= \frac{a(-1)^n}{\pi(a^2+n^2)} [e^{a\pi} - e^{-a\pi}] = \frac{2a(-1)^n}{\pi(a^2+n^2)} \sinh a\pi$$

$$b_n = \frac{1}{\pi} \int_C^{C+2\pi} f(x) \sin nx dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \sin nx dx$$

$$= \frac{1}{\pi(a^2+n^2)} [e^{ax} (a \sin nx - n \cos nx)]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(a^2+n^2)} [e^{a\pi} (a \sin n\pi - n \cos n\pi) - e^{-a\pi} (a \sin n\pi - n \cos n\pi)]$$

$$= \frac{-n(-1)^n}{\pi(a^2+n^2)} [e^{a\pi} - e^{-a\pi}] = \frac{2n(-1)^{n+1}}{\pi(a^2+n^2)} \sinh a\pi$$

Substituting values of  $a_0, a_n, b_n$  in ①

$$f(x) \approx \frac{\sinh a\pi}{\pi} \left[ \frac{1}{a} + 2a \left[ -\frac{\cos x}{(a^2+1^2)} + \frac{\cos 2x}{(a^2+2^2)} - \frac{\cos 3x}{(a^2+3^2)} + \dots \right] + 2 \left[ \frac{\sin x}{(a^2+1^2)} - \frac{2\sin 2x}{(a^2+2^2)} + \right. \right.$$

$$\left. \left. 3\sin 3x + 32 - \dots \right] \right]$$

### 1.2.1 Determination of Function Values at the Points of Discontinuity

A function satisfying Dirichlet's conditions may be expanded into Fourier series if it is discontinuous at a finite number of points.

Let the function be defined in  $(a, b)$  as

$$f(x) = \begin{cases} f_1(x), & a < x < x_o \\ f_2(x), & x_o < x < b \end{cases}$$

**1. To find  $f(x)$  at  $x = a$  or  $x = b$  (End points discontinuity)**

Since  $f(a)$  and  $f(b)$  are not defined in the interval  $(a, b)$

$$\begin{aligned} \therefore f(a) = f(b) &= \frac{1}{2}[(RHL \text{ at } x = a) + (LHL \text{ at } x = b)] \\ &= \frac{1}{2} \left[ \lim_{x \rightarrow a^+} f(x) + \lim_{x \rightarrow b^-} f(x) \right] \end{aligned}$$

**2. To find  $f(x)$  at  $x = x_o$  (Mid point discontinuity)**

Since  $f(x_o)$  is not defined in the interval  $(a, b)$

$$\begin{aligned} \therefore f(x_o) &= \frac{1}{2}[(LHL \text{ at } x = x_o) + (RHL \text{ at } x = x_o)] \\ &= \frac{1}{2} \left[ \lim_{x \rightarrow x_o^-} f(x) + \lim_{x \rightarrow x_o^+} f(x) \right] \end{aligned}$$

**Example 10** If  $f(x + 2\pi) = f(x)$ , find the Fourier series expansion of

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Hence or otherwise prove that  $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}$

**Solution:**  $f(x)$  is integrable and piecewise continuous in the interval  $(-\pi, \pi)$ .

$\therefore f(x)$  can be expanded into Fourier series given by:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \dots \dots \textcircled{1}$$

$$a_0 = \frac{1}{\pi} \int_C^{C+2\pi} f(x) dx = \frac{1}{\pi} \left[ \int_{-\pi}^0 -\pi dx + \int_0^{\pi} x dx \right] = \frac{1}{\pi} \left[ \frac{x^2}{2} \right]_0^{\pi} = -\frac{\pi}{2}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_C^{C+2\pi} f(x) \cos nx dx \\ &= \frac{1}{\pi} \left[ \int_{-\pi}^0 -\pi \cos nx dx + \int_0^{\pi} x \cos nx dx \right] \\ &= \frac{-\pi}{\pi} \left[ \frac{\sin nx}{n} \right]_{-\pi}^0 + \frac{1}{\pi} \left[ (x) \left( \frac{\sin nx}{n} \right) - (1) \left( \frac{-\cos nx}{n^2} \right) \right]_0^{\pi} \\ &= 0 + \frac{1}{\pi} \left[ \frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi} \\ &= \frac{1}{\pi} \left[ \frac{(-1)^n}{n^2} - \frac{1}{n^2} \right] \quad \because \sin nx = 0 \text{ when } x = 0 \text{ or } x = \pi \\ &= \frac{1}{\pi n^2} [(-1)^n - 1] = \begin{cases} \frac{-2}{\pi n^2} & , n \text{ is odd} \\ 0 & , n \text{ is even} \end{cases} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_C^{C+2\pi} f(x) \sin nx dx \\ &= \frac{1}{\pi} \left[ \int_{-\pi}^0 -\pi \sin nx dx + \int_0^{\pi} x \sin nx dx \right] \\ &= \frac{\pi}{\pi} \left[ \frac{\cos nx}{n} \right]_{-\pi}^0 + \frac{1}{\pi} \left[ (x) \left( \frac{-\cos nx}{n} \right) - (1) \left( \frac{-\sin nx}{n^2} \right) \right]_0^{\pi} \end{aligned}$$

$$\begin{aligned}
&= \left[ \frac{\cos nx}{n} \right]_{-\pi}^0 + \frac{1}{\pi} \left[ -\frac{x \cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^\pi \\
&= \left[ \frac{1}{n} - \frac{(-1)^n}{n} \right] - \frac{1}{\pi} \left[ \frac{\pi(-1)^n}{n} \right] \quad \because \frac{\sin nx}{n^2} = 0 \text{ when } x = 0 \text{ or } x = \pi \\
&= \frac{1}{n} [1 - 2(-1)^n] = \begin{cases} \frac{3}{n}, & n \text{ is odd} \\ -\frac{1}{n}, & n \text{ is even} \end{cases}
\end{aligned}$$

Substituting values of  $a_0, a_n, b_n$  in ①

$$f(x) \approx -\frac{\pi}{4} - \frac{2}{\pi} \left[ \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right] + \left[ \frac{3 \sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots \right]$$

Putting  $x = 0$  on both sides

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi} \left[ \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right] + 0 \dots \textcircled{2}$$

Since  $f(0)$  is not defined in the interval  $(-\pi, \pi)$

$$\begin{aligned}
\therefore f(0) &= \frac{1}{2} [(LHL \text{ at } x = 0) + (RHL \text{ at } x = 0)] \\
&= \frac{1}{2} \left[ \lim_{x \rightarrow 0^-} f(x) + \lim_{x \rightarrow 0^+} f(x) \right] \\
&= \frac{1}{2} \left[ \lim_{h \rightarrow 0} f(0 - h) + \lim_{h \rightarrow 0} f(0 + h) \right] \\
&= \frac{1}{2} [-\pi + 0] = -\frac{\pi}{2} \dots \textcircled{3}
\end{aligned}$$

Using ③ in ②, we get

$$\begin{aligned}
-\frac{\pi}{2} &= -\frac{\pi}{4} - \frac{2}{\pi} \left[ 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right] \\
\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots &= \frac{\pi^2}{8}
\end{aligned}$$

**Example 11** If  $f(x + 2\pi) = f(x)$ , find the Fourier series expansion of  $f(x) = x + x^2$  in the interval  $(-\pi, \pi)$

Hence or otherwise prove that  $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{6}$

**Solution:**  $f(x) = x + x^2$  is integrable and piecewise continuous in the interval  $(-\pi, \pi)$

$\therefore f(x)$  can be expanded into Fourier series given by:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \dots \textcircled{1}$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi} \int_C^{C+2\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) dx \\
&= \frac{1}{\pi} \left[ \frac{x^2}{2} + \frac{x^3}{3} \right]_{-\pi}^{\pi} = \frac{1}{\pi} \left[ \frac{2\pi^3}{3} \right] = \frac{2\pi^2}{3}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_C^{C+2\pi} f(x) \cos nx dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \cos nx dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\pi} \left[ (x + x^2) \left( \frac{\sin nx}{n} \right) - (1 + 2x) \left( \frac{-\cos nx}{n^2} \right) + (2) \left( \frac{-\sin nx}{n^3} \right) \right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi n^2} [(1 + 2x) \cos nx]_{-\pi}^{\pi} \because \sin nx = 0 \text{ when } x = -\pi \text{ or } x = \pi \\
&= \frac{1}{\pi n^2} [(1 + 2\pi) \cos n\pi - (1 - 2\pi) \cos n\pi] \because \cos(-n\pi) = \cos n\pi \\
&= \frac{1}{\pi n^2} [4\pi \cos n\pi] = \frac{4(-1)^n}{n^2} = \begin{cases} \frac{-4}{n^2} & , n \text{ is odd} \\ \frac{4}{n^2} & , n \text{ is even} \end{cases} \\
b_n &= \frac{1}{\pi} \int_C^{C+2\pi} f(x) \sin nx \, dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \sin nx \, dx \\
&= \frac{1}{\pi} \left[ (x + x^2) \left( \frac{-\cos nx}{n} \right) - (1 + 2x) \left( \frac{-\sin nx}{n^2} \right) + (2) \left( \frac{\cos nx}{n^3} \right) \right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi} \left[ (x + x^2) \left( \frac{-\cos nx}{n} \right) + (2) \left( \frac{\cos nx}{n^3} \right) \right]_{-\pi}^{\pi} \because \sin nx = 0 \text{ when } x = -\pi \text{ or } x = \pi \\
&= \frac{1}{\pi} \left[ (\pi + \pi^2) \left( \frac{-\cos n\pi}{n} \right) + \left( \frac{2 \cos n\pi}{n^3} \right) - (-\pi + \pi^2) \left( \frac{-\cos n\pi}{n} \right) - \left( \frac{2 \cos n\pi}{n^3} \right) \right] \\
&\qquad\qquad\qquad \because \cos(-n\pi) = \cos n\pi \\
&= \frac{-1}{\pi n} [2\pi \cos n\pi] = \frac{-2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n} = \begin{cases} \frac{2}{n} & , n \text{ is odd} \\ \frac{-2}{n} & , n \text{ is even} \end{cases}
\end{aligned}$$

Substituting values of  $a_0, a_n, b_n$  in ①

$$f(x) \approx \frac{\pi^2}{3} + 4 \left[ -\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2} + \dots \right] + 2 \left[ \frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right]$$

Putting  $x = \pi$  on both sides

$$f(\pi) \approx \frac{\pi^2}{3} + 4 \left[ -\frac{\cos \pi}{1^2} + \frac{\cos 2\pi}{2^2} - \frac{\cos 3\pi}{3^2} + \frac{\cos 4\pi}{4^2} + \dots \right] + 0 \dots \text{②}$$

Now since  $f(\pi)$  is not defined in the interval  $(-\pi, \pi)$

$$\begin{aligned}
\therefore f(\pi) &= \frac{1}{2} [(LHL \text{ at } x = \pi) + (RHL \text{ at } x = -\pi)] \\
&= \frac{1}{2} \left[ \lim_{x \rightarrow \pi^-} f(x) + \lim_{x \rightarrow -\pi^+} f(x) \right] \\
&= \frac{1}{2} \left[ \lim_{h \rightarrow 0} f(\pi - h) + \lim_{h \rightarrow 0} f(-\pi + h) \right] \\
&= \frac{1}{2} \left[ \lim_{h \rightarrow 0} \{ (\pi - h) + (\pi - h)^2 + (-\pi + h) + (-\pi + h)^2 \} \right] \\
\Rightarrow f(\pi) &= \frac{1}{2} [\pi + \pi^2 - \pi + \pi^2] = \pi^2 \dots \text{③}
\end{aligned}$$

Using ③ in ②, we get

$$\pi^2 = \frac{\pi^2}{3} + 4 \left[ 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots \right]$$

$$\Rightarrow 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{6}$$

### 1.3 Fourier Series for Arbitrary Period Length

Let  $f(x)$  be a periodic function defined in the interval  $[C, C + 2l]$ , then

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \dots\dots \textcircled{1}$$

$$a_0 = \frac{1}{l} \int_c^{c+2l} f(x) dx$$

$$a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos \frac{n\pi x}{l} dx$$

$$b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin \frac{n\pi x}{l} dx$$

**Note:** If the interval length is  $2\pi$ , putting  $2l = 2\pi$  i.e.  $l = \pi$ , then  $\textcircled{1}$  may be rewritten as  $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ , which is Fourier series expansion in the interval  $[C, C + 2\pi]$ .

$$\text{Also } a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx dx$$

**Example 12** If  $f(x + 10) = f(x)$ , find the Fourier series expansion of the function

$$f(x) = \begin{cases} 0, & -5 \leq x \leq 0 \\ 3, & 0 \leq x \leq 5 \end{cases}$$

**Solution:** Let  $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

Here interval is  $[-5, 5]$ ,  $\therefore 2l = 10 \Rightarrow l = 5$

Putting  $l = 5$  in  $\textcircled{1}$

$$f(x) \approx \frac{a_0}{2} + \sum a_n \cos \frac{n\pi x}{5} + \sum b_n \sin \frac{n\pi x}{5} \dots\dots \textcircled{1}$$

$$a_0 = \frac{1}{l} \int_c^{c+2l} f(x) dx = \frac{1}{5} \int_{-5}^5 f(x) dx = \frac{1}{5} \int_{-5}^0 0 dx + \frac{1}{5} \int_0^5 3 dx = \frac{3}{5} [x]_0^5 = 3$$

$$a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos \frac{n\pi x}{l} dx$$

$$= \frac{1}{5} \int_{-5}^5 f(x) \cos \frac{n\pi x}{5} dx$$

$$= \frac{1}{5} \int_{-5}^0 0 \cos \frac{n\pi x}{l} dx + \frac{1}{5} \int_0^5 3 \cos \frac{n\pi x}{5} dx = 0 + \frac{3}{5} \left[ \frac{5}{n\pi} \sin \frac{n\pi x}{5} \right]_0^5 = 0$$

$$b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{1}{5} \int_{-5}^5 f(x) \sin \frac{n\pi x}{5} dx$$

$$\begin{aligned}
&= \frac{1}{5} \int_{-5}^0 0 \sin \frac{n\pi x}{l} dx + \frac{1}{5} \int_0^5 3 \sin \frac{n\pi x}{5} dx \\
&= 0 - \frac{3}{5} \left[ \frac{5}{n\pi} \cos \frac{n\pi x}{5} \right]_0^5 = -\frac{3}{n\pi} [\cos n\pi - \cos 0] \\
&= -\frac{3}{n\pi} [(-1)^n - 1] = \begin{cases} \frac{6}{n\pi}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}
\end{aligned}$$

Substituting values of  $a_0, a_n, b_n$  in ①

$$\begin{aligned}
f(x) &\approx \frac{3}{2} + \frac{6}{\pi} \left[ \frac{\sin \frac{\pi x}{5}}{1} + \frac{\sin \frac{3\pi x}{5}}{3} + \frac{\sin \frac{5\pi x}{5}}{5} + \dots \right] \\
\Rightarrow f(x) &\approx \frac{3}{2} + \frac{6}{\pi} \left[ \sin \frac{\pi x}{5} + \frac{1}{3} \sin \frac{3\pi x}{5} + \frac{1}{5} \sin \pi x + \dots \right]
\end{aligned}$$

**Example 13** If  $f(x+2) = f(x)$ , find the Fourier series expansion of the function

$$f(x) = \begin{cases} \pi x, & 0 \leq x \leq 1 \\ \pi(2-x), & 1 \leq x \leq 2 \end{cases}$$

**Solution:** Let  $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

Here interval is  $[0,2]$ ,  $\therefore 2l = 2 \Rightarrow l = 1$

Putting  $l = 1$  in ①

$$f(x) \approx \frac{a_0}{2} + \sum a_n \cos n\pi x + \sum b_n \sin n\pi x \dots\dots ①$$

$$\text{Now } a_0 = \frac{1}{l} \int_c^{c+2l} f(x) dx = \int_0^2 f(x) dx = \int_0^1 \pi x dx + \int_1^2 \pi(2-x) dx$$

$$= \pi \left[ \frac{x^2}{2} \right]_0^1 + \pi \left[ 2x - \frac{x^2}{2} \right]_1^2 = \frac{\pi}{2} + \pi \left[ 4 - 2 - 2 + \frac{1}{2} \right] = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos \frac{n\pi x}{l} dx = \int_0^2 f(x) \cos n\pi x dx$$

$$= \int_0^1 \pi x \cos n\pi x dx + \int_1^2 \pi(2-x) \cos n\pi x dx$$

$$= \pi \left[ (x) \left( \frac{\sin n\pi x}{n\pi} \right) - (1) \left( \frac{-\cos n\pi x}{n^2\pi^2} \right) \right]_0^1 + \pi \left[ (2-x) \left( \frac{\sin n\pi x}{n\pi} \right) - (-1) \left( \frac{-\cos n\pi x}{n^2\pi^2} \right) \right]_1^2$$

$$= \pi \left[ \frac{x \sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2} \right]_0^1 + \pi \left[ (2-x) \left( \frac{\sin n\pi x}{n\pi} \right) - \frac{\cos n\pi x}{n^2\pi^2} \right]_1^2$$

$$= \pi \left[ 0 + \frac{(-1)^n}{n^2\pi^2} - 0 - \frac{1}{n^2\pi^2} \right] + \pi \left[ 0 - \frac{1}{n^2\pi^2} - 0 + \frac{(-1)^n}{n^2\pi^2} \right]$$

$$= \frac{2\pi}{n^2\pi^2} [(-1)^n - 1] = \begin{cases} \frac{-4}{n^2\pi}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin \frac{n\pi x}{l} dx$$

$$= \int_0^2 f(x) \sin n\pi x dx$$

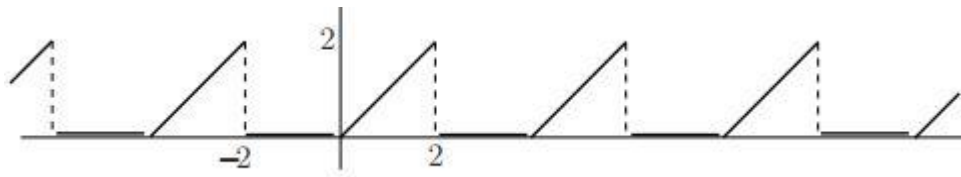
$$= \int_0^1 \pi x \sin n\pi x dx + \int_1^2 \pi(2-x) \sin n\pi x dx$$

$$\begin{aligned}
&= \pi \left[ (x) \left( -\frac{\cos n\pi x}{n\pi} \right) - (1) \left( \frac{-\sin n\pi x}{n^2\pi^2} \right) \right]_0^1 + \pi \left[ (2-x) \left( -\frac{\cos n\pi x}{n\pi} \right) - (-1) \left( \frac{-\sin n\pi x}{n^2\pi^2} \right) \right]_1^2 \\
&= \pi \left[ -\frac{x \cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2} \right]_0^1 - \pi \left[ (2-x) \left( \frac{\cos n\pi x}{n\pi} \right) + \frac{\sin n\pi x}{n^2\pi^2} \right]_1^2 \\
&= \pi \left[ -\frac{(-1)^n}{n\pi} + 0 + 0 + 0 \right] - \pi \left[ 0 + 0 - \frac{(-1)^n}{n\pi} + 0 \right] \\
&= \frac{\pi}{n\pi} [ -(-1)^n + (-1)^n ] = 0
\end{aligned}$$

Substituting values of  $a_0, a_n, b_n$  in ①

$$f(x) \approx \frac{\pi}{2} - \frac{4}{\pi} \left[ \frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \dots \right]$$

**Example 14** Find the Fourier series expansion of the periodic function shown by the graph given below in the interval  $(-2, 2)$



**Solution:** From the graph  $f(x) = \begin{cases} 0, & -2 < x < 0 \\ x, & 0 < x < 2 \end{cases}$

Clearly  $f(x)$  is integrable and piecewise continuous in the interval  $(-2, 2)$

$\therefore f(x)$  can be expanded into Fourier series given by:

$$\text{Let } f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Here interval is  $(-2, 2)$ ,  $\therefore 2l = 4 \Rightarrow l = 2$

Putting  $l = 2$  in ①

$$f(x) \approx \frac{a_0}{2} + \sum a_n \cos \frac{n\pi x}{2} + \sum b_n \sin \frac{n\pi x}{2} \dots\dots ①$$

$$a_0 = \frac{1}{l} \int_c^{c+2l} f(x) dx = \frac{1}{2} \int_{-2}^2 f(x) dx = \frac{1}{2} \int_{-2}^0 0 dx + \frac{1}{2} \int_0^2 x dx = \frac{1}{4} [x^2]_0^2 = 1$$

$$\begin{aligned}
a_n &= \frac{1}{l} \int_c^{c+2l} f(x) \cos \frac{n\pi x}{l} dx = \frac{1}{2} \int_{-2}^2 f(x) \cos \frac{n\pi x}{2} dx \\
&= \frac{1}{2} \int_{-2}^0 0 \cos \frac{n\pi x}{2} dx + \frac{1}{2} \int_0^2 x \cos \frac{n\pi x}{2} dx \\
&= 0 + \frac{1}{2} \left[ (x) \left( \frac{2}{n\pi} \sin \frac{n\pi x}{2} \right) - (1) \left( -\frac{4}{n^2\pi^2} \cos \frac{n\pi x}{2} \right) \right]_0^2 \\
&= \frac{2}{n^2\pi^2} \left[ \cos \frac{n\pi x}{2} \right]_0^2 = \frac{2}{n^2\pi^2} [(-1)^n - 1] = \begin{cases} \frac{-4}{n^2\pi^2}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{l} \int_c^{c+2l} f(x) \sin \frac{n\pi x}{l} dx = \frac{1}{2} \int_{-2}^2 f(x) \sin \frac{n\pi x}{2} dx \\
&= \frac{1}{2} \int_{-2}^0 0 \sin \frac{n\pi x}{2} dx + \frac{1}{2} \int_0^2 x \sin \frac{n\pi x}{2} dx
\end{aligned}$$

$$= 0 + \frac{1}{2} \left[ (x) \left( -\frac{2}{n\pi} \cos \frac{n\pi x}{2} \right) - (1) \left( -\frac{4}{n^2 \pi^2} \sin \frac{n\pi x}{2} \right) \right]_0^2$$

$$= \frac{-1}{n\pi} \left[ x \cos \frac{n\pi x}{2} \right]_0^2 = \frac{-1}{n\pi} [2(-1)^n] = \frac{2(-1)^{n+1}}{n\pi} = \begin{cases} \frac{2}{n\pi}, & n \text{ is odd} \\ \frac{-2}{n\pi}, & n \text{ is even} \end{cases}$$

Substituting values of  $a_0, a_n, b_n$  in ①

$$f(x) \approx \frac{1}{2} - \frac{4}{\pi^2} \left[ \frac{\cos \frac{\pi x}{2}}{1^2} + \frac{\cos \frac{3\pi x}{2}}{3^2} + \frac{\cos \frac{5\pi x}{2}}{5^2} + \dots \right] + \frac{2}{\pi} \left[ \frac{\sin \frac{\pi x}{2}}{1} - \frac{\sin \frac{2\pi x}{2}}{2} + \frac{\sin \frac{3\pi x}{2}}{3} - \frac{\sin \frac{4\pi x}{2}}{4} + \dots \right]$$

$$\Rightarrow f(x) \approx \frac{1}{2} - \frac{4}{\pi^2} \left[ \cos \frac{\pi x}{2} + \frac{1}{9} \cos \frac{3\pi x}{2} + \frac{1}{25} \cos \frac{5\pi x}{2} + \dots \right] + \frac{2}{\pi} \left[ \sin \frac{\pi x}{2} - \frac{1}{2} \sin \pi x + \frac{1}{3} \sin \frac{3\pi x}{2} \dots \right]$$

## 1.4 Fourier Series Expansion of Even Odd Functions

Computational procedure of Fourier series can be reduced to great extent, once a function is identified to be even or odd in an interval  $(-l, l)$  Characteristics of even/odd functions are given below:

|    | Even Function                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       | Odd Function                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |
|----|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 1. | A function $f(x)$ is said to be even if $f(-x) = f(x)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              | A function $f(x)$ is said to be odd if $f(-x) = -f(x)$                                                                                                                                                                                                                                                                                                                                                                                                                                                 |
| 2. | Graph of an even function is symmetrical about y -axis                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              | Graph of an odd function is symmetrical about origin                                                                                                                                                                                                                                                                                                                                                                                                                                                   |
| 3. | For an even function<br>$\int_{-c}^c f(x) dx = 2 \int_0^c f(x) dx$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  | For an odd function<br>$\int_{-c}^c f(x) dx = 0$                                                                                                                                                                                                                                                                                                                                                                                                                                                       |
| 4. | Product of two even functions is even.<br>Product of two odd functions is even                                                                                                                                                                                                                                                                                                                                                                                                                                                                      | Product of an even and an odd function is odd.                                                                                                                                                                                                                                                                                                                                                                                                                                                         |
| 5. | Fourier coefficients of an even function are:<br>$a_0 = \frac{1}{l} \int_{-c}^c f(x) dx = \frac{2}{l} \int_0^c f(x) dx$<br><br>$a_n = \frac{1}{l} \int_{-c}^c f(x) \cos \frac{n\pi x}{l} dx$<br>$= \frac{2}{l} \int_0^c f(x) \cos \frac{n\pi x}{l} dx$<br>as $f(x)$ and $\cos \frac{n\pi x}{l}$ both are even.<br><br>$b_n = \frac{1}{l} \int_{-c}^{c+2l} f(x) \sin \frac{n\pi x}{l} dx = 0$<br>as $f(x)$ is even and $\sin \frac{n\pi x}{l}$ is odd<br><br>$\therefore f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$ | Fourier coefficients of an odd function are:<br>$a_0 = \frac{1}{l} \int_{-c}^c f(x) dx = 0$<br><br>$a_n = \frac{1}{l} \int_{-c}^c f(x) \cos \frac{n\pi x}{l} dx = 0$<br>as $f(x)$ is odd and $\cos \frac{n\pi x}{l}$ is even.<br><br>$b_n = \frac{1}{l} \int_{-c}^{c+2l} f(x) \sin \frac{n\pi x}{l} dx$<br>$= \frac{2}{l} \int_0^c f(x) \sin \frac{n\pi x}{l} dx$<br>as $f(x)$ and $\sin \frac{n\pi x}{l}$ both are odd<br><br>$\therefore f(x) \approx \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ |

**Note:** Properties of Even or Odd Function comply only if interval is  $(-l, l)$  and any function in  $(0, 2l)$  does not follow the properties of even/odd functions. For example for the function  $f(x) = x^2$  in  $(0, 2\pi)$ , Fourier coefficients  $a_0, a_n, b_n$  do not follow above given rules of even/odd functions.

**Example 15** Obtain Fourier series expansion for the function  $f(x) = x^3$  in the interval  $(-\pi, \pi)$ , if  $f(x + 2\pi) = f(x)$

**Solution:**  $f(x) = x^3$  is integrable and piecewise continuous in the interval  $(-\pi, \pi)$  and also  $f(x)$  is an odd function of  $x$ .

$\therefore a_0 = a_n = 0$ ,  $f(x)$  can be expanded into Fourier series given by:

$$f(x) \approx \sum_{n=1}^{\infty} b_n \sin nx \dots\dots \textcircled{1}$$

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} x^3 \sin nx \, dx \\ &= \frac{2}{\pi} \left[ (x^3) \left( \frac{-\cos nx}{n} \right) - (3x^2) \left( \frac{-\sin nx}{n^2} \right) + (6x) \left( \frac{\cos nx}{n^3} \right) - (6) \left( \frac{\sin nx}{n^4} \right) \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[ (x^3) \left( \frac{-\cos nx}{n} \right) + (6x) \left( \frac{\cos nx}{n^3} \right) \right]_0^{\pi} \because \sin nx = 0 \text{ when } x = \pi \text{ or } 0 \\ &= \frac{2}{\pi} \left[ (\pi^3) \left( \frac{-\cos n\pi}{n} \right) + (6\pi) \left( \frac{\cos n\pi}{n^3} \right) \right] \\ &= \frac{2}{\pi} \left[ (\pi^3) \left( \frac{-(-1)^n}{n} \right) + (6\pi) \left( \frac{(-1)^n}{n^3} \right) \right] \\ &= 2(-1)^n \left[ -\frac{\pi^2}{n} + \frac{6}{n^3} \right] \\ \therefore f(x) &\approx 2 \left[ -\left( \frac{-\pi^2}{1} + \frac{6}{1^3} \right) \right] \sin x + \left( \frac{-\pi^2}{2} + \frac{6}{2^3} \right) \sin 2x - \left( \frac{-\pi^2}{3} + \frac{6}{3^3} \right) \sin 3x + \dots \end{aligned}$$

**Example 16** If  $f(x + 2\pi) = f(x)$ , obtain Fourier series expansion for the function given by

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \leq x \leq 0 \\ 1 - \frac{2x}{\pi}, & 0 \leq x \leq \pi \end{cases}$$

Hence or otherwise prove that  $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

$$\textbf{Solution: } f(-x) = \begin{cases} 1 - \frac{2x}{\pi}, & -\pi \leq -x \leq 0 \\ 1 + \frac{2x}{\pi}, & 0 \leq -x \leq \pi \end{cases} = \begin{cases} 1 - \frac{2x}{\pi}, & \pi \geq x \geq 0 \\ 1 + \frac{2x}{\pi}, & 0 \geq x \geq -\pi \end{cases} = f(x)$$

$f(-x) = f(x) \therefore f(x)$  is even function of  $x$

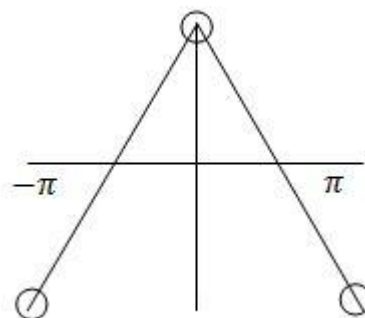
**Note:**  $f(-\pi) = -1$ ,  $f(0) = 1$ ,  $f(\pi) = -1$

Plotting graph of  $f(x)$ , we observe that it is symmetrical about y-axis as shown in given figure. Therefore the function is even.

Hence  $b_n = 0$ ,  $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \dots\dots \textcircled{1}$

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} f(x) \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} \left( 1 - \frac{2x}{\pi} \right) \, dx = \frac{2}{\pi} \left[ x - \frac{x^2}{\pi} \right]_0^{\pi} = \frac{2}{\pi} [\pi - \pi] = 0 \end{aligned}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$



$$\begin{aligned}
&= \frac{2}{\pi} \int_0^\pi \left(1 - \frac{2x}{\pi}\right) \cos nx \, dx \\
&= \frac{2}{\pi} \left[ \left(1 - \frac{2x}{\pi}\right) \left(\frac{\sin nx}{n}\right) - \left(\frac{-2}{\pi}\right) \left(\frac{-\cos nx}{n^2}\right) \right]_0^\pi \\
&= -\frac{4}{\pi^2 n^2} [\cos n\pi]_0^\pi = \frac{-4}{\pi^2 n^2} [(-1)^n - 1] = \begin{cases} \frac{8}{\pi^2 n^2}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}
\end{aligned}$$

Substituting values of  $a_0$  and  $a_n$  in ①

$$f(x) \approx \frac{8}{\pi^2} \left[ \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} \right]$$

Putting  $x = 0$  on both sides

$$\begin{aligned}
1 &= \frac{8}{\pi^2} \left[ \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right] \\
\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots &= \frac{\pi^2}{8}
\end{aligned}$$

**Example 17** If  $f(x + 2c) = f(x)$ , find the Fourier series expansion of the function given by

$$f(x) = \begin{cases} -\sin \frac{\pi x}{c}, & -c \leq x \leq 0 \\ \sin \frac{\pi x}{c}, & 0 \leq x \leq c \end{cases}$$

**Solution:**  $f(-x) = \begin{cases} \sin \frac{\pi x}{c}, & -c \leq -x \leq 0 \\ -\sin \frac{\pi x}{c}, & 0 \leq -x \leq c \end{cases} = \begin{cases} \sin \frac{\pi x}{c}, & c \geq x \geq 0 \\ -\sin \frac{\pi x}{c}, & 0 \geq x \geq -c \end{cases} = f(x)$

$$f(-x) = f(x) \therefore f(x) \text{ is an even function of } x \Rightarrow b_n = 0$$

$$\text{Also } 2l = 2c \Rightarrow l = c$$

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c} \dots\dots \text{①}$$

$$a_0 = \frac{2}{c} \int_0^c f(x) dx = \frac{2}{c} \int_0^c \sin \frac{\pi x}{c} dx = -\frac{2}{c} \frac{c}{\pi} \left[ \cos \frac{\pi x}{c} \right]_0^c = -\frac{2}{\pi} [-1 - 1] = \frac{4}{\pi}$$

$$a_n = \frac{2}{c} \int_0^c \sin \frac{\pi x}{c} \cos \frac{n\pi x}{c} dx$$

$$= \frac{1}{c} \int_0^c \left[ \sin \frac{\pi x}{c} (n+1) - \sin \frac{\pi x}{c} (n-1) \right] dx$$

$$= \frac{1}{c} \left[ -\frac{c}{\pi(n+1)} \cos \frac{\pi x}{c} (n+1) + \frac{c}{\pi(n-1)} \cos \frac{\pi x}{c} (n-1) \right]_0^c, n \neq 1$$

$$= \frac{1}{\pi} \left[ -\frac{1}{(n+1)} \cos(n+1)\pi + \frac{1}{(n-1)} \cos(n-1)\pi + \frac{1}{(n+1)} - \frac{1}{(n-1)} \right]$$

$$= \frac{1}{\pi} \left[ -\frac{(-1)^{n+1}}{(n+1)} + \frac{(-1)^{n-1}}{(n-1)} + \frac{1}{(n+1)} - \frac{1}{(n-1)} \right]$$

$$= \begin{cases} \frac{1}{\pi} \left[ -\frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right], & n \text{ is odd } (n \neq 1) \\ \frac{1}{\pi} \left[ \frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right], & n \text{ is even} \end{cases} = \begin{cases} 0, & n \text{ is odd } (n \neq 1) \\ \frac{-4}{\pi(n^2-1)}, & n \text{ is even} \end{cases}$$

$$a_1 = \frac{2}{c} \int_0^c \sin \frac{\pi x}{c} \cos \frac{\pi x}{c} dx = \frac{1}{c} \int_0^c \sin \frac{2\pi x}{c} dx = \frac{1}{c} \left[ -\frac{c}{2\pi} \cos \frac{2\pi x}{c} \right]_0^c$$

$$= -\frac{1}{2\pi} [\cos 2\pi - \cos 0] = -\frac{1}{2\pi} [1 - 1] = 0$$

Substituting values of  $a_0, a_n$  in ①

$$f(x) \approx \frac{2}{\pi} - \frac{4}{\pi} \left[ \frac{\cos \frac{2\pi x}{c}}{2^2-1} + \frac{\cos \frac{4\pi x}{c}}{4^2-1} + \frac{\cos \frac{6\pi x}{c}}{6^2-1} + \dots \right]$$

**Example 18** If  $f(x + 2\pi) = f(x)$ , obtain Fourier series expansion for the function given by  $f(x) = |x|$  in the interval  $(-\pi, \pi)$

Hence or otherwise prove that  $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

**Solution:**  $f(-x) = |-x| = |x| = f(x)$

$f(-x) = f(x) \therefore f(x)$  is even function of  $x$ .

Rewriting  $f(x)$  as  $|x| = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 \leq x < \pi \end{cases}$

Being even function of,  $b_n = 0$ ,

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \dots\dots ①$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} x dx = \frac{2}{\pi} \left[ \frac{x^2}{2} \right]_0^{\pi} = \frac{1}{\pi} [\pi^2] = \pi$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx dx$$

$$= \frac{2}{\pi} \left[ (x) \left( \frac{\sin nx}{n} \right) - (1) \left( \frac{-\cos nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi n^2} [\cos n\pi]_0^{\pi} = \frac{2}{\pi n^2} [(-1)^n - 1] = \begin{cases} \frac{-4}{\pi n^2}, n \text{ is odd} \\ 0, n \text{ is even} \end{cases}$$

Substituting values of  $a_0$  and  $a_n$  in ①

$$f(x) \approx \frac{\pi}{2} - \frac{4}{\pi} \left[ \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} \right]$$

Putting  $x = 0$  on both sides

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \left[ \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

**Example 19** Obtain Fourier series expansion for the function given by  $f(x) = |\sin x|$  in the interval  $(-\pi, \pi)$

**Solution:**  $f(x) = |\sin x|$  is even function of  $x$

$$\therefore b_n = 0, \quad f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \dots\dots ①$$

Rewriting  $f(x)$  as  $|\sin x| = \begin{cases} -\sin x, & -\pi < x < 0 \\ \sin x, & 0 \leq x < \pi \end{cases}$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$\begin{aligned}
&= \frac{2}{\pi} \int_0^{\pi} \sin x \, dx = \frac{-2}{\pi} [\cos x]_0^{\pi} = \frac{4}{\pi} \\
a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx \, dx \\
&= \frac{1}{\pi} \int_0^{\pi} (\sin(n+1)x - \sin(n-1)x) \, dx \\
&= \frac{1}{\pi} \left[ -\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right]_0^{\pi} \\
&= \frac{1}{\pi} \left[ -\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{\cos 0}{n+1} - \frac{\cos 0}{n-1} \right] \\
&= \frac{1}{\pi} \left[ -\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right] \\
&= \begin{cases} \frac{1}{\pi} \left[ -\frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right], & n \text{ is odd } (n \neq 1) \\ \frac{1}{\pi} \left[ \frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right], & n \text{ is even} \end{cases} = \begin{cases} 0, & n \text{ is odd } (n \neq 1) \\ \frac{-4}{\pi(n^2-1)}, & n \text{ is even} \end{cases} \\
a_1 &= \frac{2}{\pi} \int_0^{\pi} \sin x \cos x \, dx = \frac{1}{\pi} \int_0^{\pi} \sin 2x \, dx \\
&= \frac{-1}{2\pi} [\cos 2x]_0^{\pi} = -\frac{1}{2\pi} [\cos 2\pi - \cos 0] = -\frac{1}{2\pi} [1 - 1] = 0
\end{aligned}$$

Substituting values of  $a_0, a_n$  in ①

$$\begin{aligned}
f(x) &= \frac{2}{\pi} - \frac{4}{\pi} \left[ \frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \dots \right] \\
\Rightarrow f(x) &= \frac{2}{\pi} - \frac{4}{\pi} \left[ \frac{1}{3} \cos 2x + \frac{1}{15} \cos 4x + \frac{1}{35} \cos 6x + \dots \right]
\end{aligned}$$

**Example 20** Obtain Fourier series expansion for the function given by  $f(x) = |\cos x|$  in the interval  $(-\pi, \pi)$

**Solution:**  $f(x) = |\cos x|$  is even function of  $x$

$$\therefore b_n = 0, \quad f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \dots\dots \textcircled{1}$$

$$\text{Rewriting } f(x) \text{ as } |\cos x| = \begin{cases} -\cos x, & -\pi < x < -\frac{\pi}{2} \\ \cos x, & -\frac{\pi}{2} \leq x < \frac{\pi}{2} \\ -\cos x, & \frac{\pi}{2} \leq x < \pi \end{cases}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) \, dx$$

$$= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos x \, dx + \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} -\cos x \, dx = \frac{2}{\pi} [\sin x]_0^{\frac{\pi}{2}} - \frac{2}{\pi} [\sin x]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{2}{\pi} \left[ \sin \frac{\pi}{2} - \sin 0 - \sin \pi + \sin \frac{\pi}{2} \right] = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos x \cos nx \, dx + \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} -\cos x \cos nx \, dx$$

$$= \frac{1}{\pi} \int_0^{\frac{\pi}{2}} (\cos(n+1)x + \cos(n-1)x) \, dx - \frac{1}{\pi} \int_{\frac{\pi}{2}}^{\pi} (\cos(n+1)x + \cos(n-1)x) \, dx$$

$$\begin{aligned}
&= \frac{1}{\pi} \left[ \frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right]_0^{\frac{\pi}{2}} - \frac{1}{\pi} \left[ \frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right]_{\frac{\pi}{2}}^{\pi} \\
&= \frac{1}{\pi} \left[ \frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1} + \frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1} \right], (n \neq 1) \\
&= \frac{2}{\pi} \left[ \frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1} \right], (n \neq 1) \\
&= \begin{cases} 0, & n \text{ is odd } (n \neq 1) \\ \frac{2}{\pi} \left[ -\frac{1}{n+1} + \frac{1}{n-1} \right], & n = 2, 6, 10, \dots \\ \frac{2}{\pi} \left[ \frac{1}{n+1} - \frac{1}{n-1} \right], & n = 4, 8, 12, \dots \end{cases} = \begin{cases} 0, & n \text{ is odd } (n \neq 1) \\ \frac{4}{\pi(n^2-1)}, & n = 2, 6, 10, \dots \\ \frac{-4}{\pi(n^2-1)}, & n = 4, 8, 12, \dots \end{cases} \\
a_1 &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos x \, dx = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos x \cos x \, dx + \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} -\cos x \cos x \, dx \\
&= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos^2 x \, dx - \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} \cos^2 x \, dx \\
&= \frac{1}{\pi} \int_0^{\frac{\pi}{2}} (1 + \cos 2x) \, dx - \frac{1}{\pi} \int_{\frac{\pi}{2}}^{\pi} (1 + \cos 2x) \, dx \\
&= \frac{1}{\pi} \left[ x + \frac{\sin 2x}{2} \right]_0^{\frac{\pi}{2}} - \frac{1}{\pi} \left[ x + \frac{\sin 2x}{2} \right]_{\frac{\pi}{2}}^{\pi} \\
&= \frac{1}{\pi} \left[ \frac{\pi}{2} + \frac{\sin \pi}{2} - 0 - 0 - \pi - \frac{\sin 2\pi}{2} + \frac{\pi}{2} + \frac{\sin \pi}{2} \right] = 0
\end{aligned}$$

Substituting values of  $a_0, a_n$  in ①

$$\begin{aligned}
f(x) &= \frac{2}{\pi} + \frac{4}{\pi} \left[ \frac{\cos 2x}{2^2-1} - \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} - \dots \right] \\
\Rightarrow f(x) &= \frac{2}{\pi} + \frac{4}{\pi} \left[ \frac{1}{3} \cos 2x - \frac{1}{15} \cos 4x + \frac{1}{35} \cos 6x + \dots \right]
\end{aligned}$$

### Exercise 1A

1. If  $f(x + 2\pi) = f(x)$ , find the Fourier series expansion of the function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \frac{\pi x}{4}, & 0 < x < \pi \end{cases}$$

Hence or otherwise prove that  $1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

2. If  $f(x + 2\pi) = f(x)$ , find the Fourier series expansion of the function

$$f(x) = \begin{cases} -\pi - x, & \pi \leq x \leq 0 \\ \pi + x, & 0 \leq x \leq \pi \end{cases}$$

Hence or otherwise prove that  $1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

3. Find the Fourier series expansion of the function  $f(x) = x - x^2$  in  $(-\pi, \pi)$ .

Hence deduce that  $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$

4. Find the Fourier series expansion of the function  $f(x) = \frac{(\pi-x)^2}{4}$  in  $(0, 2\pi)$ .

5. If  $f(x + 2\pi) = f(x)$ , find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ \frac{1}{2}, & x = 0 \\ x, & 0 < x < 1 \end{cases}$$

6. Assuming  $f(x) = x$  to be periodic with a period  $2\pi$ , find Fourier series expansion of  $f(x)$  in the interval  $(-\pi, \pi)$ .

7. If  $f(x + 2\pi) = f(x)$ , find the Fourier series expansion of the function

$$f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}$$

$$\text{Hence deduce that } 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

8. If  $f(x + 1) = f(x)$ , find Fourier series expansion of  $f(x) = x$  in the interval  $(0,1)$ .

9. If  $f(x + 4) = f(x)$ , find Fourier series expansion of  $f(x) = |x|$  in the interval  $(-2,2)$ .

10. If  $f(x + 2) = f(x)$ , find the Fourier series expansion of the function

$$f(x) = \begin{cases} -2, & -1 < x < 0 \\ 2, & 0 < x < 1 \end{cases}$$

11. If  $f(x + 2\pi) = f(x)$ , find the Fourier series expansion of the function

$$f(x) = \begin{cases} \pi + x, & -\pi \leq x \leq -\frac{\pi}{2} \\ \frac{\pi}{2}, & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

12. find the Fourier series expansion of the function

$$f(x) = \begin{cases} \pi x, & 0 \leq x < 1 \\ 0, & x = 1 \\ \pi(x - 2), & 1 < x \leq 2 \end{cases}$$

$$\text{Hence deduce that } 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

## Answers

$$1. f(x) \approx \frac{\pi^2}{16} - \frac{1}{2} \left[ \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right] + \frac{\pi}{4} \left[ \frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right]$$

$$2. f(x) \approx \frac{\pi}{2} - \frac{4}{\pi} \left[ \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right] + 4 \left[ \frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots \right]$$

$$3. f(x) \approx \frac{-\pi^2}{3} + 4 \left[ \frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots \right] + 2 \left[ \frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right]$$

$$4. f(x) \approx \frac{\pi^2}{12} + \left[ \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \dots \right]$$

$$5. f(x) \approx \frac{3}{4} - \frac{2}{\pi^2} \left[ \frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \dots \right] - \frac{1}{\pi} \left[ \frac{\sin \pi x}{1} + \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} + \dots \right]$$

$$6. f(x) \approx 2 \left[ \frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right]$$

$$7. f(x) \approx \frac{4k}{\pi} \left[ \frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right]$$

$$8. f(x) \approx \frac{1}{2} - \frac{1}{\pi} \left[ \frac{\sin 2\pi x}{1} + \frac{\sin 4\pi x}{2} + \frac{\sin 6\pi x}{3} + \dots \right]$$

$$9. f(x) \approx 1 - \frac{8}{\pi^2} \left[ \frac{\cos \frac{\pi x}{2}}{1^2} + \frac{\cos \frac{3\pi x}{2}}{3^2} + \frac{\cos \frac{5\pi x}{2}}{5^2} + \dots \right]$$

$$10. f(x) \approx \frac{8}{\pi} \left[ \frac{\sin \pi x}{1} + \frac{\sin 3\pi x}{3} + \frac{\sin 5\pi x}{5} + \dots \right]$$

$$11. f(x) \approx \frac{3\pi}{8} + \frac{1}{\pi} \left[ \frac{2\cos x}{1^2} - \frac{\cos 2x}{1^2} + \frac{2\cos 3x}{3^2} - \frac{\cos 6x}{3^2} + \frac{2\cos 5x}{5^2} - \frac{\cos 10x}{5^2} + \dots \right]$$

$$12. f(x) \approx 2 \left[ \frac{\sin \pi x}{1} - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} + \dots \right]$$

## 1.6 Half Range Fourier Series in the Interval $(0, l)$

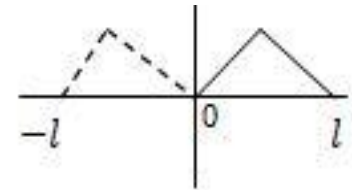
Sometimes it is required to expand a function  $f(x)$  in the range  $(0, \pi)$  or more generally in  $(0, l)$  into a sine series or cosine series.

If it is required to expand  $f(x)$  in  $(0, l)$ , it is immaterial what the function may be outside the range  $0 < x < l$ , we are free to choose the function in  $(-l, 0)$ .

### 1.6.1 Half Range Cosine Series

To develop into Cosine series, we extend  $f(x)$  in  $(-l, 0)$  by reflecting it in  $y$  - axis as shown in adjoining figure, so that  $f(-x) = f(x)$ , function becomes even function and  $b_n = 0$

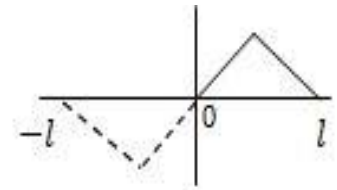
$$\therefore f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$



### 1.6.2 Half Range Sine Series

To develop into Sine series, we extend  $f(x)$  in  $(-l, 0)$ , by reflecting it in origin, so that  $f(-x) = -f(x)$ , function becomes odd function and  $a_0 = a_n = 0$

$$\therefore f(x) \approx \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$



**Example 21** Obtain half range Fourier Cosine series for  $f(x) = 2x - 1$  in the interval  $(0, 1)$ .

**Solution:** To develop  $f(x) = 2x - 1$  into Cosine series, extending  $f(x)$  in  $(-1, 0)$  by reflecting it in  $y$  - axis, so that  $f(-x) = f(x)$ , function becomes even function and  $b_n = 0$

$$\therefore f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \dots\dots \textcircled{1}$$

Here  $2l = 2 \therefore l = 1$

$$a_0 = \frac{2}{l} \int_0^l f(x) dx = 2 \int_0^1 (2x - 1) dx = 0$$

$$a_n = 2 \int_0^1 (2x - 1) \cos n\pi x dx$$

$$\begin{aligned}
&= 2 \left[ (2x-1) \left( \frac{\sin n\pi x}{n\pi} \right) - (2) \left( \frac{-\cos n\pi x}{n^2\pi^2} \right) \right]_0^1 \\
&= \frac{4}{n^2\pi^2} [\cos n\pi - \cos 0] \\
&= \frac{4}{n^2\pi^2} [(-1)^n - 1] = \begin{cases} \frac{-8}{n^2\pi^2}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}
\end{aligned}$$

Substituting values of  $a_0, a_n$  in ①

$$f(x) \approx -\frac{8}{\pi^2} \left[ \frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \dots \right]$$

**Example 22** Obtain half range Fourier Cosine series for  $f(x) = x \sin x$  in the interval  $(0, \pi)$  and show that  $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots = \frac{\pi-2}{4}$

**Solution:** To develop  $f(x) = x \sin x$  into Cosine series, extending  $f(x)$  in  $(-\pi, 0)$  by reflecting it in  $y$ -axis, so that  $f(-x) = f(x)$ , function becomes even function and  $b_n = 0$

$$\therefore f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \dots\dots \textcircled{1}$$

$$\begin{aligned}
a_0 &= \frac{2}{\pi} \int_0^{\pi} f(x) dx \\
&= \frac{2}{\pi} \int_0^{\pi} x \sin x dx = \frac{2}{\pi} [(x)(-\cos x) - (1)(-\sin x)]_0^{\pi} \\
&= \frac{-2}{\pi} [x \cos x]_0^{\pi} = 2 \\
a_n &= \frac{2}{\pi} \int_0^{\pi} x \sin x \cdot \cos nx dx \\
&= \frac{1}{\pi} \int_0^{\pi} x [\sin(n+1)x - \sin(n-1)x] dx \\
&= \frac{1}{\pi} \left[ (x) \left( \frac{-\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)} \right) - (1) \left( \frac{-\sin(n+1)}{(n+1)^2} + \frac{\sin(n-1)}{(n-1)^2} \right) \right]_0^{\pi} \\
&= \frac{1}{\pi} \left[ -\frac{x \cos(n+1)x}{(n+1)} + \frac{x \cos(n-1)x}{n-1} \right]_0^{\pi} \\
&= \frac{\pi}{\pi} \left[ -\frac{(-1)^{n+1}}{(n+1)} + \frac{(-1)^{n-1}}{(n-1)} \right] \\
&= (-1)^{n+1} \left[ \frac{n+1-n+1}{(n-1)(n+1)} \right] \quad \because (-1)^{n-1} = (-1)^{n+1} \\
&= \frac{2(-1)^{n+1}}{(n-1)(n+1)}, \quad n \neq 1
\end{aligned}$$

$$\begin{aligned}
a_1 &= \frac{2}{\pi} \int_0^{\pi} x \sin x \cos x dx = \frac{1}{\pi} \int_0^{\pi} x \sin 2x dx \\
&= \frac{1}{\pi} \left[ (x) \left( \frac{-\cos 2x}{2} \right) - (1) \left( \frac{-\sin 2x}{4} \right) \right]_0^{\pi} \\
&= \frac{-1}{2\pi} [x \cos 2x]_0^{\pi} = \frac{-1}{2\pi} [\pi] = -\frac{1}{2}
\end{aligned}$$

$$\therefore f(x) = 1 - \frac{1}{2} \cos x + 2 \left[ -\frac{\cos 2x}{1.3} + \frac{\cos 3x}{2.4} - \frac{\cos 4x}{3.5} + \frac{\cos 5x}{4.6} - \frac{\cos 6x}{5.7} + \dots \right]$$

Putting  $x = \frac{\pi}{2}$  on both sides

$$\begin{aligned}\frac{\pi}{2} &= 1 + 2 \left[ \frac{1}{1.3} + 0 - \frac{1}{3.5} + 0 + \frac{1}{5.7} + \dots \right] \\ \Rightarrow \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots &= \frac{\pi-2}{4}\end{aligned}$$

**Example 23** Obtain half range Fourier Sine and Cosine series for the function given by

$$f(x) = \begin{cases} x, & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

**Solution: Fourier Sine series**

To develop  $f(x)$  into Sine series, extending  $f(x)$  in  $(-\pi, 0)$  by reflecting in origin, so that  $f(-x) = -f(x)$ , function becomes odd function and  $a_0 = a_n = 0$

$$\therefore f(x) \approx \sum_{n=1}^{\infty} b_n \sin nx \dots\dots \textcircled{1}$$

$$\begin{aligned}b_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \\ &= \frac{2}{\pi} \left[ \int_0^{\frac{\pi}{2}} x \sin nx \, dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x) \sin nx \, dx \right] \\ &= \frac{2}{\pi} \left[ (x) \left( \frac{-\cos nx}{n} \right) - (1) \left( \frac{-\sin nx}{n^2} \right) \right]_0^{\frac{\pi}{2}} + \frac{2}{\pi} \left[ (\pi - x) \left( \frac{-\cos nx}{n} \right) - (-1) \left( \frac{-\sin nx}{n^2} \right) \right]_{\frac{\pi}{2}}^{\pi} \\ &= \frac{2}{\pi} \left[ -\frac{x \cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{\frac{\pi}{2}} + \frac{2}{\pi} \left[ -(\pi - x) \frac{\cos nx}{n} - \frac{\sin nx}{n^2} \right]_{\frac{\pi}{2}}^{\pi} \\ &= \frac{2}{\pi} \left[ -\frac{\pi}{2n} \cos \frac{n\pi}{2} + \frac{1}{n^2} \sin \frac{n\pi}{2} + \frac{\pi}{2n} \cos \frac{n\pi}{2} + \frac{1}{n^2} \sin \frac{n\pi}{2} \right] \\ &= \frac{2}{\pi} \left[ \frac{2}{n^2} \sin \frac{n\pi}{2} \right] = \frac{4}{\pi n^2} \sin \frac{n\pi}{2} = \begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{4}{\pi n^2}, & n = 1, 5, 9, \dots \\ \frac{-4}{\pi n^2}, & n = 3, 7, 11, \dots \end{cases}\end{aligned}$$

Substituting  $b_n$  in  $\textcircled{1}$

$$f(x) = \frac{4}{\pi} \left[ \sin x - \frac{1}{9} \sin 3x + \frac{1}{25} \sin 5x - \frac{1}{49} \sin 7x \dots \right]$$

**Fourier Cosine series**

To develop  $f(x)$  into Cosine series, extending  $f(x)$  in  $(-\pi, 0)$  by reflecting it in  $y$ -axis, so that  $f(-x) = f(x)$ , function becomes even function and  $b_n = 0$

$$\therefore f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \dots\dots \textcircled{2}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) \, dx = \frac{2}{\pi} \left[ \int_0^{\frac{\pi}{2}} x \, dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x) \, dx \right] = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$\begin{aligned}
&= \frac{2}{\pi} \left[ \int_0^{\frac{\pi}{2}} x \cos nx \, dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x) \cos nx \, dx \right] \\
&= \frac{2}{\pi} \left[ (x) \left( \frac{\sin nx}{n} \right) - (1) \left( \frac{-\cos nx}{n^2} \right) \right]_0^{\frac{\pi}{2}} + \frac{2}{\pi} \left[ (\pi - x) \left( \frac{\sin nx}{n} \right) - (-1) \left( \frac{-\cos nx}{n^2} \right) \right]_{\frac{\pi}{2}}^{\pi} \\
&= \frac{2}{\pi} \left[ \frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\frac{\pi}{2}} + \frac{2}{\pi} \left[ (\pi - x) \frac{\sin nx}{n} - \frac{\cos nx}{n^2} \right]_{\frac{\pi}{2}}^{\pi} \\
&= \frac{2}{\pi} \left[ \frac{\pi}{2n} \sin \frac{n\pi}{2} + \frac{1}{n^2} \cos \frac{n\pi}{2} - \frac{1}{n^2} - \frac{1}{n^2} \cos n\pi - \frac{\pi}{2n} \sin \frac{n\pi}{2} + \frac{1}{n^2} \cos \frac{n\pi}{2} \right] \\
&= \frac{2}{\pi n^2} \left[ 2 \cos \frac{n\pi}{2} - 1 - (-1)^n \right] \\
&= \frac{2}{\pi n^2} \begin{cases} 0, & n \text{ is odd} \\ 2(-1) - 1 - 1, & n = 2, 6, 10, \dots \\ 2(1) - 1 - 1, & n = 4, 8, 12, \dots \end{cases} = \begin{cases} 0, & n \text{ is odd} \\ \frac{-8}{\pi n^2}, & n = 2, 6, 10, \dots \\ 0, & n = 4, 8, 12, \dots \end{cases}
\end{aligned}$$

Substituting  $a_0, a_n$  in ②

$$\begin{aligned}
f(x) &= \frac{\pi}{4} - \frac{8}{\pi} \left[ \frac{\cos 2x}{4} + \frac{\cos 6x}{36} + \frac{\cos 10x}{100} + \dots \right] \\
&= \frac{\pi}{4} - \frac{2}{\pi} \left[ \frac{\cos 2x}{1} + \frac{\cos 6x}{9} + \frac{\cos 10x}{25} + \dots \right]
\end{aligned}$$

**Example 24** Obtain half range Fourier Sine series for  $f(x) = \begin{cases} \frac{2kx}{l}, & 0 \leq x \leq \frac{l}{2} \\ \frac{2k}{l}(l - x), & \frac{l}{2} \leq x \leq l \end{cases}$

**Solution:** To develop  $f(x)$  into Sine series, extending  $f(x)$  in  $(-l, 0)$  by reflecting in origin, so that  $f(-x) = -f(x)$ , function becomes odd function and  $a_0 = a_n = 0$

$$\therefore f(x) \approx \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \dots\dots \textcircled{1}$$

$$\begin{aligned}
b_n &= \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx \\
&= \frac{2}{l} \int_0^{\frac{l}{2}} \frac{2kx}{l} \sin \frac{n\pi x}{l} \, dx + \frac{2}{l} \int_{\frac{l}{2}}^l \frac{2k}{l}(l - x) \sin \frac{n\pi x}{l} \, dx \\
&= \frac{4k}{l^2} \left[ (x) \left( -\frac{l}{n\pi} \cos \frac{n\pi x}{l} \right) - (1) \left( -\frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l} \right) \right]_0^{\frac{l}{2}} + \\
&\quad \frac{4k}{l^2} \left[ (l - x) \left( -\frac{l}{n\pi} \cos \frac{n\pi x}{l} \right) - (-1) \left( -\frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l} \right) \right]_{\frac{l}{2}}^l \\
&= \frac{4k}{l^2} \left[ -\frac{lx}{n\pi} \cos \frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l} \right]_0^{\frac{l}{2}} + \frac{4k}{l^2} \left[ -(l - x) \frac{l}{n\pi} \cos \frac{n\pi x}{l} - \frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l} \right]_{\frac{l}{2}}^l \\
&= \frac{4k}{l^2} \left[ -\frac{l^2}{2n\pi} \cos \frac{n\pi}{2} + \frac{l^2}{n^2\pi^2} \sin \frac{n\pi}{2} + \frac{l^2}{2n\pi} \cos \frac{n\pi}{2} + \frac{l^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right]
\end{aligned}$$

$$= \frac{8k}{n^2\pi^2} \left[ \sin \frac{n\pi}{2} \right] = \begin{cases} \frac{8k}{n^2\pi^2}, & n = 1, 5, 9, \dots \\ \frac{-8k}{n^2\pi^2}, & n = 3, 7, 11, \dots \\ 0, & n \text{ is even} \end{cases}$$

Substituting values of  $b_n$  in ①

$$f(x) \approx \frac{8k}{n^2\pi^2} \left[ \sin \frac{\pi x}{l} - \frac{1}{9} \sin \frac{3\pi x}{l} + \frac{1}{25} \sin \frac{5\pi x}{l} - \frac{1}{49} \sin \frac{7\pi x}{l} + \dots \right]$$

## 1.7 Practical Harmonic Analysis

In many engineering and scientific problems,  $f(x)$  is not given directly, rather set of discrete values of function are given in the form  $(x_i, y_i), i = 1, 2, 3, \dots, m$  where  $x_i$ 's are equispaced. The process of obtaining  $f(x)$  in terms of Fourier series from given set of values  $(x_i, y_i)$ , is known as practical harmonic analysis.

In a given interval  $(0, 2l)$ ,  $f(x)$  is represented in terms of harmonics as shown below:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

Where  $n = 1, 2, 3$  give 1<sup>st</sup>, 2<sup>nd</sup> and 3<sup>rd</sup> harmonics respectively.

$\therefore \left( a_1 \cos \frac{\pi x}{l} + b_1 \sin \frac{\pi x}{l} \right)$  is the first harmonic

$\left( a_2 \cos \frac{2\pi x}{l} + b_2 \sin \frac{2\pi x}{l} \right)$  is the second harmonic

$\left( a_3 \cos \frac{3\pi x}{l} + b_3 \sin \frac{3\pi x}{l} \right)$  is the third harmonic

$\vdots$

$\left( a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$  is the  $n^{\text{th}}$  harmonic

Fourier coefficient  $a_0$  is computed using the relation

2 [Mean value of  $y$  in the interval  $(0, 2l)$ ]

$$\therefore a_0 = \frac{2}{m} \sum_{i=1}^m y_i, \text{ where } m \text{ denotes number of observations}$$

Similarly  $a_n$  and  $b_n$  can be found out using the relations

$$a_n = 2 \left[ \text{Mean value of } y \cos \frac{n\pi x}{l} \text{ in the interval } (0, 2l) \right] = \frac{2}{m} \sum_{i=1}^m y_i \cos \frac{n\pi x_i}{l}$$

$$b_n = 2 \left[ \text{Mean value of } y \sin \frac{n\pi x}{l} \text{ in the interval } (0, 2l) \right] = \frac{2}{m} \sum_{i=1}^m y_i \sin \frac{n\pi x_i}{l}$$

Also when interval length is  $2\pi$ , putting  $2l = 2\pi$  i.e.  $l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{2}{m} \sum_{i=1}^m y_i, \quad a_n = \frac{2}{m} \sum_{i=1}^m y_i \cos nx_i, \quad b_n = \frac{2}{m} \sum_{i=1}^m y_i \sin nx_i$$

- The amplitude of first harmonic is given by  $\sqrt{a_1^2 + b_1^2}$  and similarly amplitudes of second and third harmonics are given by  $\sqrt{a_2^2 + b_2^2}$  and  $\sqrt{a_3^2 + b_3^2}$  respectively.
- For  $f(x)$  in discrete form, values of Fourier coefficients  $a_0, a_n$  and  $b_n$  have been computed using trapezoidal rule for definite integration.

**Example 25** The following values of ‘y’ give the displacement of a machine part for the rotation  $x$  of a flywheel. Express ‘y’ in Fourier series up to third harmonic.

|     |           |            |             |             |             |             |             |
|-----|-----------|------------|-------------|-------------|-------------|-------------|-------------|
| $x$ | $0^\circ$ | $60^\circ$ | $120^\circ$ | $180^\circ$ | $240^\circ$ | $300^\circ$ | $360^\circ$ |
| $y$ | 1.98      | 2.15       | 2.77        | -0.22       | -0.31       | 1.43        | 1.98        |

**Solution:** Here number of observations ( $m$ ) are 6, period length is  $2\pi \therefore [y]_{0^\circ} \equiv [y]_{360^\circ}$

$$\text{Let } f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\therefore y \approx \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + (a_3 \cos 3x + b_3 \sin 3x) \dots \textcircled{1}$$

$$a_0 = \frac{2}{m} \sum_{i=1}^m y_i, \quad a_n = \frac{2}{m} \sum_{i=1}^m y_i \cos nx_i, \quad b_n = \frac{2}{m} \sum_{i=1}^m y_i \sin nx_i$$

| $x_i$       | $y_i$ | $\cos x_i$ | $\sin x_i$ | $\cos 2x_i$ | $\sin 2x_i$ | $\cos 3x_i$ | $\sin 3x_i$ |
|-------------|-------|------------|------------|-------------|-------------|-------------|-------------|
| $0^\circ$   | 19.8  | 1.0        | 0          | 1.0         | 0           | 1.0         | 0           |
| $60^\circ$  | 2.15  | 0.5        | 0.866      | -0.5        | 0.866       | -1.0        | 0           |
| $120^\circ$ | 2.77  | -0.5       | 0.866      | -0.5        | -0.866      | 1.0         | 0           |
| $180^\circ$ | -0.22 | -1         | 0          | 1.0         | 0           | -1.0        | 0           |
| $240^\circ$ | -0.31 | -0.5       | -0.866     | -0.5        | 0.866       | 1.0         | 0           |
| $300^\circ$ | 1.43  | 0.5        | -0.866     | -0.5        | -0.866      | -1.0        | 0           |

$$a_0 = \frac{2}{6} \sum_{i=1}^6 y_i = \frac{2}{6} [1.98 + 2.15 + 2.77 - 0.22 - 0.31 + 1.4] = 2.6$$

$$a_1 = \frac{2}{6} \sum_{i=1}^6 y_i \cos x_i = \frac{2}{6} [(1.98)(1) + (2.15)(0.5) + \dots + (1.43)(0.5)] = 0.92$$

$$b_1 = \frac{2}{6} \sum_{i=1}^6 y_i \sin x_i = \frac{2}{6} [(1.98)(0) + (2.15)(0.866) + \dots + (1.43)(-0.866)] = 1.097$$

$$a_2 = \frac{2}{6} \sum_{i=1}^6 y_i \cos 2x_i = \frac{2}{6} [(1.98)(1) + (2.15)(-0.5) + \dots + (1.43)(-0.5)] = -0.42$$

$$b_2 = \frac{2}{6} \sum_{i=1}^6 y_i \sin 2x_i = \frac{2}{6} [(1.98)(0) + (2.15)(0.866) + \dots + (1.43)(-0.866)] = -0.681$$

$$a_3 = \frac{2}{6} \sum_{i=1}^6 y_i \cos 3x_i = \frac{2}{6} [(1.98)(1) + (2.15)(-1) + \dots + (1.43)(-1)] = 0.36$$

$$b_3 = \frac{2}{6} \sum_{i=1}^6 y_i \sin 3x_i = \frac{2}{6} [(1.98)(0) + (2.15)(0) + \dots + (1.43)(0)] = 0$$

Substituting values of  $a_0, a_n, b_n$  in  $\textcircled{1}$  where  $n = 1, 2, 3$

$$y \approx 1.3 + (0.92 \cos x + 1.097 \sin x) - (0.42 \cos 2x + 0.681 \sin 2x) + 0.36 \cos 3x + \dots$$

**Example 26** Experimental values of  $y$  corresponding to  $x$  are tabulated below:

|     |     |                 |                  |                  |                  |                  |       |                  |                  |                  |                   |                   |        |
|-----|-----|-----------------|------------------|------------------|------------------|------------------|-------|------------------|------------------|------------------|-------------------|-------------------|--------|
| $x$ | 0   | $\frac{\pi}{6}$ | $\frac{2\pi}{6}$ | $\frac{3\pi}{6}$ | $\frac{4\pi}{6}$ | $\frac{5\pi}{6}$ | $\pi$ | $\frac{7\pi}{6}$ | $\frac{8\pi}{6}$ | $\frac{9\pi}{6}$ | $\frac{10\pi}{6}$ | $\frac{11\pi}{6}$ | $2\pi$ |
| $y$ | 298 | 356             | 373              | 337              | 254              | 155              | 80    | 51               | 60               | 93               | 147               | 221               | 298    |

Express  $y$  in Fourier series up to second harmonic.

**Solution:** Here number of observations ( $m$ ) are 12, period length is  $2\pi \therefore [y]_0 \equiv [y]_{2\pi}$

$$\text{Let } y \approx \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots \textcircled{1}$$

$$a_0 = \frac{2}{m} \sum_{i=1}^m y_i, \quad a_n = \frac{2}{m} \sum_{i=1}^m y_i \cos nx_i, \quad b_n = \frac{2}{m} \sum_{i=1}^m y_i \sin nx_i$$

| $x_i$             | $y_i$ | $\cos x_i$ | $\sin x_i$ | $\cos 2x_i$ | $\sin 2x_i$ |
|-------------------|-------|------------|------------|-------------|-------------|
| 0                 | 298   | 1          | 0          | 1           | 0           |
| $\frac{\pi}{6}$   | 356   | 0.866      | 0.5        | 0.5         | 0.866       |
| $\frac{2\pi}{6}$  | 373   | 0.5        | 0.866      | -0.5        | 0.866       |
| $\frac{3\pi}{6}$  | 337   | 0          | 1          | -1          | 0           |
| $\frac{4\pi}{6}$  | 254   | -0.5       | 0.866      | -0.5        | -0.866      |
| $\frac{5\pi}{6}$  | 155   | -0.866     | 0.5        | 0.5         | -0.866      |
| $\pi$             | 80    | -1         | 0          | 1           | 1           |
| $\frac{7\pi}{6}$  | 51    | -0.866     | -0.5       | 0.5         | 0.866       |
| $\frac{8\pi}{6}$  | 60    | -0.5       | -0.866     | -0.5        | 0.866       |
| $\frac{9\pi}{6}$  | 93    | 0          | -1         | -1          | 0           |
| $\frac{10\pi}{6}$ | 147   | 0.5        | -0.866     | -0.5        | -0.866      |
| $\frac{11\pi}{6}$ | 221   | 0.866      | -0.5       | 0.5         | -0.866      |

$$a_0 = \frac{2}{12} \sum_{i=1}^{12} y_i = \frac{1}{6} [298 + 356 + \dots + 221] = 404.17$$

$$a_1 = \frac{2}{12} \sum_{i=1}^{12} y_i \cos x_i = \frac{1}{6} [(298)(1) + (356)(0.866) + \dots + (221)(0.866)] = 107.048$$

$$b_1 = \frac{2}{12} \sum_{i=1}^{12} y_i \sin x_i = \frac{1}{6} [(298)(0) + (356)(0.5) + \dots + (221)(-0.5)] = 121.203$$

$$a_2 = \frac{2}{12} \sum_{i=1}^{12} y_i \cos 2x_i = \frac{1}{6} [(298)(1) + (356)(0.5) + \dots + (221)(0.5)] = -13$$

$$b_2 = \frac{2}{12} \sum_{i=1}^{12} y_i \sin 2x_i = \frac{1}{6} [(298)(0) + (356)(0.866) + \dots + (221)(-0.866)] = 9.093$$

Substituting values of  $a_0, a_1, b_1, a_2, b_2$  in  $\textcircled{1}$

$$y \approx 202.09 + (107.048 \cos x + 121.203 \sin x) + (-13 \cos 2x + 9.093 \sin 2x) + \dots$$

**Example 27** The following table connects values of  $x$  and  $y$  for a statistical input:

|     |   |    |    |    |    |    |
|-----|---|----|----|----|----|----|
| $x$ | 0 | 1  | 2  | 3  | 4  | 5  |
| $y$ | 9 | 18 | 24 | 28 | 26 | 20 |

Express  $y$  in Fourier series up to first harmonic. Also find amplitude of the first harmonic.

**Solution:** Here  $m = 6$  , Also putting  $2l = 6 \Rightarrow l = 3 \therefore [y]_{x=0} \equiv [y]_{x=6}$  , if  $y$  is periodic

$$\therefore y \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{3} + b_n \sin \frac{n\pi x}{3} \right)$$

$$\Rightarrow y \approx \frac{a_0}{2} + \left( a_1 \cos \frac{\pi x}{3} + b_1 \sin \frac{\pi x}{3} \right) + \dots \quad \text{.....①}$$

$$a_0 = \frac{2}{6} \sum_{i=1}^6 y_i, \quad a_1 = \frac{2}{6} \sum_{i=1}^6 y_i \cos \frac{\pi x_i}{3}, \quad b_1 = \frac{2}{6} \sum_{i=1}^6 y_i \sin \frac{\pi x_i}{3}$$

| $x_i$ | $y_i$ | $\cos \frac{\pi x_i}{3}$ | $\sin \frac{\pi x_i}{3}$ |
|-------|-------|--------------------------|--------------------------|
| 0     | 9     | 1                        | 0                        |
| 1     | 18    | 0.5                      | 0.866                    |
| 2     | 24    | -0.5                     | 0.866                    |
| 3     | 28    | -1                       | 0                        |
| 4     | 26    | -0.5                     | -0.866                   |
| 5     | 20    | 0.5                      | -0.866                   |

$$a_0 = \frac{2}{6} \sum_{i=1}^6 y_i = \frac{1}{3} [9 + 18 + 24 + 28 + 26 + 20] = 41.67$$

$$a_1 = \frac{2}{6} \sum_{i=1}^6 y_i \cos \frac{\pi x_i}{3} = \frac{1}{3} [(9)(1) + (18)(0.5) + \dots + (20)(0.5)] = -8.33$$

$$b_1 = \frac{2}{6} \sum_{i=1}^6 y_i \sin \frac{\pi x_i}{3} = \frac{1}{3} [(9)(0) + (18)(0.866) + \dots + (20)(-0.866)] = -1.15$$

Substituting values of  $a_0, a_1, b_1$  in ①

$$\Rightarrow y \approx 20.835 - \left( 8.33 \cos \frac{\pi x}{3} + 1.15 \sin \frac{\pi x}{3} \right) + \dots$$

The amplitude of first harmonic is given by  $\sqrt{(-8.33)^2 + (-1.15)^2} = 8.41$

**Example 28** The following table gives the variation of a periodic current over a period ‘ $T$ ’

| Time(t)<br>Sec    | 0    | T/6  | T/3  | T/2 | 2T/3  | 5T/6  | T    |
|-------------------|------|------|------|-----|-------|-------|------|
| Current(A)<br>Amp | 1.98 | 1.30 | 1.05 | 1.3 | -0.88 | -0.25 | 1.98 |

Show that there is a direct current part of 0.75 amp in the variable current. Also obtain the amplitude of the first harmonic.

**Solution:** Let  $A \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{l}$

Here  $m = 6$ , Also  $2l = T \Rightarrow l = \frac{T}{2}$

$$\therefore A \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2n\pi t}{T} + \sum_{n=1}^{\infty} b_n \sin \frac{2n\pi t}{T}$$

$$\Rightarrow A \approx \frac{a_0}{2} + a_1 \cos \frac{2\pi t}{T} + b_1 \sin \frac{2\pi t}{T} \quad \text{for the first harmonic...①}$$

$$a_0 = \frac{2}{m} \sum A, \quad a_1 = \frac{2}{m} \sum A \cos \frac{2\pi t}{T}, \quad b_1 = \frac{2}{m} \sum A \sin \frac{2\pi t}{T}$$

| Time(t)<br>sec | Current(A)<br>amp | $\cos \frac{2\pi t}{T}$ | $\sin \frac{2\pi t}{T}$ |
|----------------|-------------------|-------------------------|-------------------------|
| 0              | 1.98              | 1                       | 0                       |
| T/6            | 1.3               | 0.5                     | 0.866                   |
| T/3            | 1.05              | -0.5                    | 0.866                   |
| T/2            | 1.3               | -1                      | 0                       |
| 2T/3           | -0.88             | -0.5                    | -0.866                  |
| 5T/6           | -0.25             | 0.5                     | -0.866                  |

$$a_0 = \frac{2}{6} \sum A = \frac{1}{3} [1.98 + 1.3 + 1.05 + 1.3 - 0.88 - 0.25] = 1.5$$

$$a_1 = \frac{2}{6} \sum A \cos \frac{2\pi t}{T} = \frac{1}{3} [(1.98)(1) + (1.3)(0.5) + \dots + (-0.25)(0.5)] = 0.373$$

$$b_1 = \frac{2}{6} \sum A \sin \frac{2\pi t}{T} = \frac{1}{3} [(1.98)(0) + (1.3)(0.866) + \dots + (-0.25)(-0.866)] = 1.005$$

Substituting values of  $a_0, a_1, b_1$  in ①

$$\therefore A \approx 0.75 + 0.373 \cos \frac{2\pi t}{T} + 1.005 \sin \frac{2\pi t}{T}$$

Here  $\frac{a_0}{2}$  represents the direct current part and the amplitude of the first harmonic is given by  $\sqrt{a_1^2 + b_1^2}$

$\therefore A$  has a direct current part of 0.75 amp

The amplitude of first harmonic is given by  $\sqrt{(0.373)^2 + (1.005)^2} = \sqrt{1.1491} = 1.072$

### 1.7.1 Harmonic Analysis for Half Range Series

If it is required to express  $f(x)$  given in discrete form  $(x_i, y_i), i = 1, 2, 3, \dots, m$ , taken in the interval  $(0, l)$  into half range sine or cosine series, we extend  $f(x)$  in  $(-l, 0)$  to make it odd or even respectively.

#### Sine Series

To develop  $f(x)$  into sine series, extend  $f(x)$  in the interval  $(-l, 0)$  by reflecting in origin, so that  $f(-x) = -f(x)$ , function becomes odd function and  $a_0 = a_n = 0$

$$\therefore f(x) \approx \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$b_n = 2 \left[ \text{Mean value of } y \sin \frac{n\pi x}{l} \text{ in the interval } (0, l) \right] = \frac{2}{m} \sum_{i=1}^m y_i \sin \frac{n\pi x_i}{l}$$

**Note:** To express  $f(x)$  into sine series,  $y_1$  must be zero, otherwise it cannot be reflected in origin.

#### Cosine Series

To develop  $f(x)$  into cosine series, extend  $f(x)$  in the interval  $(-l, 0)$  by reflecting in y-axis, so that  $f(-x) = f(x)$ , function becomes even function and  $b_n = 0$

$$\therefore f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$a_0 = 2 \left[ \text{Mean value of } y \text{ in the interval } (0, l) \right]$$

$$= \frac{2}{m} \left[ \frac{y_1 + y_m}{2} + y_2 + y_3 + \dots + y_{m-1} \right] \quad \dots \text{ using trapezoidal rule}$$

$$a_n = 2 \left[ \text{Mean value of } y \cos \frac{n\pi x}{l} \text{ in the interval } (0, l) \right]$$

$$= \frac{2}{m} \left[ \frac{y_1 \cos \frac{n\pi x}{l} + y_m \cos \frac{n\pi x}{l}}{2} + y_2 \cos \frac{n\pi x}{l} + y_3 \cos \frac{n\pi x}{l} + \dots + y_{m-1} \cos \frac{n\pi x}{l} \right]$$

$$n = 1, 2, 3 \dots$$

**Example 29** The turning moment ‘ $M$ ’ units of a crank shaft of a steam engine are given for a series of values of the crank angle ‘ $\theta$ ’ in degrees. Obtain first three terms of sine series to represent  $M$ . Also verify the value  $M$  from obtained function at  $\theta = 60^\circ$

| $\theta$ | $0^\circ$ | $30^\circ$ | $60^\circ$ | $90^\circ$ | $120^\circ$ | $150^\circ$ |
|----------|-----------|------------|------------|------------|-------------|-------------|
| $M$      | 0         | 5224       | 8097       | 7850       | 5499        | 2656        |

**Solution:** Assuming  $M$  periodic, to represent into sine series (half range series), extending  $M$  in the interval  $(-180^\circ, 0)$  by reflecting in origin, so that  $M(-\theta) = -M(\theta)$ , function becomes odd function and  $a_0 = a_n = 0$

$$\therefore M \approx \sum_{n=1}^{\infty} b_n \sin \frac{n\pi\theta}{l}$$

$$\text{Here } m = 6, \text{ Also } 2l = 2\pi \Rightarrow l = \pi$$

$$\Rightarrow M \approx \sum_{n=1}^{\infty} b_n \sin n\theta$$

$$\Rightarrow M \approx b_1 \sin \theta + b_2 \sin 2\theta + b_3 \sin 3\theta + \dots$$

$$b_n = \frac{2}{6} \sum M \sin n\theta, n = 1, 2, 3 \dots$$

| $\theta$    | $M$  | $\sin \theta$ | $\sin 2\theta$ | $\sin 3\theta$ |
|-------------|------|---------------|----------------|----------------|
| $0^\circ$   | 0    | 0             | 0              | 0              |
| $30^\circ$  | 5224 | 0.5           | 0.866          | 1              |
| $60^\circ$  | 8097 | 0.866         | 0.866          | 0              |
| $90^\circ$  | 7850 | 1             | 0              | -1             |
| $120^\circ$ | 5499 | 0.866         | -0.866         | 0              |
| $150^\circ$ | 2656 | 0.5           | -0.866         | 1              |

$$b_1 = \frac{2}{6} \sum M \sin \theta = \frac{1}{3} [(0)(0) + (5224)(0.5) + \dots + (2656)(0.5)] = 7850$$

$$b_2 = \frac{2}{6} \sum M \sin 2\theta = \frac{1}{3} [(0)(0) + (5224)(0.866) + \dots + (2656)(-0.866)] = 1500$$

$$b_3 = \frac{2}{6} \sum M \sin 3\theta = \frac{1}{3} [(0)(0) + (5224)(1) + \dots + (2656)(1)] = 0$$

$$\therefore M \approx 7850 \sin \theta + 1500 \sin 2\theta + 0 + \dots$$

$$\text{When } \theta = 60^\circ, M \approx 7850 \sin 60^\circ + 1500 \sin 120^\circ + 0 + \dots$$

$$\approx 7850(0.866) + 1500(0.866) + \dots$$

$$\approx 8097.1$$

**Example 30** Obtain half range Fourier cosine series for the data given below:

|     |   |   |    |    |    |   |
|-----|---|---|----|----|----|---|
| $x$ | 0 | 1 | 2  | 3  | 4  | 5 |
| $y$ | 4 | 8 | 11 | 15 | 12 | 7 |

Also check value of  $y$  at  $x = 2$  from the obtained cosine series.

**Solution:** Assuming  $y$  periodic, to represent it into half range cosine series, extending  $y$  in the interval  $(-6,0)$  by reflecting it in  $y$ -axis, so that  $y(-x) = y(x)$ , function becomes even function and  $b_n = 0$

$$\therefore y \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$\text{Here } m = 6, \text{ Also } 2l = 12 \Rightarrow l = 6$$

$$\Rightarrow y \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{6}$$

$$\Rightarrow y \approx \frac{a_0}{2} + a_1 \cos \frac{\pi x}{6} + a_2 \cos \frac{2\pi x}{6} + a_3 \cos \frac{3\pi x}{6} + \dots$$

$$a_0 = \frac{2}{m} \left[ \frac{y_1 + y_m}{2} + y_2 + y_3 + \dots + y_{m-1} \right]$$

$$a_n = \frac{2}{m} \left[ \frac{y_1 \cos \frac{n\pi x}{l} + y_m \cos \frac{n\pi x}{l}}{2} + y_2 \cos \frac{n\pi x}{l} + y_3 \cos \frac{n\pi x}{l} + \dots + y_{m-1} \cos \frac{n\pi x}{l} \right]$$

$$n = 1, 2, 3 \dots$$

| $x_i$ | $y_i$ | $\cos \frac{\pi x}{6}$ | $\cos \frac{2\pi x}{6}$ | $\cos \frac{3\pi x}{6}$ | $y \cos \frac{\pi x}{6}$ | $y \cos \frac{2\pi x}{6}$ | $y \cos \frac{3\pi x}{6}$ |
|-------|-------|------------------------|-------------------------|-------------------------|--------------------------|---------------------------|---------------------------|
| 0     | 4     | 1                      | 1                       | 1                       | 4                        | 4                         | 4                         |
| 1     | 8     | 0.866                  | 0.5                     | 0                       | 0.6928                   | 4                         | 0                         |
| 2     | 11    | 0.5                    | -0.5                    | -1                      | 5.5                      | -5.5                      | -11                       |
| 3     | 15    | 0                      | -1                      | 0                       | 0                        | -15                       | 0                         |
| 4     | 12    | -0.5                   | -0.5                    | 1                       | -6                       | -6                        | 12                        |
| 5     | 7     | -0.866                 | 0.5                     | 0                       | -6.062                   | 3.5                       | 0                         |

$$a_0 = \frac{2}{6} \left[ \frac{4+7}{2} + 8 + 11 + 15 + 12 \right] = \frac{1}{3} [51.5] = 17.2$$

$$a_1 = \frac{2}{6} \left[ \frac{4-6.062}{2} + 0.6928 + 5.5 + 0 - 6 \right] = \frac{1}{3} [-0.8382] = -0.2794$$

$$a_2 = \frac{2}{6} \left[ \frac{4+3.5}{2} + 4 - 5.5 - 15 - 6 \right] = \frac{1}{3} [-18.75] = -6.25$$

$$a_3 = \frac{2}{6} \left[ \frac{4+0}{2} + 0 - 11 + 0 + 12 \right] = \frac{1}{3} [3] = 1$$

$$\therefore y \approx 8.6 - 0.2794 \cos \frac{\pi x}{6} - 6.25 \cos \frac{2\pi x}{6} + \cos \frac{3\pi x}{6} + \dots$$

$$\text{When } x = 2, y \approx 8.6 - 0.2794 \cos \frac{2\pi}{6} - 6.25 \cos \frac{4\pi}{6} + \cos \frac{6\pi}{6} + \dots$$

$$\approx 8.6 - 0.2794(0.5) - 6.25(-0.5) - 1 + \dots$$

$$\approx 10.5853$$

## Exercise 1B

1. Find half range Fourier cosine series expansion of the function

$$f(x) = (\pi - x) \text{ in the interval } (0, \pi)$$

2. Find half range Fourier cosine series expansion of the function

$$f(x) = \sin \frac{\pi x}{l} \text{ in the interval } (0, 1)$$

3. Find half range Fourier sine series expansion of the function

$$f(x) = e^x \text{ in the interval } (0, 1)$$

4. Find the Fourier sine series expansion of the function

$$f(x) = \begin{cases} \frac{1}{4} - x, & 0 < x < \frac{1}{2} \\ x - \frac{3}{4}, & \frac{1}{2} < x < 1 \end{cases}$$

5. The following values of 'y' give the displacement of a machine part for the rotation  $x$  of a flywheel. Express 'y' in Fourier series.

|     |           |            |            |            |             |             |             |             |             |             |             |             |
|-----|-----------|------------|------------|------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $x$ | $0^\circ$ | $30^\circ$ | $60^\circ$ | $90^\circ$ | $120^\circ$ | $150^\circ$ | $180^\circ$ | $210^\circ$ | $240^\circ$ | $270^\circ$ | $300^\circ$ | $330^\circ$ |
| $y$ | 70        | 886        | 1293       | 1400       | 1307        | 814         | -70         | -886        | -1293       | -1400       | -1307       | -814        |

Express  $y$  in Fourier series up to third harmonics.

6.  $y$  is a discrete function of  $x$ , given as below:

|     |           |            |             |             |             |             |             |
|-----|-----------|------------|-------------|-------------|-------------|-------------|-------------|
| $x$ | $0^\circ$ | $60^\circ$ | $120^\circ$ | $180^\circ$ | $240^\circ$ | $300^\circ$ | $360^\circ$ |
| $y$ | 0.8       | 0.6        | 0.4         | 0.7         | 0.9         | 1.1         | 0.8         |

Express  $y$  in Fourier series up to first harmonic.

7. Obtain Fourier sine series of  $y$  for the data given below:

|     |   |    |    |   |   |   |   |
|-----|---|----|----|---|---|---|---|
| $x$ | 0 | 1  | 2  | 3 | 4 | 5 | 6 |
| $y$ | 0 | 10 | 15 | 8 | 5 | 3 | 0 |

8. Following values of  $y$  give the displacement of a certain machine part for the movement  $x$  of another part:

|     |   |                 |                  |                  |                  |                  |       |
|-----|---|-----------------|------------------|------------------|------------------|------------------|-------|
| $x$ | 0 | $\frac{\pi}{6}$ | $\frac{2\pi}{6}$ | $\frac{3\pi}{6}$ | $\frac{4\pi}{6}$ | $\frac{5\pi}{6}$ | $\pi$ |
| $y$ | 0 | 9.2             | 14.4             | 17.8             | 17.3             | 11.7             | 0     |

Express  $y$  in Fourier series up to second harmonic.

## Answers

- $f(x) \approx \frac{\pi}{2} + \frac{4}{\pi} \left[ \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right]$
- $f(x) \approx \frac{2}{\pi} - \frac{4}{\pi} \left[ \frac{\cos \frac{2\pi x}{l}}{1.3} + \frac{\cos \frac{4\pi x}{l}}{3.5} + \frac{\cos \frac{6\pi x}{l}}{5.7} + \dots \right]$

3.  $f(x) \approx 2\pi \left[ \frac{1(1+e)}{1+\pi^2} \sin \pi x + \frac{2(1-e)}{1+4\pi^2} \sin 2\pi x + \frac{3(1+e)}{1+9\pi^2} \sin 3\pi x + \dots \right]$
4.  $f(x) \approx \frac{1}{\pi} \left[ \frac{1}{1} \left( 1 - \frac{4}{\pi} \right) \sin \pi x + \frac{1}{3} \left( 1 + \frac{4}{3\pi} \right) \sin 3\pi x + \frac{1}{5} \left( 1 - \frac{4}{5\pi} \right) \sin 5\pi x + \dots \right]$
5.  $y \approx 42 \cos x + 1501 \sin x + 28 \cos 3x + 100 \sin 3x + \dots$
6.  $y \approx 0.75 + 0.1 \cos x - 0.29 \sin x + \dots$
7.  $y \approx 10.607 \sin \frac{\pi x}{6} + 4.907 \sin \frac{2\pi x}{6} + 1.667 \sin \frac{3\pi x}{6} + \dots$
8.  $y \approx 11.73 - 7.73 \cos 2x - 1.56 \sin 2x - 2.83 \cos 4x + 0.115 \sin 4x + \dots$

# FOURIER TRANSFORMS

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## 2.1 Introduction

The Fourier series expresses any periodic function into a sum of sinusoids. The Fourier transform is the extension of this idea to non-periodic functions by taking the limiting form of Fourier series when the fundamental period is made very large (infinite). Fourier transform finds its applications in astronomy, signal processing, linear time invariant (LTI) systems etc.

### Some useful results in computation of the Fourier transforms:

$$1. \int_0^{\infty} e^{-ax} \sin \lambda x \, dx = \frac{\lambda}{a^2 + \lambda^2}$$

$$2. \int_0^{\infty} e^{-ax} \cos \lambda x \, dx = \frac{a}{a^2 + \lambda^2}$$

$$3. \int_0^{\infty} \frac{\sin \lambda x}{x} \, dx = \frac{\pi}{2}, \lambda > 0$$

$$\text{When } \lambda = 1, \int_0^{\infty} \frac{\sin x}{x} \, dx = \frac{\pi}{2}$$

$$4. \sin ax = \frac{e^{iax} - e^{-iax}}{2i}$$

$$5. \cos ax = \frac{e^{iax} + e^{-iax}}{2}$$

$$6. \int_0^{\infty} e^{-a^2 x^2} \, dx = \frac{\sqrt{\pi}}{2a}$$

$$\text{When } a = 1, \int_0^{\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}$$

$$7. \text{ Heaviside Step Function or Unit step function } H(t) \text{ or } U(t) = \begin{cases} 0, & \text{when } t < 0 \\ 1, & \text{when } t \geq 0 \end{cases}$$

At  $t = 0$ ,  $H(t)$  is sometimes taken as 0.5 or it may not have any specific value.

Shifting at  $t = a$

$$H(t - a) \text{ or } U(t - a) = \begin{cases} 0, & \text{when } t < a \\ 1, & \text{when } t \geq a \end{cases}$$

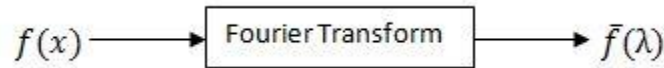
8. Dirac Delta Function or Unit Impulse Function is defined as  $\delta(t - a) = 0$ ,  $t \neq a$  such that  $\int_0^{\infty} \delta(t - a) dt = 1$ ,  $a \geq 0$ . It is zero everywhere except one point 'a'. Delta function is sometimes thought of having infinite value at  $t = a$ . The delta function can be viewed as the derivative of the Heaviside step function

### Dirichlet's Conditions for Existence of Fourier Transform

Fourier transform can be applied to any function  $f(x)$  if it satisfies the following conditions:

1.  $f(x)$  is absolutely integrable i.e.  $\int_{-\infty}^{\infty} |f(x)| dx$  is convergent.
2. The function  $f(x)$  has a finite number of maxima and minima.
3.  $f(x)$  has only a finite number of discontinuities in any finite

## 2.2 Fourier Transform, Inverse Fourier Transform and Fourier Integral



The Fourier transform of  $f(x)$ ,  $-\infty < x < \infty$ , denoted by  $\bar{f}(\lambda)$  where  $\lambda \in \mathbb{N}$ , is given by

$$F\{f(x)\} \equiv \bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} f(x) dx \quad \dots \textcircled{1}$$

Also inverse Fourier transform of  $\bar{f}(\lambda)$  gives  $f(x)$  as:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda x} \bar{f}(\lambda) d\lambda \quad \dots \textcircled{2}$$

Rewriting  $\textcircled{1}$  as  $\bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda t} f(t) dt$  and using in  $\textcircled{2}$ , Fourier integral representation of  $f(x)$  is given by:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\lambda(t-x)} f(t) dt d\lambda$$

### 2.2.1 Fourier Sine Transform (F.S.T.)

Fourier Sine transform of  $f(x)$ ,  $0 < x < \infty$ , denoted by  $\bar{f}_s(\lambda)$ , is given by

$$F_s\{f(x)\} \equiv \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \lambda x dx \quad \dots \textcircled{3}$$

Also inverse Fourier Sine transform of  $\bar{f}_s(\lambda)$  gives  $f(x)$  as:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \bar{f}_s(\lambda) \sin \lambda x d\lambda \quad \dots \textcircled{4}$$

Rewriting  $\textcircled{3}$  as  $\bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin \lambda t dt$  and using in  $\textcircled{4}$ , Fourier sine integral representation of  $f(x)$  is given by:

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(t) \sin \lambda t \sin \lambda x dt d\lambda$$

### 2.2.2 Fourier Cosine Transform (F.C.T.)

Fourier Cosine transform of  $f(x)$ ,  $0 < x < \infty$ , denoted by  $\bar{f}_c(\lambda)$ , is given by

$$F_c\{f(x)\} \equiv \bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \lambda x dx \dots \textcircled{5}$$

Also inverse Fourier Cosine transform of  $\bar{f}_c(\lambda)$  gives  $f(x)$  as:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}_c(\lambda) \cos \lambda x d\lambda \dots \textcircled{6}$$

Rewriting  $\textcircled{5}$  as  $\bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \lambda t dt$  and using in  $\textcircled{6}$ , Fourier cosine integral representation of  $f(x)$  is given by:

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \cos \lambda t \cos \lambda x dt d\lambda$$

#### Remark:

- Parameter  $\lambda$  may be taken as  $p$ ,  $s$  or  $\omega$  as per usual notations.
- Fourier transform of  $f(x)$  may be given by  $\bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-i\lambda x} f(x) dx$ ,  
then Inverse Fourier transform of  $\bar{f}(\lambda)$  is given by  $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{i\lambda x} \bar{f}(\lambda) d\lambda$
- Sometimes Fourier transform of  $f(x)$  is taken as  $\bar{f}(\lambda) = \int_{-\infty}^\infty e^{i\lambda x} f(x) dx$ ,  
thereby Inverse Fourier transform is given by  $f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-i\lambda x} \bar{f}(\lambda) d\lambda$   
Similarly if Fourier Sine transform is taken as  $\bar{f}_s(\lambda) = \int_0^\infty f(x) \sin \lambda x dx$ ,  
then Inverse Sine transform is given by  $f(x) = \frac{2}{\pi} \int_0^\infty \bar{f}_s(\lambda) \sin \lambda x d\lambda$

Similar is the case with Fourier Cosine transform.

**Example 1** State giving reasons whether the Fourier transforms of the following functions exist: i.  $\sin \frac{1}{x}$  ii.  $e^x$  iii.  $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$

**Solution:** i. The graph of  $\sin \frac{1}{x}$  oscillates infinite number of times at  $x = n\pi, n \in \mathbb{Z}$

$\therefore f(x) \sin \frac{1}{x}$  is having infinite number of maxima and minima in the interval  $(-\infty, \infty)$ . Hence Fourier transform of  $f(x) = \sin \frac{1}{x}$  does not exist.

ii. For  $f(x) = e^x$ ,  $\int_{-\infty}^\infty |e^x| dx$  is not convergent. Hence Fourier transform of  $e^x$  does not exist.

iii.  $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$  is having infinite number of maxima and minima in the interval  $(-\infty, \infty)$ . Hence Fourier transform of  $f(x)$  does not exist.

**Example 2** Find Fourier Sine transform of

i.  $\frac{1}{x}$       ii.  $2e^{-3x} + 3e^{-2x}$

**Solution:** i. By definition, we have  $F_s\{f(x)\} \equiv \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \lambda x \, dx$

$$\therefore \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x} \sin \lambda x \, dx = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$$

ii. By definition,  $F_s\{f(x)\} \equiv \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \lambda x \, dx$

$$\begin{aligned} \therefore \bar{f}_s(\lambda) &= \sqrt{\frac{2}{\pi}} \int_0^\infty (2e^{-3x} + 3e^{-2x}) \sin \lambda x \, dx \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty 2e^{-3x} \sin \lambda x \, dx + \sqrt{\frac{2}{\pi}} \int_0^\infty 3e^{-2x} \sin \lambda x \, dx \\ &= \sqrt{\frac{2}{\pi}} \left[ \frac{2e^{-3x}}{9+\lambda^2} (-3 \sin \lambda x - \lambda \cos \lambda x) \right]_0^\infty + \sqrt{\frac{2}{\pi}} \left[ \frac{3e^{-2x}}{4+\lambda^2} (-2 \sin \lambda x - \lambda \cos \lambda x) \right]_0^\infty \\ &= \sqrt{\frac{2}{\pi}} \left[ 0 + \frac{2\lambda}{9+\lambda^2} \right] + \sqrt{\frac{2}{\pi}} \left[ 0 + \frac{3\lambda}{4+\lambda^2} \right] = \sqrt{\frac{2}{\pi}} \left[ \frac{2\lambda}{9+\lambda^2} + \frac{3\lambda}{4+\lambda^2} \right] = \sqrt{\frac{2}{\pi}} \left[ \frac{5\lambda^3+35\lambda}{(9+\lambda^2)(4+\lambda^2)} \right] \end{aligned}$$

**Example 3** Find Fourier transform of Delta function  $\delta(x - a)$

$$\begin{aligned} \text{Solution: } F\{\delta(x - a)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{i\lambda x} \cdot \delta(x - a) \, dx \\ &= \frac{1}{\sqrt{2\pi}} e^{i\lambda a} \end{aligned}$$

$\therefore \int_{-\infty}^\infty f(t) \delta(t - a) dt = f(a)$  by virtue of fundamental property of Delta function where  $f(t)$  is any differentiable function.

**Example 4** Show that Fourier sine and cosine transforms of  $x^{n-1}$  are  $\frac{[n]}{\lambda^n} \sin \frac{n\pi}{2}$  and

$$\frac{[n]}{\lambda^n} \cos \frac{n\pi}{2} \text{ respectively.}$$

**Solution:** By definition,  $\int_0^\infty e^{-t} t^{n-1} dt = [n]$

Putting  $t = i\lambda x$  so that  $dt = i\lambda dx$

$$\Rightarrow \int_0^\infty e^{-i\lambda x} (i\lambda x)^{n-1} i\lambda \, dx = [n]$$

$$\Rightarrow \int_0^\infty x^{n-1} e^{-i\lambda x} dx = \frac{[n] i^{-n}}{\lambda^n}$$

$$\Rightarrow \int_0^\infty x^{n-1} (\cos \lambda x - i \sin \lambda x) dx = \frac{[n]}{\lambda^n} \left( \cos \frac{n\pi}{2} - i \sin \frac{n\pi}{2} \right)$$

$$\because i^{-n} = \left( \cos \frac{n\pi}{2} - i \sin \frac{n\pi}{2} \right)$$

$$\Rightarrow \int_0^\infty x^{n-1} \cos \lambda x dx - i \int_0^\infty x^{n-1} \sin \lambda x dx = \frac{[n]}{\lambda^n} \cos \frac{n\pi}{2} - i \frac{[n]}{\lambda^n} \sin \frac{n\pi}{2}$$

Equating real and imaginary parts, we get

$$\int_0^\infty x^{n-1} \cos \lambda x dx = \frac{[n]}{\lambda^n} \cos \frac{n\pi}{2} \text{ and } \int_0^\infty x^{n-1} \sin \lambda x dx = \frac{[n]}{\lambda^n} \sin \frac{n\pi}{2}$$

$$\Rightarrow \bar{f}_c(\lambda) = \frac{[n]}{\lambda^n} \cos \frac{n\pi}{2} \text{ and } \bar{f}_s(\lambda) = \frac{[n]}{\lambda^n} \sin \frac{n\pi}{2}$$

**Example 5** Find Fourier Cosine transform of  $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$

**Solution:** By definition, we have  $F_c\{f(x)\} \equiv \bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \lambda x dx$

$$\begin{aligned} \therefore \bar{f}_c(\lambda) &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \lambda x dx \\ &= \sqrt{\frac{2}{\pi}} \left[ \int_0^1 x \cos \lambda x dx + \int_1^2 (2 - x) \cos \lambda x dx + \int_2^\infty 0 \cdot \cos \lambda x dx \right] \\ &= \sqrt{\frac{2}{\pi}} \left[ \left[ (x) \left( \frac{\sin \lambda x}{\lambda} \right) - (1) \left( -\frac{\cos \lambda x}{\lambda^2} \right) \right]_0^1 + \left[ (2 - x) \left( \frac{\sin \lambda x}{\lambda} \right) - (-1) \left( -\frac{\cos \lambda x}{\lambda^2} \right) \right]_1^2 \right] \\ &= \sqrt{\frac{2}{\pi}} \left[ \frac{\sin \lambda}{\lambda} + \frac{\cos \lambda}{\lambda^2} - \frac{1}{\lambda^2} - \frac{\cos 2\lambda}{\lambda^2} - \frac{\sin \lambda}{\lambda} + \frac{\cos \lambda}{\lambda^2} \right] = \sqrt{\frac{2}{\pi}} \left[ \frac{2 \cos \lambda - \cos 2\lambda - 1}{\lambda^2} \right] \end{aligned}$$

**Example 6** Find Fourier Sine and Cosine transform of  $f(x) = e^{-x}$  and hence show that

$$\int_0^\infty \frac{\cos mx}{1+x^2} dx = \frac{\pi}{2} e^{-m} = \int_0^\infty \frac{x \sin mx}{1+x^2} dx$$

**Solution:** To find Fourier Sine transform

$$\begin{aligned} F_s\{f(x)\} &\equiv \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \lambda x dx \\ \Rightarrow \bar{f}_s(\lambda) &= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin \lambda x dx = \sqrt{\frac{2}{\pi}} \left( \frac{\lambda}{1+\lambda^2} \right) \dots\dots\dots \textcircled{1} \end{aligned}$$

Taking inverse Fourier Sine transform of  $\textcircled{1}$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}_s(\lambda) \sin \lambda x d\lambda$$

$$\Rightarrow f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\lambda}{1+\lambda^2} \sin \lambda x d\lambda \dots \textcircled{2}$$

Substituting  $f(x) = e^{-x}$  in  $\textcircled{2}$

$$\Rightarrow e^{-x} = \frac{2}{\pi} \int_0^{\infty} \frac{\lambda \sin \lambda x}{1+\lambda^2} d\lambda$$

Replacing  $x$  by  $m$  on both sides

$$\Rightarrow e^{-m} = \frac{2}{\pi} \int_0^{\infty} \frac{\lambda \sin \lambda m}{1+\lambda^2} d\lambda$$

Now by property of definite integrals  $\int_a^b f(x) dx = \int_a^b f(y) dy$

$$\therefore \frac{\pi}{2} e^{-m} = \int_0^{\infty} \frac{x \sin mx}{1+x^2} dx \dots \textcircled{3}$$

Similarly taking Fourier Cosine transform of  $f(x) = e^{-x}$

$$F_c\{f(x)\} \equiv \bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \lambda x dx$$

$$\Rightarrow \bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \cos \lambda x dx = \sqrt{\frac{2}{\pi}} \left( \frac{1}{1+\lambda^2} \right) \dots \textcircled{4}$$

Taking inverse Fourier Cosine transform of  $\textcircled{4}$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \bar{f}_c(\lambda) \cos \lambda x d\lambda$$

$$\Rightarrow f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{1}{1+\lambda^2} \cos \lambda x d\lambda \dots \textcircled{5}$$

Substituting  $f(x) = e^{-x}$  in  $\textcircled{5}$

$$\Rightarrow e^{-x} = \frac{2}{\pi} \int_0^{\infty} \frac{\cos \lambda x}{1+\lambda^2} d\lambda$$

Replacing  $x$  by  $m$  on both sides

$$\Rightarrow e^{-m} = \frac{2}{\pi} \int_0^{\infty} \frac{\cos \lambda m}{1+\lambda^2} d\lambda$$

Again by property of definite integrals  $\int_a^b f(x) dx = \int_a^b f(y) dy$

$$\therefore \frac{\pi}{2} e^{-m} = \int_0^{\infty} \frac{\cos mx}{1+x^2} dx \dots \textcircled{6}$$

From  $\textcircled{3}$  and  $\textcircled{6}$ , we get

$$\int_0^{\infty} \frac{\cos mx}{1+x^2} dx = \frac{\pi}{2} e^{-m} = \int_0^{\infty} \frac{x \sin mx}{1+x^2} dx$$

**Example 7** Find Fourier transform of  $f(x) = \begin{cases} 1-x^2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

and hence evaluate  $\int_0^\infty \left( \frac{x \cos x - \sin x}{x^3} \right) \cos \frac{x}{2} dx$

**Solution:** Fourier transform of  $f(x)$  is given by

$$\begin{aligned}
 F\{f(x)\} &\equiv \bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} f(x) dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-1}^1 (1 - x^2) e^{i\lambda x} dx \\
 &= \frac{1}{\sqrt{2\pi}} \left[ (1 - x^2) \left( \frac{e^{i\lambda x}}{i\lambda} \right) - (-2x) \left( \frac{e^{i\lambda x}}{i^2 \lambda^2} \right) + (-2) \left( \frac{e^{i\lambda x}}{i^3 \lambda^3} \right) \right]_{-1}^1 \\
 &= \frac{1}{\sqrt{2\pi}} \left[ \frac{2e^{i\lambda}}{i^2 \lambda^2} - \frac{2e^{i\lambda}}{i^3 \lambda^3} + \frac{2e^{-i\lambda}}{i^2 \lambda^2} + \frac{2e^{-i\lambda}}{i^3 \lambda^3} \right] \\
 &= \sqrt{\frac{2}{\pi}} \left[ -\frac{e^{i\lambda} + e^{-i\lambda}}{\lambda^2} + \frac{e^{i\lambda} - e^{-i\lambda}}{i\lambda^3} \right] \quad \because i^2 = -1 \text{ and } i^3 = -i \\
 &= \sqrt{\frac{2}{\pi}} \left[ -\frac{2 \cos \lambda}{\lambda^2} + \frac{2 \sin \lambda}{\lambda^3} \right] \\
 \therefore \bar{f}(\lambda) &= \frac{2\sqrt{2}}{\sqrt{\pi}} \left( \frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3} \right) \dots\dots \textcircled{1}
 \end{aligned}$$

Taking inverse Fourier transform of  $\textcircled{1}$

$$\begin{aligned}
 f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda x} \bar{f}(\lambda) d\lambda \\
 \Rightarrow f(x) &= \frac{2}{\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} \left( \frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3} \right) d\lambda \\
 \Rightarrow f(x) &= \frac{2}{\pi} \int_{-\infty}^{\infty} (\cos \lambda x - i \sin \lambda x) \left( \frac{\lambda \cos \lambda - \sin \lambda}{\lambda^3} \right) d\lambda \quad \because e^{-i\lambda x} = \cos \lambda x - i \sin \lambda x \\
 \Rightarrow f(x) &= \frac{2}{\pi} \int_{-\infty}^{\infty} \left[ \cos \lambda x \left( \frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3} \right) - i \sin \lambda x \left( \frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3} \right) \right] d\lambda \dots\dots \textcircled{2}
 \end{aligned}$$

Substituting  $f(x) = \begin{cases} 1 - x^2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$  in  $\textcircled{2}$

$$\Rightarrow \begin{cases} 1 - x^2, & |x| < 1 \\ 0, & |x| > 1 \end{cases} = \frac{2}{\pi} \int_{-\infty}^{\infty} \left[ \cos \lambda x \left( \frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3} \right) - i \sin \lambda x \left( \frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3} \right) \right] d\lambda$$

Equating real parts on both sides, we get

$$\int_{-\infty}^{\infty} \cos \lambda x \left( \frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3} \right) d\lambda = \begin{cases} \frac{\pi}{2} (1 - x^2), & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

Putting  $x = \frac{1}{2}$  on both sides

$$\begin{aligned}
 \int_{-\infty}^{\infty} \cos \frac{\lambda}{2} \left( \frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3} \right) d\lambda &= \frac{\pi}{2} \left( 1 - \frac{1}{4} \right) \\
 \Rightarrow 2 \int_0^{\infty} \cos \frac{\lambda}{2} \left( \frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3} \right) d\lambda &= \frac{3\pi}{8} \quad \because \cos \frac{\lambda}{2} \left( \frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3} \right) \text{ is an even function of } \lambda
 \end{aligned}$$

Now by property of definite integrals  $\int_a^b f(x)dx = \int_a^b f(y)dy$

$$\therefore \int_0^\infty \left( \frac{x \cos x - \sin x}{x^3} \right) \cos \frac{x}{2} dx = -\frac{3\pi}{16}$$

**Example 8** Find the Fourier cosine transform of  $f(x) = \frac{1}{1+x^2}$

**Solution:** To find Fourier cosine transform

$$\begin{aligned} F_c\{f(x)\} &\equiv \bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \lambda x dx \\ &\Rightarrow \bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{1+x^2} \cos \lambda x dx \dots\dots \textcircled{1} \end{aligned}$$

To evaluate the integral given by  $\textcircled{1}$

Let  $g(x) = e^{-x} \dots\dots \textcircled{2}$

$$\begin{aligned} F_c\{g(x)\} &\equiv \bar{g}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos \lambda x dx \\ &\Rightarrow \bar{g}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos \lambda x dx \\ &= \sqrt{\frac{2}{\pi}} \left[ \frac{e^{-x}}{1+\lambda^2} (-\cos \lambda x + \lambda \sin \lambda x) \right]_0^\infty \\ &\Rightarrow \bar{g}_c(\lambda) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\lambda^2} \end{aligned}$$

Again taking Inverse Fourier cosine transform

$$\begin{aligned} g(x) &= \frac{2}{\pi} \int_0^\infty \frac{1}{1+\lambda^2} \cos \lambda x d\lambda \\ &\Rightarrow g(\lambda) = \frac{2}{\pi} \int_0^\infty \frac{1}{1+x^2} \cos \lambda x dx \\ &\Rightarrow \int_0^\infty \frac{1}{1+x^2} \cos \lambda x dx = \frac{\pi}{2} g(\lambda) \dots\dots \textcircled{3} \end{aligned}$$

Using  $\textcircled{2}$  in  $\textcircled{3}$ , we get

$$\Rightarrow \int_0^\infty \frac{1}{1+x^2} \cos \lambda x dx = \frac{\pi}{2} e^{-\lambda} \dots\dots \textcircled{4}$$

Using  $\textcircled{4}$  in  $\textcircled{1}$ , we get

$$\bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{1+x^2} \cos \lambda x dx = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} e^{-\lambda} = \sqrt{\frac{\pi}{2}} e^{-\lambda}$$

**Example 9** Find the Fourier sine transform of  $f(x) = \frac{e^{-ax}}{x}$  and use it to evaluate  $\int_0^{\infty} \tan^{-1}\left(\frac{x}{a}\right) \sin x dx$

**Solution:** To find Fourier sine transform

$$F_s\{f(x)\} \equiv \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \lambda x dx$$

$$\Rightarrow \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{x} \sin \lambda x dx$$

To evaluate the integral, differentiating both sides with respect to  $\lambda$

$$\frac{d}{d\lambda} \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{x} (\cos \lambda x) x dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos \lambda x dx = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + \lambda^2}$$

Now integrating both sides with respect to  $\lambda$

$$\bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int \frac{a}{a^2 + \lambda^2} d\lambda$$

$$\Rightarrow \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{\lambda}{a}\right) + c$$

when  $\lambda = 0$ ,  $\bar{f}_s(\lambda) = 0$ ,  $\Rightarrow c = 0$

$$\therefore \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{\lambda}{a}\right)$$

Again taking Inverse Fourier Sine transform

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \tan^{-1}\left(\frac{\lambda}{a}\right) \sin \lambda x d\lambda$$

Substituting  $f(x) = \frac{e^{-ax}}{x}$  on both sides

$$\frac{e^{-ax}}{x} = \frac{2}{\pi} \int_0^{\infty} \tan^{-1}\left(\frac{\lambda}{a}\right) \sin \lambda x d\lambda$$

Putting  $x=1$  on both sides

$$\frac{\pi}{2} e^{-a} = \int_0^{\infty} \tan^{-1}\left(\frac{\lambda}{a}\right) \sin \lambda d\lambda$$

$$\Rightarrow \int_0^{\infty} \tan^{-1}\left(\frac{\lambda}{a}\right) \sin x dx = \frac{\pi}{2} e^{-a}$$

**Example 10** If  $t > 0$  Show that i.  $\int_0^{\infty} \frac{\cos \lambda t}{\lambda^2 + a^2} d\lambda = \frac{\pi}{2a} e^{-at}, a > 0$

$$\text{ii. } \int_0^\infty \frac{\lambda \sin \lambda t}{\lambda^2 + a^2} d\lambda = \frac{\pi}{2} e^{at}, a \leq 0$$

**Solution:** i. Let  $f(t) = \frac{\pi}{2a} e^{-at}, a > 0, t > 0$

Taking Fourier cosine transform of  $f(t)$ , we get

$$\begin{aligned} F_c\{f(t)\} &\equiv \bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \lambda t dt \\ &= \frac{\pi}{2a} \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-at} \cos \lambda t dt \\ &= \frac{1}{a} \sqrt{\frac{\pi}{2}} \frac{a}{a^2 + \lambda^2} \end{aligned}$$

Also inverse Fourier cosine transform of  $\bar{f}_c(\lambda)$  gives  $f(t)$  as:

$$\begin{aligned} f(t) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}_c(\lambda) \cos \lambda t d\lambda \\ &= \frac{1}{a} \sqrt{\frac{\pi}{2}} \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{a}{a^2 + \lambda^2} \cos \lambda t d\lambda \\ \Rightarrow f(t) &= \int_0^\infty \frac{\cos \lambda t}{\lambda^2 + a^2} d\lambda \\ \therefore \int_0^\infty \frac{\cos \lambda t}{\lambda^2 + a^2} d\lambda &= \frac{\pi}{2a} e^{-at}, a > 0 \end{aligned}$$

ii. Again let  $g(t) = \frac{\pi}{2} e^{at}, a \leq 0, t > 0$

Taking Fourier sine transform of  $g(t)$ , we get

$$\begin{aligned} F_s\{g(t)\} &\equiv \bar{g}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(t) \sin \lambda t dt \\ &= \frac{\pi}{2} \sqrt{\frac{2}{\pi}} \int_0^\infty e^{at} \sin \lambda t dt, a \leq 0 \\ &= \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-at} \sin \lambda t dt, a > 0 \\ &= \sqrt{\frac{\pi}{2}} \frac{\lambda}{a^2 + \lambda^2} \end{aligned}$$

Also inverse Fourier sine transform of  $\bar{g}_s(\lambda)$  gives  $g(t)$  as:

$$g(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{g}_s(\lambda) \sin \lambda t d\lambda$$

$$\begin{aligned}
&= \sqrt{\frac{\pi}{2}} \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\lambda}{a^2 + \lambda^2} \sin \lambda t \, d\lambda \\
&\Rightarrow g(t) = \int_0^\infty \frac{\lambda \sin \lambda t}{\lambda^2 + a^2} d\lambda \\
\therefore \int_0^\infty \frac{\lambda \sin \lambda t}{\lambda^2 + a^2} d\lambda &= \frac{\pi}{2} e^{at}, a \leq 0
\end{aligned}$$

**Example 11** Prove that Fourier transform of  $e^{-\frac{x^2}{2}}$  is self reciprocal.

**Solution:** Fourier transform of  $f(x)$  is given by

$$\begin{aligned}
F\{f(x)\} &\equiv \bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{i\lambda x} f(x) dx \\
\therefore F\left\{e^{-\frac{x^2}{2}}\right\} &= \bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{x^2}{2}} e^{i\lambda x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{x^2}{2} + i\lambda x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{\frac{-1}{2}(x^2 - 2i\lambda x)} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{\frac{-1}{2}(x^2 - 2i\lambda x + (i\lambda)^2 - (i\lambda)^2)} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{\frac{-1}{2}(x - i\lambda)^2 + \frac{i^2 \lambda^2}{2}} dx \\
&= \frac{e^{-\frac{\lambda^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{\frac{-1}{2}(x - i\lambda)^2} dx \\
&= \frac{e^{-\frac{\lambda^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{z^2}{2}} dz \quad \text{By putting } z = (x - i\lambda) \\
&= \frac{2e^{-\frac{\lambda^2}{2}}}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{z^2}{2}} dz \quad e^{-\frac{z^2}{2}} \text{ being even function of } z \\
&= \frac{2e^{-\frac{\lambda^2}{2}}}{\sqrt{2\pi}} \int_0^\infty e^{-\left(\frac{z}{\sqrt{2}}\right)^2} dz
\end{aligned}$$

$$\text{Put } \frac{z}{\sqrt{2}} = t \Rightarrow dz = \sqrt{2} \, dt$$

$$\begin{aligned}
\therefore \bar{f}(\lambda) &= \frac{2\sqrt{2}e^{-\frac{\lambda^2}{2}}}{\sqrt{2\pi}} \int_0^\infty e^{-t^2} dt \\
&= \frac{2e^{-\frac{\lambda^2}{2}}}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = e^{-\frac{\lambda^2}{2}} \quad \because \int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}
\end{aligned}$$

$$\therefore F\left\{e^{-\frac{x^2}{2}}\right\} = e^{-\frac{\lambda^2}{2}}$$

Hence we see that Fourier transform of  $e^{\frac{-x^2}{2}}$  is given by  $e^{\frac{-\lambda^2}{2}}$ . Variable  $x$  is transformed to  $\lambda$ .  $\therefore$  We can say that Fourier transform of  $e^{\frac{-x^2}{2}}$  is self reciprocal.

**Example 12** Find Fourier Cosine transform of  $e^{-x^2}$ .

**Solution:** By definition,  $F_c\{f(x)\} \equiv \bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \lambda x \, dx$

$$\begin{aligned}
 \Rightarrow \bar{f}_c(\lambda) &= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cos \lambda x \, dx \\
 &= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \left( \frac{e^{i\lambda x} + e^{-i\lambda x}}{2} \right) dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_0^\infty (e^{-x^2} e^{i\lambda x} + e^{-x^2} e^{-i\lambda x}) dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_0^\infty (e^{-x^2 + i\lambda x} + e^{-x^2 - i\lambda x}) dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \left( e^{-\left(x^2 - 2\left(\frac{i\lambda}{2}\right)x + \left(\frac{i\lambda}{2}\right)^2 - \left(\frac{i\lambda}{2}\right)^2\right)} + e^{-\left(x^2 + 2\left(\frac{i\lambda}{2}\right)x + \left(\frac{i\lambda}{2}\right)^2 - \left(\frac{i\lambda}{2}\right)^2\right)} \right) dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \left( e^{-\left(x - \frac{i\lambda}{2}\right)^2 + \frac{i^2\lambda^2}{4}} + e^{-\left(x + \frac{i\lambda}{2}\right)^2 + \frac{i^2\lambda^2}{4}} \right) dx \\
 &= \frac{e^{\frac{-\lambda^2}{4}}}{\sqrt{2\pi}} \left[ \int_0^\infty e^{-\left(x - \frac{i\lambda}{2}\right)^2} dx + \int_0^\infty e^{-\left(x + \frac{i\lambda}{2}\right)^2} dx \right] \\
 &= \frac{e^{\frac{-\lambda^2}{4}}}{\sqrt{2\pi}} \left[ \frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{2} \right] = \frac{\sqrt{\pi} e^{\frac{-\lambda^2}{4}}}{\sqrt{2\pi}} \\
 \Rightarrow \bar{f}_c(\lambda) &= \frac{1}{\sqrt{2}} e^{\frac{-\lambda^2}{4}}
 \end{aligned}$$

Or

Fourier Cosine transform of  $e^{-x^2}$  can also be found using the method given below:

$$\bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cos \lambda x \, dx \dots \textcircled{1}$$

Differentiating both sides with respect to  $\lambda$

$$\Rightarrow \frac{d}{d\lambda} \bar{f}_c(\lambda) = -\sqrt{\frac{2}{\pi}} \int_0^\infty x e^{-x^2} \sin \lambda x \, dx$$

$$\begin{aligned}
&= \sqrt{\frac{2}{\pi}} \left[ \sin \lambda x \cdot \frac{e^{-x^2}}{2} \right]_0^\infty - \sqrt{\frac{2}{\pi}} \int_0^\infty \lambda \cos \lambda x \cdot \frac{e^{-x^2}}{2} dx \\
&= 0 - \frac{\lambda}{2} \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cos \lambda x dx \quad \dots \textcircled{2} \\
&\Rightarrow \frac{d}{d\lambda} \bar{f}_c(\lambda) = -\frac{\lambda}{2} \bar{f}_c(\lambda) \quad \text{using } \textcircled{1} \text{ in } \textcircled{2} \\
&\Rightarrow \frac{\frac{d}{d\lambda} \bar{f}_c(\lambda)}{\bar{f}_c(\lambda)} = -\frac{\lambda}{2}
\end{aligned}$$

Integrating both sides with respect to  $\lambda$

$$\begin{aligned}
&\Rightarrow \log \bar{f}_c(\lambda) = -\frac{\lambda^2}{4} + \log k, \text{ where } \log k \text{ is the constant of integration} \\
&\Rightarrow \bar{f}_c(\lambda) = K e^{-\frac{\lambda^2}{4}} \dots \textcircled{3} \\
&\Rightarrow \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cos \lambda x dx = k e^{-\frac{\lambda^2}{4}}
\end{aligned}$$

Putting  $\lambda = 0$  on both sides

$$\begin{aligned}
&\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} dx = k \\
&\Rightarrow k = \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2}} \dots \textcircled{4}
\end{aligned}$$

Using  $\textcircled{4}$  in  $\textcircled{3}$ , we get

$$\bar{f}_c(\lambda) = \frac{1}{\sqrt{2}} e^{-\frac{\lambda^2}{4}}$$

**Example 13** Find Fourier transform of  $x e^{-ax^2}$ ,  $a > 0$

**Solution:** By definition,  $F\{x e^{-ax^2}\} = \bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty x e^{-ax^2} e^{i\lambda x} dx$

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty x e^{-ax^2 + i\lambda x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty x e^{-a\left(x^2 - 2\left(\frac{i\lambda}{2a}\right)x + \left(\frac{i\lambda}{2a}\right)^2 - \left(\frac{i\lambda}{2a}\right)^2\right)} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty x e^{-a\left(x - \frac{i\lambda}{2a}\right)^2 + \frac{i^2\lambda^2}{4a}} dx \\
&= \frac{e^{-\frac{\lambda^2}{4a}}}{\sqrt{2\pi}} \left[ \int_{-\infty}^\infty \left(x - \frac{i\lambda}{2a}\right) e^{-a\left(x - \frac{i\lambda}{2a}\right)^2} dx + \frac{i\lambda}{2a} \int_{-\infty}^\infty e^{-a\left(x - \frac{i\lambda}{2a}\right)^2} dx \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{e^{-\frac{\lambda^2}{4a}}}{\sqrt{2\pi}} \left[ \int_{-\infty}^{\infty} t e^{-at^2} dt + \frac{i\lambda}{2a} \int_{-\infty}^{\infty} e^{-at^2} dt \right], \text{ Putting } \left(x - \frac{i\lambda}{2a}\right) = t \\
&= \frac{e^{-\frac{\lambda^2}{4a}}}{\sqrt{2\pi}} \left[ 0 + \frac{i\lambda}{a} \int_0^{\infty} e^{-at^2} dt \right] \\
&\quad \because t e^{-at^2} \text{ is odd function and } e^{-at^2} \text{ is even function in } t \\
&= \frac{e^{-\frac{\lambda^2}{4a}}}{\sqrt{2\pi}} \cdot \frac{i\lambda}{a} \int_0^{\infty} e^{-(\sqrt{a}t)^2} dt \\
&= \frac{e^{-\frac{\lambda^2}{4a}}}{\sqrt{2\pi}} \cdot \frac{i\lambda}{a\sqrt{a}} \int_0^{\infty} e^{-z^2} dz, \text{ Putting } \sqrt{a}t = z \\
&= \frac{e^{-\frac{\lambda^2}{4a}}}{\sqrt{2\pi}} \cdot \frac{i\lambda}{a\sqrt{a}} \cdot \frac{\sqrt{\pi}}{2} \quad \because \int_0^{\infty} e^{-z^2} dz = \frac{\sqrt{\pi}}{2} \\
&\Rightarrow \bar{f}(\lambda) = \frac{i\lambda e^{-\frac{\lambda^2}{4a}}}{2a\sqrt{2a}}
\end{aligned}$$

**Example 14** Find Fourier cosine integral representation of  $f(x) = \begin{cases} x^2, & 0 < x < a \\ 0, & x > a \end{cases}$

**Solution:** Taking Fourier Cosine transform of  $f(x) = \begin{cases} x^2, & 0 < x < a \\ 0, & x > a \end{cases}$

$$\begin{aligned}
F_c\{f(x)\} &\equiv \bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \lambda x dx \\
&\Rightarrow \bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^a x^2 \cos \lambda x dx \\
&= \sqrt{\frac{2}{\pi}} \left[ (x^2) \left( \frac{\sin \lambda x}{\lambda} \right) - (2x) \left( \frac{-\cos \lambda x}{\lambda^2} \right) + (2) \left( \frac{-\sin \lambda x}{\lambda^3} \right) \right]_0^a \\
&\Rightarrow \bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \left[ \left( \frac{a^2}{\lambda} - \frac{2}{\lambda^3} \right) \sin \lambda a + \frac{2a}{\lambda^2} \cos \lambda a \right]
\end{aligned}$$

Now taking Inverse Fourier Cosine transform

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left[ \left( \frac{a^2}{\lambda} - \frac{2}{\lambda^3} \right) \sin \lambda a + \frac{2a}{\lambda^2} \cos \lambda a \right] \cos \lambda x d\lambda$$

This is the required Fourier cosine integral representation of  $f(x) = \begin{cases} x^2, & 0 < x < a \\ 0, & x > a \end{cases}$

**Example 15** If  $f(x) = \begin{cases} \sin x, & 0 < x < \pi \\ 0, & \text{otherwise} \end{cases}$ , prove that

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{2\cos \lambda x + \cos(\pi+x)\lambda + \cos(\pi-x)\lambda}{1-\lambda^2} d\lambda . \text{ Hence evaluate } \int_0^\infty \frac{\cos \frac{\pi t}{2}}{1-t^2} dt$$

**Solution:** Given  $f(x) = \begin{cases} \sin x, & 0 < x < \pi \\ 0, & \text{otherwise} \end{cases}$

To find Fourier cosine integral representation of  $f(x)$ , taking Fourier Cosine transform of  $f(x)$

$$\begin{aligned} \bar{f}_c(\lambda) &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \lambda x dx = \sqrt{\frac{2}{\pi}} \int_0^\pi \sin x \cos \lambda x dx \\ &= \sqrt{\frac{1}{2\pi}} \int_0^\pi (\sin(\lambda+1)x - \sin(\lambda-1)x) dx \\ &= \sqrt{\frac{1}{2\pi}} \left[ -\frac{\cos(\lambda+1)x}{(\lambda+1)} + \frac{\cos(\lambda-1)x}{(\lambda-1)} \right]_0^\pi \\ &= \sqrt{\frac{1}{2\pi}} \left[ -\frac{\cos(\lambda+1)\pi}{(\lambda+1)} + \frac{\cos(\lambda-1)\pi}{(\lambda-1)} + \frac{1}{(\lambda+1)} - \frac{1}{(\lambda-1)} \right] \\ &= \sqrt{\frac{1}{2\pi}} \left[ \frac{\cos \lambda \pi}{(\lambda+1)} - \frac{\cos \lambda \pi}{(\lambda-1)} + \frac{1}{(\lambda+1)} - \frac{1}{(\lambda-1)} \right] \\ &= \sqrt{\frac{1}{2\pi}} \left[ \frac{(\lambda-1) \cos \lambda \pi - (\lambda+1) \cos \lambda \pi}{(\lambda+1)(\lambda-1)} + \frac{\lambda-1-\lambda-1}{(\lambda+1)(\lambda-1)} \right] \\ \Rightarrow \bar{f}_c(\lambda) &= \sqrt{\frac{1}{2\pi}} \left[ \frac{-2 \cos \lambda \pi - 2}{\lambda^2 - 1} \right] = \sqrt{\frac{2}{\pi}} \left[ \frac{1 + \cos \lambda \pi}{1 - \lambda^2} \right] \end{aligned}$$

Taking Inverse Fourier Cosine transform,  $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}_c(\lambda) \cos \lambda x d\lambda$

$$\begin{aligned} \Rightarrow f(x) &= \frac{2}{\pi} \int_0^\infty \left[ \frac{1 + \cos \lambda \pi}{1 - \lambda^2} \right] \cos \lambda x d\lambda \\ &= \frac{1}{\pi} \int_0^\infty \frac{2 \cos \lambda x + 2 \cos \lambda \pi \cos \lambda x}{1 - \lambda^2} d\lambda \\ &= \frac{1}{\pi} \int_0^\infty \frac{2 \cos \lambda x + \cos(\pi+x)\lambda + \cos(\pi-x)\lambda}{1 - \lambda^2} d\lambda \\ \Rightarrow f(x) &= \frac{1}{\pi} \int_0^\infty \frac{2 \cos \lambda x + \cos(\pi+x)\lambda + \cos(\pi-x)\lambda}{1 - \lambda^2} d\lambda \end{aligned}$$

Putting  $f(x) = \begin{cases} \sin x, & 0 < x < \pi \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow \frac{1}{\pi} \int_0^\infty \frac{2 \cos \lambda x + \cos(\pi+x)\lambda + \cos(\pi-x)\lambda}{1 - \lambda^2} d\lambda = \begin{cases} \sin x, & 0 < x < \pi \\ 0, & \text{otherwise} \end{cases}$$

Putting  $x = \frac{\pi}{2}$  on both sides

$$\Rightarrow \frac{1}{\pi} \int_0^\infty \frac{2\cos\frac{\pi\lambda}{2} + \cos\left(\pi+\frac{\pi}{2}\right)\lambda + \cos\left(\pi-\frac{\pi}{2}\right)\lambda}{1-\lambda^2} d\lambda = 1$$

$$\Rightarrow \int_0^\infty \frac{\cos\frac{\pi\lambda}{2}}{1-\lambda^2} d\lambda = \frac{\pi}{2} \Rightarrow \int_0^\infty \frac{\cos\frac{\pi t}{2}}{1-t^2} dt = \frac{\pi}{2}$$

**Example 16** Solve the integral equation  $\int_0^\infty f(x)\cos\lambda x dx = \begin{cases} 1-\lambda, & 0 \leq \lambda \leq 1 \\ 0, & \lambda > 1 \end{cases}$

Hence deduce that  $\int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$

**Solution:** Given that  $\int_0^\infty f(x)\cos\lambda x dx = \begin{cases} 1-\lambda, & 0 \leq \lambda \leq 1 \\ 0, & \lambda > 1 \end{cases} \dots\dots ①$

$$\Rightarrow \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos\lambda x dx = \begin{cases} \sqrt{\frac{2}{\pi}}(1-\lambda), & 0 \leq \lambda \leq 1 \\ 0, & \lambda > 1. \end{cases}$$

$$\Rightarrow \bar{f}_c(\lambda) = \begin{cases} \sqrt{\frac{2}{\pi}}(1-\lambda), & 0 \leq \lambda \leq 1 \\ 0, & \lambda > 1. \end{cases}$$

Taking Inverse Fourier Cosine transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}_c(\lambda) \cos \lambda x d\lambda$$

$$\begin{aligned} \Rightarrow f(x) &= \frac{2}{\pi} \int_0^1 (1-\lambda) \cos \lambda x d\lambda \\ &= \frac{2}{\pi} \left[ (1-\lambda) \left( \frac{\sin \lambda x}{x} \right) - (-1) \left( \frac{-\cos \lambda x}{x^2} \right) \right]_0^1 \\ &= \frac{2}{\pi} \left[ -\frac{\cos x}{x^2} + \frac{1}{x^2} \right] = \frac{2}{\pi} \left[ \frac{1-\cos x}{x^2} \right] = \frac{2}{\pi} \frac{2\sin^2 \frac{x}{2}}{x^2} \\ \Rightarrow f(x) &= \frac{4\sin^2 \frac{x}{2}}{\pi x^2} \dots\dots ② \end{aligned}$$

Using ② in ①, we get

$$\int_0^\infty \frac{4\sin^2 \frac{x}{2}}{\pi x^2} \cos \lambda x dx = \begin{cases} 1-\lambda, & 0 \leq \lambda \leq 1 \\ 0, & \lambda > 1 \end{cases}$$

Putting  $\lambda = 0$  on both sides

$$\Rightarrow \frac{4}{\pi} \int_0^\infty \frac{\sin^2 \frac{x}{2}}{x^2} dx = 1$$

$$\Rightarrow \int_0^\infty \frac{\sin^2 \frac{x}{2}}{x^2} dx = \frac{\pi}{4}$$

Putting  $\frac{x}{2} = t$ ,  $dx = 2dt$

$$\therefore \int_0^{\infty} \frac{\sin^2 t}{4t^2} 2dt = \frac{\pi}{4} \Rightarrow \int_0^{\infty} \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$$

**Example 17** Find the function  $f(x)$  if its Cosine transform is given by:

$$(i) \frac{\sin a\lambda}{\lambda} \quad (ii) \begin{cases} \frac{1}{\sqrt{2\pi}} \left(a - \frac{\lambda}{2}\right), & \lambda < 2a \\ 0, & \lambda \geq 2a \end{cases}$$

**Solution:** (i) Given that  $\bar{f}_c(\lambda) = \frac{\sin a\lambda}{\lambda}$

Taking Inverse Fourier Cosine transform

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \bar{f}_c(\lambda) \cos \lambda x d\lambda$$

$$\begin{aligned} \Rightarrow f(x) &= \frac{2}{\pi} \int_0^{\infty} \frac{\sin a\lambda}{\lambda} \cos \lambda x d\lambda \\ &= \frac{1}{2} \cdot \frac{2}{\pi} \int_0^{\infty} \frac{2 \sin a\lambda \cos \lambda x}{\lambda} d\lambda \\ &= \frac{1}{\pi} \int_0^{\infty} \frac{\sin(a+x)\lambda}{\lambda} d\lambda + \frac{1}{\pi} \int_0^{\infty} \frac{\sin(a-x)\lambda}{\lambda} d\lambda \end{aligned}$$

Now  $0 < x < \infty \therefore a + x > 0$

$$\Rightarrow f(x) = \begin{cases} \frac{1}{\pi} \left[ \frac{\pi}{2} + \frac{\pi}{2} \right], & a - x > 0 \text{ i.e. } x < a \\ \frac{1}{\pi} \left[ \frac{\pi}{2} - \frac{\pi}{2} \right], & a - x < 0 \text{ i.e. } x > a \end{cases} \quad \because \int_0^{\infty} \frac{\sin \lambda x}{x} dx = \frac{\pi}{2}, \lambda > 0$$

$$\Rightarrow f(x) = \begin{cases} 1, & x < a \\ 0, & x > a \end{cases}$$

(ii) Given that  $\bar{f}_c(\lambda) = \begin{cases} \frac{1}{\sqrt{2\pi}} \left(a - \frac{\lambda}{2}\right), & \lambda < 2a \\ 0, & \lambda \geq 2a \end{cases}$

Taking Inverse Fourier Cosine transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \bar{f}_c(\lambda) \cos \lambda x d\lambda$$

$$\begin{aligned} \Rightarrow f(x) &= \sqrt{\frac{2}{\pi}} \int_0^{2a} \frac{1}{\sqrt{2\pi}} \left(a - \frac{\lambda}{2}\right) \cos \lambda x d\lambda \\ &= \frac{1}{\pi} \int_0^{2a} \left(a - \frac{\lambda}{2}\right) \cos \lambda x d\lambda \\ &= \frac{1}{\pi} \left[ \left(a - \frac{\lambda}{2}\right) \left(\frac{\sin \lambda x}{x}\right) - \left(-\frac{1}{2}\right) \left(-\frac{\cos \lambda x}{x^2}\right) \right]_0^{2a} \end{aligned}$$

$$= \frac{1}{\pi} \left[ -\frac{\cos 2ax}{2x^2} + \frac{1}{2x^2} \right] = \frac{1}{2\pi x^2} [1 - \cos 2ax] = \frac{\sin^2 ax}{\pi x^2}$$

**Example 18** Find the function  $f(x)$  if its Sine transform is given by:

$$(i) e^{-a\lambda} \quad (ii) \frac{\lambda}{1+\lambda^2}$$

**Solution:** (i) Given that  $\bar{f}_s(\lambda) = e^{-a\lambda}$

Taking Inverse Fourier Sine transform

$$f(x) = \frac{2}{\pi} \int_0^\infty \bar{f}_s(\lambda) \sin \lambda x d\lambda$$

$$\Rightarrow f(x) = \frac{2}{\pi} \int_0^\infty e^{-a\lambda} \sin \lambda x d\lambda = \frac{2}{\pi} \cdot \frac{x}{a^2 + x^2}$$

$$(ii) \text{ Given that } \bar{f}_s(\lambda) = \frac{\lambda}{1+\lambda^2}$$

Taking Inverse Fourier Sine transform

$$f(x) = \frac{2}{\pi} \int_0^\infty \bar{f}_s(\lambda) \sin \lambda x d\lambda$$

$$\Rightarrow f(x) = \frac{2}{\pi} \int_0^\infty \frac{\lambda}{1+\lambda^2} \sin \lambda x d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\lambda^2}{\lambda(1+\lambda^2)} \sin \lambda x d\lambda = \frac{2}{\pi} \int_0^\infty \frac{(1+\lambda^2)-1}{\lambda(1+\lambda^2)} \sin \lambda x d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\sin \lambda x}{\lambda} d\lambda - \frac{2}{\pi} \int_0^\infty \frac{\sin \lambda x}{\lambda(1+\lambda^2)} d\lambda$$

$$\Rightarrow f(x) = 1 - \frac{2}{\pi} \int_0^\infty \frac{\sin \lambda x}{\lambda(1+\lambda^2)} d\lambda \quad \dots\dots \textcircled{1}$$

$$\because \int_0^\infty \frac{\sin \lambda x}{\lambda} d\lambda = \frac{\pi}{2}, x > 0$$

Differentiating with respect to  $x$

$$\Rightarrow f'(x) = 0 - \frac{2}{\pi} \int_0^\infty \frac{\lambda \cos \lambda x}{\lambda(1+\lambda^2)} d\lambda$$

$$\Rightarrow f'(x) = -\frac{2}{\pi} \int_0^\infty \frac{\cos \lambda x}{(1+\lambda^2)} d\lambda \quad \dots\dots \textcircled{2}$$

$$\text{Also } f''(x) = \frac{2}{\pi} \int_0^\infty \frac{\lambda \sin \lambda x}{(1+\lambda^2)} d\lambda = f(x)$$

$$\Rightarrow f''(x) - f(x) = 0 \dots\dots \textcircled{3}$$

This is a linear differential equation with constant coefficients

$$\textcircled{3} \text{ may be written as } (D^2 - 1)f(x) = 0$$

$$\text{Auxiliary equation is } m^2 - 1 = 0$$

$$\Rightarrow m = \pm 1$$

Solution of ③ is given by

$$f(x) = c_1 e^x + c_2 e^{-x} \dots \text{④}$$

$$\Rightarrow f'(x) = c_1 e^x - c_2 e^{-x} \dots \text{⑤}$$

Now from ①,  $f(x) = 1$ , at  $x = 0$

Using in ④, we get  $c_1 + c_2 = 1 \dots \text{⑥}$

Again from ②,  $f'(x) = -\frac{2}{\pi} \int_0^\infty \frac{1}{(1+\lambda^2)} d\lambda$ , at  $x = 0$

$$\Rightarrow f'(x) = -\frac{2}{\pi} [\tan^{-1} \lambda]_0^\infty = -1 \text{ at } x = 0$$

Using in ⑤, we get  $c_1 - c_2 = -1 \dots \text{⑦}$

Solving ⑥ and ⑦, we get  $c_1 = 0$ ,  $c_2 = 1$

Using in ④, we get  $f(x) = e^{-x}$

**Note:** Solution of the differential equation  $f''(x) - f(x) = 0$  may be written directly

as  $f(x) = e^{-x}$

**Example 19** Find the Fourier transform of the function  $f(x) = e^{-a|x|}$ ,  $-\infty < x < \infty$

**Solution:**  $f(x) = \begin{cases} e^{ax}, & x < 0 \\ e^{-ax}, & x \geq 0 \end{cases}$

Fourier transform of  $f(x)$  is given by  $F\{f(x)\} \equiv \bar{f}(\lambda) = \int_{-\infty}^\infty e^{i\lambda x} f(x) dx$

$$\Rightarrow \bar{f}(\lambda) = \int_{-\infty}^0 e^{ax} e^{i\lambda x} dx + \int_0^\infty e^{-ax} e^{i\lambda x} dx$$

$$= \int_{-\infty}^0 e^{x(a+i\lambda)} dx + \int_0^\infty e^{-x(a-i\lambda)} dx$$

$$= \left[ \frac{e^{x(a+i\lambda)}}{(a+i\lambda)} \right]_{-\infty}^0 - \left[ \frac{e^{-x(a-i\lambda)}}{(a-i\lambda)} \right]_0^\infty$$

$$\Rightarrow \bar{f}(\lambda) = \frac{1}{a+i\lambda} + \frac{1}{a-i\lambda} = \frac{2a}{a^2+\lambda^2}$$

$$\therefore F\{e^{-a|x|}\} = \frac{2a}{a^2+\lambda^2}$$

**Result:**

$$F\{e^{-a|x|}\} = \frac{2a}{a^2+\lambda^2} \Rightarrow F^{-1}\left[\frac{2a}{a^2+\lambda^2}\right] = e^{-a|x|}$$

**Example 20** Find  $F^{-1} \left[ \frac{1}{(9+\lambda^2)(4+\lambda^2)} \right]$

$$\begin{aligned}
 \text{Solution: } F^{-1} \left[ \frac{1}{(9+\lambda^2)(4+\lambda^2)} \right] &= \frac{1}{5} F^{-1} \left[ -\frac{1}{9+\lambda^2} + \frac{1}{4+\lambda^2} \right] \\
 &= \frac{1}{5} F^{-1} \left[ -\frac{1}{3^2+\lambda^2} + \frac{1}{2^2+\lambda^2} \right] \\
 &= \frac{-1}{30} F^{-1} \left[ \frac{6}{9+\lambda^2} \right] + \frac{1}{20} F^{-1} \left[ \frac{4}{4+\lambda^2} \right] \\
 &= \frac{-1}{30} e^{-3|x|} + \frac{1}{20} e^{-2|x|} \quad \because F^{-1} \left[ \frac{2a}{a^2+\lambda^2} \right] = e^{-a|x|}
 \end{aligned}$$

**Example 21** Find the Fourier transform of the function  $f(x) = e^{-ax}U(x)$ ,  $a > 0$   
where  $U(x)$  represents unit step function

$$\text{Solution: } f(x) = e^{-ax} \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases} = \begin{cases} 0, & x < 0 \\ e^{-ax}, & x \geq 0 \end{cases}$$

Fourier transform of  $f(x)$  is given by  $F\{f(x)\} \equiv \bar{f}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} f(x) dx$

$$\begin{aligned}
 \Rightarrow \bar{f}(\lambda) &= \int_0^{\infty} e^{-ax} e^{i\lambda x} dx \\
 &= \int_0^{\infty} e^{-x(a-i\lambda)} dx \\
 &= - \left[ \frac{e^{-x(a-i\lambda)}}{(a-i\lambda)} \right]_0^{\infty}
 \end{aligned}$$

$$\Rightarrow \bar{f}(\lambda) = \frac{1}{a-i\lambda}$$

$$\therefore F\{f(x)\} = \frac{1}{a-i\lambda}$$

$$\text{or } F\{e^{-ax}U(x)\} = \frac{1}{a-i\lambda}$$

**Result:** 
$$F\{e^{-ax}U(x)\} = \frac{1}{a-i\lambda} \Rightarrow F^{-1} \left[ \frac{1}{a-i\lambda} \right] = e^{-ax}U(x) = e^{-ax}H(x)$$

**Note:** If Fourier transform of  $f(x) = e^{-ax}U(x)$  is taken as  $\int_{-\infty}^{\infty} e^{-i\lambda x} e^{-ax}U(x) dx$ , then  $F^{-1} \left[ \frac{1}{a+i\lambda} \right] = e^{-ax}U(x) = e^{-ax}H(x)$

**Example 22** Find the inverse transform of the following functions:

$$\text{i. } \frac{1}{2-3i\lambda-\lambda^2} \quad \text{ii. } \frac{1}{8+6i\lambda-\lambda^2} \quad \text{iii. } \frac{5}{6-5i\lambda-\lambda^2}$$

$$\text{Solution: i. } F^{-1} \left[ \frac{1}{2-3i\lambda-\lambda^2} \right] = F^{-1} \left[ \frac{1}{(1-i\lambda)(2-i\lambda)} \right] = F^{-1} \left[ \frac{1}{(1-i\lambda)} - \frac{1}{(2-i\lambda)} \right]$$

$$\begin{aligned}
&= F^{-1} \left[ \frac{1}{(1-i\lambda)} \right] - F^{-1} \left[ \frac{1}{(2-i\lambda)} \right] \\
&= e^{-x}H(x) - e^{-2x}H(x) \quad \because F^{-1} \left[ \frac{1}{a-i\lambda} \right] = e^{-ax}H(x) \\
&\Rightarrow F^{-1} \left[ \frac{1}{2-3i\lambda-\lambda^2} \right] = \begin{cases} (e^{-x} - e^{-2x}), & x \geq 0 \\ 0, & x < 0 \end{cases} \\
\text{ii. } F^{-1} \left[ \frac{1}{8+6i\lambda-\lambda^2} \right] &= F^{-1} \left[ \frac{1}{(4+i\lambda)(2+i\lambda)} \right] = F^{-1} \left[ \frac{1}{(4+i\lambda)} - \frac{1}{(2+i\lambda)} \right] \\
&= F^{-1} \left[ \frac{1}{(4+i\lambda)} \right] - F^{-1} \left[ \frac{1}{(2+i\lambda)} \right] \\
&= e^{-4x}H(x) - e^{-2x}H(x) \quad \because F^{-1} \left[ \frac{1}{a+i\lambda} \right] = e^{-ax}H(x) \\
&\Rightarrow F^{-1} \left[ \frac{1}{8+6i\lambda-\lambda^2} \right] = \begin{cases} (e^{-x} - e^{-2x}), & x \geq 0 \\ 0, & x < 0 \end{cases} \\
\text{iii. } F^{-1} \left[ \frac{5}{6-5i\lambda-\lambda^2} \right] &= 5 F^{-1} \left[ \frac{1}{(2-i\lambda)(3-i\lambda)} \right] = 5F^{-1} \left[ \frac{1}{(2-i\lambda)} - \frac{1}{(3-i\lambda)} \right] \\
&= 5F^{-1} \left[ \frac{1}{(2-i\lambda)} \right] - 5F^{-1} \left[ \frac{1}{(3-i\lambda)} \right] \\
&= 5e^{-2x}H(x) - 5e^{-3x}H(x) \quad \because F^{-1} \left[ \frac{1}{a-i\lambda} \right] = e^{-ax}H(x) \\
&\Rightarrow F^{-1} \left[ \frac{5}{6-5i\lambda-\lambda^2} \right] = \begin{cases} 5(e^{-2x} - e^{-3x}), & x \geq 0 \\ 0, & x < 0 \end{cases}
\end{aligned}$$

**Example 23** Find the Fourier transform of  $f(x) = \frac{1}{2-ix}$

**Solution:** We know  $F^{-1} \left[ \frac{1}{a-i\lambda} \right] = e^{-ax}H(x)$

$$\begin{aligned}
&\Rightarrow F^{-1} \left[ \frac{1}{2-i\lambda} \right] = e^{-2x}H(x) \\
&\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2-i\lambda} e^{-i\lambda x} d\lambda = e^{-2x}H(x)
\end{aligned}$$

Interchanging  $x$  and  $\lambda$ , we get

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2-ix} e^{-i\lambda x} dx &= e^{-2\lambda}H(\lambda) \\
&= \begin{cases} 0, & \lambda < 0 \\ e^{-2\lambda}, & \lambda \geq 0 \end{cases} \\
\Rightarrow \int_{-\infty}^{\infty} \frac{1}{2-ix} e^{-i\lambda x} dx &= \begin{cases} 0, & \lambda < 0 \\ 2\pi e^{-2\lambda}, & \lambda \geq 0 \end{cases}
\end{aligned}$$

$$\Rightarrow F\left\{\frac{1}{2-ix}\right\} = \begin{cases} 0, & \lambda < 0 \\ 2\pi e^{-2\lambda}, & \lambda \geq 0 \end{cases}$$

## 2.3 Properties of Fourier Transforms

**Linearity:** If  $\bar{f}(\lambda)$  and  $\bar{g}(\lambda)$  are Fourier transforms of  $f(x)$  and  $g(x)$  respectively, then

$$F\{af(x) + bg(x)\} = a\bar{f}(\lambda) + b\bar{g}(\lambda)$$

**Proof:** 
$$\begin{aligned} F\{af(x) + bg(x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)] e^{i\lambda x} dx \\ &= a \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\lambda x} dx + b \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{i\lambda x} dx \\ &= a\bar{f}(\lambda) + b\bar{g}(\lambda) \end{aligned}$$

**Change of scale:** If  $\bar{f}(\lambda)$  is Fourier transforms of  $f(x)$ , then  $F\{f(ax)\} = \frac{1}{a} \bar{f}\left(\frac{\lambda}{a}\right)$

**Proof:** 
$$\begin{aligned} F\{f(ax)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) \cdot e^{i\lambda x} dx \\ \text{Putting } ax &= t \Rightarrow adx = dt \\ \therefore F\{f(ax)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{i\lambda \frac{t}{a}} \cdot \frac{dt}{a} = \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{i\left(\frac{\lambda}{a}\right)t} dt \\ &= \frac{1}{a} \bar{f}\left(\frac{\lambda}{a}\right) \end{aligned}$$

**Shifting Property:** If  $\bar{f}(\lambda)$  is Fourier transforms of  $f(x)$ , then  $F\{f(x-a)\} = e^{i\lambda a} \bar{f}(\lambda)$

**Proof:** 
$$\begin{aligned} F\{f(x-a)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) \cdot e^{i\lambda x} dx \\ \text{Putting } (x-a) &= t \Rightarrow dx = dt \\ \therefore F\{f(x-a)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{i\lambda(t+a)} dt \\ &= e^{i\lambda a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{i\lambda t} dt = e^{i\lambda a} \bar{f}(\lambda) \end{aligned}$$

**Modulation Theorem:** If  $\bar{f}(\lambda)$  is Fourier transforms of  $f(x)$ , then

- i.  $F\{f(x) \cos ax\} = \frac{1}{2} \{\bar{f}(\lambda + a) + \bar{f}(\lambda - a)\}$
- ii.  $F_s[f(x) \cos ax] = \frac{1}{2} \{\bar{f}_s(\lambda + a) + \bar{f}_s(\lambda - a)\}$
- iii.  $F_c[f(x) \sin ax] = \frac{1}{2} \{\bar{f}_s(\lambda + a) - \bar{f}_s(\lambda - a)\}$
- iv.  $F_c[f(x) \cos ax] = \frac{1}{2} \{\bar{f}_c(\lambda + a) + \bar{f}_c(\lambda - a)\}$

$$\text{v. } F_s[f(x) \sin ax] = \frac{1}{2} \{ \bar{f}_c(\lambda - a) - \bar{f}_c(\lambda + a) \}$$

**Proof:** i.  $F\{f(x) \cos ax\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos ax \cdot e^{i\lambda x} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \frac{e^{iax} + e^{-iax}}{2} e^{i\lambda x} dx$$

$$= \frac{1}{2} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(\lambda+a)x} dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(\lambda-a)x} dx \right]$$

$$= \frac{1}{2} \{ \bar{f}(\lambda + a) + \bar{f}(\lambda - a) \}$$

ii.  $F_s[f(x) \cos ax] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos ax \sin \lambda x dx$

$$= \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) [\sin(\lambda + a)x + \sin(\lambda - a)x] dx$$

$$= \frac{1}{2} \left[ \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin(\lambda + a)x dx + \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin(\lambda - a)x dx \right]$$

$$= \frac{1}{2} \{ \bar{f}_s(\lambda + a) + \bar{f}_s(\lambda - a) \}$$

iii.  $F_c[f(x) \sin ax] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin ax \cos \lambda x dx$

$$= \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) [\sin(\lambda + a)x - \sin(\lambda - a)x] dx$$

$$= \frac{1}{2} \left[ \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin(\lambda + a)x dx - \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin(\lambda - a)x dx \right]$$

$$= \frac{1}{2} \{ \bar{f}_s(\lambda + a) - \bar{f}_s(\lambda - a) \}$$

iv.  $F_c[f(x) \cos ax] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos ax \cos \lambda x dx$

$$= \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) [\cos(\lambda + a)x + \cos(\lambda - a)x] dx$$

$$= \frac{1}{2} \left[ \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(\lambda + a)x dx + \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(\lambda - a)x dx \right]$$

$$= \frac{1}{2} \{ \bar{f}_c(\lambda + a) + \bar{f}_c(\lambda - a) \}$$

v.  $F_s[f(x) \sin ax] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin ax \sin \lambda x dx$

$$\begin{aligned}
&= \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) [\cos(\lambda - a)x - \cos(\lambda + a)x] dx \\
&= \frac{1}{2} \left[ \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\lambda - a)x dx - \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\lambda + a)x dx \right] \\
&= \frac{1}{2} \{ \bar{f}_c(\lambda - a) - \bar{f}_c(\lambda + a) \}
\end{aligned}$$

**Convolution theorem:** Convolution of two functions  $f(x)$  and  $g(x)$  is defined as

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du$$

If  $\bar{f}(\lambda)$  and  $\bar{g}(\lambda)$  are Fourier transforms of  $f(x)$  and  $g(x)$  respectively, then Convolution theorem for Fourier transforms states that

$$F\{f(x) * g(x)\} = F\{f(x)\} \cdot F\{g(x)\} \equiv \bar{f}(\lambda) \cdot \bar{g}(\lambda)$$

**Proof:** By definition  $\bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} f(x) dx$  and  $\bar{g}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} g(x) dx$

$$\text{Now } f(x) * g(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du$$

$$\therefore F\{f(x) * g(x)\} = \int_{-\infty}^{\infty} e^{i\lambda x} \left[ \int_{-\infty}^{\infty} f(u)g(x-u)du \right] dx$$

Changing the order of integration, we get

$$\therefore F\{f * g\} = \int_{-\infty}^{\infty} f(u) \left[ \int_{-\infty}^{\infty} e^{i\lambda x} g(x-u) dx \right] du$$

Putting  $x - u = t \Rightarrow dx = dt$  in the inner integral, we get

$$\begin{aligned}
F\{f * g\} &= \int_{-\infty}^{\infty} f(u) \left[ \int_{-\infty}^{\infty} e^{i\lambda(u+t)} g(t) dt \right] du \\
&= \int_{-\infty}^{\infty} e^{i\lambda u} f(u) \left[ \int_{-\infty}^{\infty} e^{i\lambda t} g(t) dt \right] du \\
&= \int_{-\infty}^{\infty} e^{i\lambda u} f(u) \bar{g}(\lambda) du \\
&= \bar{g}(\lambda) \int_{-\infty}^{\infty} e^{i\lambda u} f(u) du \\
&= \bar{f}(\lambda) \cdot \bar{g}(\lambda)
\end{aligned}$$

**Example 24** Find the Fourier transform of  $e^{-x^2}$ . Hence find Fourier transforms of

$$\text{i. } e^{-ax^2}, a > 0 \quad \text{ii. } e^{-\frac{x^2}{2}} \quad \text{iii. } e^{2(x-3)^2} \quad \text{iv. } e^{-x^2} \cos 2x$$

**Solution:**  $F\{e^{-x^2}\} = \bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} e^{i\lambda x} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2 + i\lambda x} dx$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x^2 - 2\left(\frac{i\lambda}{2}\right)x + \left(\frac{i\lambda}{2}\right)^2 - \left(\frac{i\lambda}{2}\right)^2\right)} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x - \frac{i\lambda}{2}\right)^2 + \frac{i^2\lambda^2}{4}} dx \\
&= \frac{e^{-\frac{\lambda^2}{4}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x - \frac{i\lambda}{2}\right)^2} dx \\
&= \frac{e^{-\frac{\lambda^2}{4}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2} dz \quad \text{By putting } z = \left(x - \frac{i\lambda}{2}\right) \\
&= \frac{2e^{-\frac{\lambda^2}{4}}}{\sqrt{2\pi}} \int_0^{\infty} e^{-z^2} dz \quad e^{-z^2} \text{ being even function of } z \\
\therefore \bar{f}(\lambda) &= \frac{2e^{-\frac{\lambda^2}{4}}}{\sqrt{2\pi}} \cdot \frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2}} e^{-\frac{\lambda^2}{4}} \dots\dots \textcircled{1}
\end{aligned}$$

$$\therefore \text{We have } F\{f(x)\} = \bar{f}(\lambda) = \frac{1}{\sqrt{2}} e^{-\frac{\lambda^2}{4}} \text{ if } f(x) = e^{-x^2}$$

$$\begin{aligned}
\text{i. Now } F\{e^{-ax^2}\} &= F\{e^{(\sqrt{a}\sqrt{x})^2}\} \\
&= \frac{1}{\sqrt{a}} \bar{f}\left(\frac{\lambda}{\sqrt{a}}\right) \quad \text{By change of scale property} \dots \textcircled{2}
\end{aligned}$$

$$\therefore F\{e^{-ax^2}\} = \frac{1}{\sqrt{a}} \cdot \frac{1}{\sqrt{2}} e^{-\frac{1}{4}\left(\frac{\lambda}{\sqrt{a}}\right)^2} = \frac{1}{\sqrt{2a}} e^{-\frac{\lambda^2}{4a}} \quad \text{Using } \textcircled{1} \text{ in } \textcircled{2}$$

$$\text{ii. Putting } a = \frac{1}{2} \text{ in i.}$$

$$F\left\{e^{-\frac{x^2}{2}}\right\} = \frac{1}{\sqrt{2 \cdot \frac{1}{2}}} \cdot e^{-\frac{\lambda^2}{2}} = e^{-\frac{\lambda^2}{2}}$$

$$\text{iii. To find } F\{e^{-2(x-3)^2}\}, \text{ Put } a = 2 \text{ in i.}$$

$$F\{e^{-2x^2}\} = \frac{1}{2} e^{-\frac{\lambda^2}{8}}$$

$$\therefore F\{e^{-2(x-3)^2}\} = e^{3i\lambda} \cdot \frac{1}{2} e^{-\frac{\lambda^2}{8}} \therefore \text{By shifting property } F\{f(x-k)\} = e^{i\lambda k} \bar{f}(\lambda)$$

$$\text{iv. To find Fourier transform of } F\{e^{-x^2} \cos 2x\}$$

$$F\{f(x) \cos ax\} = \frac{1}{2} \bar{f}(\lambda + a) + \bar{f}(\lambda - a) \quad \text{By modulation theorem}$$

$$\text{Now } F\{e^{-x^2}\} \equiv \bar{f}(\lambda) = \frac{1}{\sqrt{2}} e^{-\frac{\lambda^2}{4}}.$$

$$\therefore F\{e^{-x^2} \cos 2x\} = \frac{1}{2} \left[ \frac{1}{\sqrt{2}} e^{-\frac{(\lambda+2)^2}{4}} + \frac{1}{\sqrt{2}} e^{-\frac{(\lambda-2)^2}{4}} \right]$$

**Example 25** Using Convolution theorem, find  $F^{-1} \left[ \frac{1}{12-7i\lambda-\lambda^2} \right]$

**Solution:**  $F^{-1} \left[ \frac{1}{12-7i\lambda-\lambda^2} \right] = F^{-1} \left[ \frac{1}{(4-i\lambda)(3-i\lambda)} \right] = F^{-1} \left[ \frac{1}{(4-i\lambda)} \cdot \frac{1}{(3-i\lambda)} \right]$

Now by Convolution theorem

$$F\{f(x) * g(x)\} = \bar{f}(\lambda) \cdot \bar{g}(\lambda) \Rightarrow F^{-1}[\bar{f}(\lambda) \cdot \bar{g}(\lambda)] = f(x) * g(x)$$

$$\begin{aligned} \therefore F^{-1} \left[ \frac{1}{(4-i\lambda)} \cdot \frac{1}{(3-i\lambda)} \right] &= F^{-1} \left[ \frac{1}{(4-i\lambda)} \right] * F^{-1} \left[ \frac{1}{(3-i\lambda)} \right] \\ &= e^{-4x} H(x) * e^{-3x} H(x) \quad \because F^{-1} \left[ \frac{1}{a-i\lambda} \right] = e^{-ax} H(x) \\ &= \int_{-\infty}^{\infty} e^{-4u} H(u) e^{-3(x-u)} H(x-u) du \\ &\quad \because f(x) * g(x) = \int_{-\infty}^{\infty} f(u) g(x-u) du \\ &= e^{-3x} \int_{-\infty}^{\infty} e^{-u} H(u) H(x-u) du \end{aligned}$$

$$\text{Now } H(u)H(x-u) = \begin{cases} 1, & u \geq 0, x-u \geq 0, \quad \text{i.e. } 0 \leq u \leq x \\ 0, & u < 0, x-u < 0, \quad \text{i.e. } u < 0 \text{ and } u > x \end{cases}$$

$$\begin{aligned} \therefore F^{-1} \left[ \frac{1}{12-7i\lambda-\lambda^2} \right] &= e^{-3x} \int_0^x e^{-u} du = -e^{-3x} [e^{-u}]_0^x = -e^{-3x} [e^{-x} - 1], x \geq 0 \\ &= e^{-3x} - e^{-4x}, x \geq 0 \end{aligned}$$

$$\Rightarrow F^{-1} \left[ \frac{1}{12-7i\lambda-\lambda^2} \right] = \begin{cases} e^{-3x} - e^{-4x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

**Example 26** Find the inverse Fourier transforms of  $\frac{e^{3i\lambda}}{2-i\lambda}$

**Solution:** i. We know that  $F^{-1} \left[ \frac{1}{a-i\lambda} \right] = e^{-ax} H(x)$

$$\therefore F^{-1} \left[ \frac{1}{2-i\lambda} \right] = e^{-2x} H(x)$$

Now By shifting property  $F\{f(x-k)\} = e^{i\lambda k} \bar{f}(\lambda)$

$$\Rightarrow F^{-1}[e^{i\lambda k} \bar{f}(\lambda)] = f(x-k)$$

$$\therefore F^{-1} \left[ \frac{e^{3i\lambda}}{2-i\lambda} \right] = e^{-2(x-3)} H(x-3)$$

## 2.4 Fourier Transforms of Derivatives

Let  $u(x, t)$  be a function of two independent variables  $x$  and  $t$ , such that Fourier transform of  $u(x, t)$  is denoted by  $\bar{u}(\lambda, t)$  i.e  $\bar{u}(\lambda, t) = \int_{-\infty}^{\infty} e^{i\lambda x} u(x, t) dx$

Again let  $u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots \rightarrow 0$  as  $x \rightarrow \pm\infty$ ,

Then Fourier transforms of  $\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots$  with respect to  $x$  are given by:

$$\begin{aligned} 1. \quad F\left\{\frac{\partial u}{\partial x}\right\} &= \int_{-\infty}^{\infty} e^{i\lambda x} \frac{\partial u}{\partial x} dx = \left[e^{i\lambda x} u\right]_{-\infty}^{\infty} - i\lambda \int_{-\infty}^{\infty} e^{i\lambda x} u dx = -i\lambda \bar{u}(\lambda, t) \\ F\left\{\frac{\partial^2 u}{\partial x^2}\right\} &= \int_{-\infty}^{\infty} e^{i\lambda x} \frac{\partial^2 u}{\partial x^2} dx = \left[e^{i\lambda x} \frac{\partial u}{\partial x}\right]_{-\infty}^{\infty} - i\lambda \int_{-\infty}^{\infty} e^{i\lambda x} \frac{\partial u}{\partial x} dx = (-i\lambda)^2 \bar{u}(\lambda, t) \\ &\vdots \\ F\left\{\frac{\partial^n u}{\partial x^n}\right\} &= (-i\lambda)^n \bar{u}(\lambda, t) \end{aligned}$$

2. Fourier sine transform of  $\frac{\partial^2 u}{\partial x^2}$  is given by:

$$\begin{aligned} F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} &= \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} \sin \lambda x dx = \left[\sin \lambda x \frac{\partial u}{\partial x}\right]_0^{\infty} - \lambda \int_0^{\infty} \cos \lambda x \frac{\partial u}{\partial x} dx \\ &= 0 - \lambda [\cos \lambda x \cdot u(x, t)]_0^{\infty} - \lambda^2 \int_0^{\infty} \sin \lambda x \frac{\partial^2 u}{\partial x^2} dx \end{aligned}$$

$$\therefore F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \lambda u(0, t) - \lambda^2 \bar{u}_s(\lambda, t)$$

3. Fourier cosine transform of  $\frac{\partial^2 u}{\partial x^2}$  is given by:

$$\begin{aligned} F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} &= \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} \cos \lambda x dx = \left[\cos \lambda x \frac{\partial u}{\partial x}\right]_0^{\infty} + \lambda \int_0^{\infty} \sin \lambda x \frac{\partial u}{\partial x} dx \\ &= -\left[\frac{\partial u}{\partial x}\right]_{x=0} + \lambda [\sin \lambda x \cdot u(x, t)]_0^{\infty} - \lambda^2 \int_0^{\infty} \cos \lambda x \frac{\partial^2 u}{\partial x^2} dx \end{aligned}$$

$$\therefore F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -\left[\frac{\partial u}{\partial x}\right]_{x=0} - \lambda^2 \bar{u}_c(\lambda, t)$$

4. Fourier transforms of  $\frac{\partial u}{\partial t}$  with respect to  $x$  are given by:

$$F\left\{\frac{\partial u}{\partial t}\right\} = \int_{-\infty}^{\infty} e^{i\lambda x} \frac{\partial u}{\partial t} dx = \frac{d}{dt} \int_{-\infty}^{\infty} e^{i\lambda x} u(x, t) dx$$

$$\therefore F\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt} \bar{u}(\lambda, t)$$

$$\text{Similarly } F_s\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt} \bar{u}_s(\lambda, t)$$

$$F_c\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt} \bar{u}_c(\lambda, t)$$

## 2.5 Applications of Fourier Transforms to boundary value problems

Partial differential equation together with boundary and initial conditions can be easily solved using Fourier transforms. In one dimensional boundary value problems, the partial differential equations can easily be transformed into an ordinary differential equation by applying a suitable transform and solution to boundary value problem is obtained by applying inverse transform. In two dimensional problems, it is sometimes required to apply the transforms twice and the desired solution is obtained by double inversion.

### **Algorithm to solve partial differential equations with boundary values:**

1. Apply the suitable transform to given partial differential equation. For this check the range of  $x$ 
  - i. If  $-\infty < x < \infty$ , then apply Fourier transform.
  - ii. If  $0 < x < \infty$ , then check initial value conditions
    - a) If value of  $u(0, t)$  is given, then apply Fourier sine transform
    - b) If value of  $\left[\frac{\partial u}{\partial x}\right]_{x=0}$  is given, then apply Fourier cosine transform

An ordinary differential equation will be formed after applying the transform.

2. Solve the differential equation using usual methods.
3. Apply Boundary value conditions to evaluate arbitrary constants.
4. Apply inverse transform to get the required expression for  $u(x, t)$ .

**Example 27** The temperature  $u(x, t)$  at any point of an infinite bar satisfies the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty, t > 0 \text{ and the initial temperature along the length of the bar is given by } u(x, 0) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

Determine the expression for  $u(x, t)$ .

**Solution:** As range of  $x$  is  $(-\infty, \infty)$ , applying Fourier transform to both sides of the given equation :

$$F\left\{\frac{\partial u}{\partial t}\right\} = F\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$
$$\Rightarrow \frac{d}{dt}\bar{u}(\lambda, t) = -\lambda^2\bar{u}(\lambda, t) \because F\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}\bar{u}(\lambda, t) \text{ and } F\left\{\frac{\partial^2 u}{\partial x^2}\right\} = (-i\lambda)^2 \bar{u}(\lambda, t)$$

Rearranging the ordinary differential equation in variable separable form:

$$\Rightarrow \frac{d\bar{u}}{\bar{u}} = -\lambda^2 dt \dots \textcircled{1} \quad \text{where } \bar{u} \approx \bar{u}(\lambda, t)$$

Solving  $\textcircled{1}$  using usual methods of variable separable differential equations

$$\log \bar{u} = -\lambda^2 t + \log A$$

$$\Rightarrow \log \frac{\bar{u}}{A} = -\lambda^2 t$$

$$\Rightarrow \bar{u}(\lambda, t) = A e^{-\lambda^2 t} \dots \textcircled{2}$$

Putting  $t = 0$  on both sides

$$\Rightarrow \bar{u}(\lambda, 0) = A \dots \textcircled{3}$$

Now given that  $u(x, 0) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$

Taking Fourier transform on both sides, we get

$$\begin{aligned} \Rightarrow \bar{u}(\lambda, 0) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, 0) e^{i\lambda x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^1 e^{i\lambda x} dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{i\lambda} [e^{i\lambda x}]_{-1}^1 \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{i\lambda} [e^{i\lambda} - e^{-i\lambda}] = \frac{1}{\sqrt{2\pi}} \frac{2i}{i\lambda} \left[ \frac{e^{i\lambda} - e^{-i\lambda}}{2i} \right] \end{aligned}$$

$$\Rightarrow \bar{u}(\lambda, 0) = \frac{2}{\sqrt{2\pi}} \frac{\sin \lambda}{\lambda} \dots \textcircled{4}$$

From  $\textcircled{3}$  and  $\textcircled{4}$ , we get

$$A = \frac{2}{\sqrt{2\pi}} \frac{\sin \lambda}{\lambda} \dots \textcircled{5}$$

Using  $\textcircled{5}$  in  $\textcircled{2}$ , we get

$$\bar{u}(\lambda, t) = \frac{2}{\sqrt{2\pi}} \frac{\sin \lambda}{\lambda} e^{-\lambda^2 t}$$

Taking Inverse Fourier transform

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda x} \bar{u}(\lambda, t) d\lambda \\ \Rightarrow u(x, t) &= \frac{2}{2\pi} \int_{-\infty}^{\infty} \frac{\sin \lambda}{\lambda} e^{-\lambda^2 t} e^{-i\lambda x} d\lambda \\ \Rightarrow u(x, t) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \lambda}{\lambda} e^{-\lambda^2 t} (\cos \lambda x - i \sin \lambda x) d\lambda \\ \Rightarrow u(x, t) &= \frac{2}{\pi} \int_0^{\infty} e^{-\lambda^2 t} \left( \frac{\sin \lambda \cos \lambda x}{\lambda} \right) d\lambda \quad \because \left( \frac{\sin \lambda \sin \lambda x}{\lambda} \right) \text{ is odd function of } \lambda \end{aligned}$$

**Example 28** Using Fourier transform, solve the equation  $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ ,  $0 < x < \infty$ ,  $t > 0$  subject to conditions:

- i.  $u(0, t) = 0, \quad t > 0$
- ii.  $u(x, 0) = e^{-x}, \quad x > 0$
- iii.  $u$  and  $\frac{\partial u}{\partial x}$  both tend to zero as  $x \rightarrow \pm\infty$

**Solution:** As range of  $x$  is  $(0, \infty)$ , and also value of  $u(0, t)$  is given in initial value conditions, applying Fourier sine transform to both sides of the given equation:

$$\begin{aligned}
 F_s \left\{ \frac{\partial u}{\partial t} \right\} &= k F_s \left\{ \frac{\partial^2 u}{\partial x^2} \right\} \\
 \Rightarrow \frac{d}{dt} \bar{u}_s(\lambda, t) &= k \lambda u(0, t) - k \lambda^2 \bar{u}_s(\lambda, t) \\
 \therefore F_s \left\{ \frac{\partial u}{\partial t} \right\} &= \frac{d}{dt} \bar{u}_s(\lambda, t) \text{ and } F_s \left\{ \frac{\partial^2 u}{\partial x^2} \right\} = \lambda u(0, t) - \lambda^2 \bar{u}_s(\lambda, t) \\
 \Rightarrow \frac{d}{dt} \bar{u}_s(\lambda, t) &= -k \lambda^2 \bar{u}_s(\lambda, t) \quad \therefore u(0, t) = 0
 \end{aligned}$$

Rearranging the ordinary differential equation in variable separable form:

$$\Rightarrow \frac{d\bar{u}}{\bar{u}} = -k\lambda^2 dt \dots \textcircled{1} \quad \text{where } \bar{u} \approx \bar{u}_s(\lambda, t)$$

Solving  $\textcircled{1}$  using usual methods of variable separable differential equations

$$\log \bar{u} = -k\lambda^2 t + \log A$$

$$\Rightarrow \log \frac{\bar{u}}{A} = -k\lambda^2 t$$

$$\Rightarrow \bar{u}_s(\lambda, t) = A e^{-k\lambda^2 t} \dots \textcircled{2}$$

Putting  $t = 0$  on both sides

$$\Rightarrow \bar{u}_s(\lambda, 0) = A \dots \textcircled{3}$$

Now given that  $u(x, 0) = e^{-x}$

Taking Fourier sine transform on both sides, we get

$$\Rightarrow \bar{u}_s(\lambda, 0) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x, 0) \sin \lambda x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin \lambda x \, dx$$

$$\Rightarrow \bar{u}_s(\lambda, 0) = \sqrt{\frac{2}{\pi}} \frac{\lambda}{1+\lambda^2} \dots \textcircled{4}$$

From  $\textcircled{3}$  and  $\textcircled{4}$ , we get

$$A = \sqrt{\frac{2}{\pi}} \frac{\lambda}{1+\lambda^2} \dots \textcircled{5}$$

Using  $\textcircled{5}$  in  $\textcircled{2}$ , we get

$$\bar{u}_s(\lambda, t) = \sqrt{\frac{2}{\pi}} \frac{\lambda}{1+\lambda^2} e^{-k\lambda^2 t}$$

Taking Inverse Fourier sine transform

$$u(x, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{u}_s(\lambda, t) \sin \lambda x d\lambda$$

$$\Rightarrow u(x, t) = \frac{2}{\pi} \int_0^\infty \frac{\lambda}{1+\lambda^2} e^{-k\lambda^2 t} \sin \lambda x d\lambda$$

**Example 29** The temperature  $u(x, t)$  in a semi-infinite rod  $0 < x < \infty$  is determined by the differential equation  $\frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2}$  subject to conditions:

- i.  $u = 0$ , when  $t = 0$ ,  $x \geq 0$
- ii.  $\frac{\partial u}{\partial x} = -k$  (a constant), when  $x = 0, t > 0$

**Solution:** As range of  $x$  is  $(0, \infty)$ , and also value of  $\left[\frac{\partial u}{\partial x}\right]_{x=0}$  is given in initial value conditions, applying Fourier cosine transform to both sides of the equation:

$$F_c \left\{ \frac{\partial u}{\partial t} \right\} = 2 F_c \left\{ \frac{\partial^2 u}{\partial x^2} \right\}$$

$$\Rightarrow \frac{d}{dt} \bar{u}_c(\lambda, t) = -2 \left[ \frac{\partial u}{\partial x} \right]_{x=0} - 2\lambda^2 \bar{u}_c(\lambda, t)$$

$$\because F_c \left\{ \frac{\partial u}{\partial t} \right\} = \frac{d}{dt} \bar{u}_c(\lambda, t) \text{ and } F_c \left\{ \frac{\partial^2 u}{\partial x^2} \right\} = - \left[ \frac{\partial u}{\partial x} \right]_{x=0} - \lambda^2 \bar{u}_c(\lambda, t)$$

$$\Rightarrow \frac{d}{dt} \bar{u}_c(\lambda, t) = 2k - 2\lambda^2 \bar{u}_c(\lambda, t)$$

$$\Rightarrow \frac{d\bar{u}}{dt} + 2\lambda^2 \bar{u} = 2k \dots \textcircled{1} \quad \text{where } \bar{u} \approx \bar{u}_c(\lambda, t)$$

This is a linear differential equation of the form  $\frac{dy}{dx} + Py = Q$

where  $P = 2\lambda^2$ ,  $Q = 2k$

Integrating Factor (IF) =  $e^{\int P dt} = e^{\int 2\lambda^2 dt} = e^{2\lambda^2 t}$

Solution of  $\textcircled{1}$  is given by

$$\bar{u} \cdot e^{2\lambda^2 t} = \int 2k \cdot e^{2\lambda^2 t} dt + A$$

$$\Rightarrow \bar{u}.e^{2\lambda^2 t} = \frac{2ke^{2\lambda^2 t}}{2\lambda^2} + A$$

$$\Rightarrow \bar{u}_c(\lambda, t) = \frac{k}{\lambda^2} + Ae^{-2\lambda^2 t} \dots \textcircled{2}$$

Putting  $t = 0$  on both sides

$$\Rightarrow \bar{u}_c(\lambda, 0) = \frac{k}{\lambda^2} + A \dots \textcircled{3}$$

Now given that  $u(x, 0) = 0$

Taking Fourier cosine transform on both sides, we get

$$\Rightarrow \bar{u}_c(\lambda, 0) = \int_0^\infty u(x, 0) \cos \lambda x \, dx = 0$$

$$\Rightarrow \bar{u}_c(\lambda, 0) = 0 \dots \textcircled{4}$$

From  $\textcircled{3}$  and  $\textcircled{4}$ , we get

$$A = -\frac{k}{\lambda^2} \dots \textcircled{5}$$

Using  $\textcircled{5}$  in  $\textcircled{2}$ , we get

$$\bar{u}_c(\lambda, t) = \frac{k}{\lambda^2} (1 - e^{-2\lambda^2 t})$$

Taking Inverse Fourier cosine transform

$$u(x, t) = \frac{2}{\pi} \int_0^\infty \bar{u}_c(\lambda, t) \cos \lambda x \, d\lambda$$

$$\Rightarrow u(x, t) = \frac{2k}{\pi} \int_0^\infty \left( \frac{1 - e^{-2\lambda^2 t}}{\lambda^2} \right) \cos \lambda x \, d\lambda$$

**Example 30** Using Fourier transforms, solve the equation  $\frac{\partial y}{\partial t} = k \frac{\partial^2 y}{\partial x^2}$ ,  $x > 0$ ,  $t > 0$  subject to conditions:

- i.  $y = \alpha$ , when  $x = 0$ ,  $t > 0$
- ii.  $y = 0$ , when  $t = 0$ ,  $x > 0$

**Solution:** As range of  $x$  is  $(0, \infty)$ , and also value of  $y(0, t)$  is given in initial value conditions, applying Fourier sine transform to both sides of the given equation:

$$F_s \left\{ \frac{\partial y}{\partial t} \right\} = k F_s \left\{ \frac{\partial^2 y}{\partial x^2} \right\}$$

$$\Rightarrow \frac{d}{dt} \bar{y}_s(\lambda, t) = k \lambda y(0, t) - k \lambda^2 \bar{y}_s(\lambda, t)$$

$$\because F_s \left\{ \frac{\partial y}{\partial t} \right\} = \frac{d}{dt} \bar{y}_s(\lambda, t) \text{ and } F_s \left\{ \frac{\partial^2 y}{\partial x^2} \right\} = \lambda y(0, t) - \lambda^2 \bar{y}_s(\lambda, t)$$

$$\Rightarrow \frac{d}{dt} \bar{y}_s(\lambda, t) = k\alpha\lambda - k\lambda^2 \bar{y}_s(\lambda, t) \quad \because y(0, t) = \alpha$$

$$\Rightarrow \frac{d\bar{y}}{dt} + k\lambda^2 \bar{y} = k\alpha\lambda \dots \textcircled{1} \quad \text{where } \bar{y} \approx \bar{y}_s(\lambda, t)$$

This is a linear differential equation of the form  $\frac{dy}{dx} + Py = Q$

where  $P = k\lambda^2$ ,  $Q = k\alpha\lambda$

$$\text{Integrating Factor (IF)} = e^{\int P dt} = e^{\int k\lambda^2 dt} = e^{k\lambda^2 t}$$

Solution of  $\textcircled{1}$  is given by

$$\bar{y} \cdot e^{k\lambda^2 t} = \int k\alpha\lambda \cdot e^{k\lambda^2 t} dt + A$$

$$\Rightarrow \bar{y} \cdot e^{k\lambda^2 t} = \frac{k\alpha\lambda e^{k\lambda^2 t}}{k\lambda^2} + A$$

$$\Rightarrow \bar{y}_s(\lambda, t) = \frac{\alpha}{\lambda} + A e^{-k\lambda^2 t} \dots \textcircled{2}$$

Putting  $t = 0$  on both sides

$$\Rightarrow \bar{y}_c(\lambda, 0) = \frac{\alpha}{\lambda} + A \dots \textcircled{3}$$

Now given that  $y(x, 0) = 0$

Taking Fourier sine transform on both sides, we get

$$\Rightarrow \bar{y}_s(\lambda, 0) = \int_0^\infty y(x, 0) \sin \lambda x dx = 0$$

$$\Rightarrow \bar{y}_s(\lambda, 0) = 0 \dots \textcircled{4}$$

From  $\textcircled{3}$  and  $\textcircled{4}$ , we get

$$A = -\frac{\alpha}{\lambda} \dots \textcircled{5}$$

Using  $\textcircled{5}$  in  $\textcircled{2}$ , we get

$$\bar{y}_s(\lambda, t) = \frac{\alpha}{\lambda} (1 - e^{-k\lambda^2 t})$$

Taking Inverse Fourier sine transform

$$y(x, t) = \frac{2}{\pi} \int_0^\infty \bar{y}_s(\lambda, t) \sin \lambda x d\lambda$$

$$\Rightarrow y(x, t) = \frac{2\alpha}{\pi} \int_0^\infty \left( \frac{1 - e^{-k\lambda^2 t}}{\lambda} \right) \sin \lambda x d\lambda$$

**Example 31** An infinite string is initially at rest and its initial displacement is given by  $f(x)$ ,  $-\infty < x < \infty$ . Determine the displacement  $y(x, t)$  of the string.

**Solution:** The equation of the vibrating string is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Initial conditions are

- i.  $\left. \frac{\partial y}{\partial t} \right|_{t=0} = 0$
- ii.  $y(x, 0) = f(x)$

Taking Fourier transform on both sides

$$\begin{aligned} F\left\{\frac{\partial^2 y}{\partial t^2}\right\} &= c^2 F\left\{\frac{\partial^2 y}{\partial x^2}\right\} \\ \Rightarrow \frac{d^2}{dt^2} \bar{y}(\lambda, t) &= -c^2 \lambda^2 \bar{y}(\lambda, t) \quad \text{where } F\{y(x, t)\} \equiv \bar{y}(\lambda, t) \\ \Rightarrow \frac{d^2 \bar{y}}{dt^2} + c^2 \lambda^2 \bar{y} &= 0 \dots \textcircled{1} \quad \text{where } \bar{y} \approx \bar{y}(\lambda, t) \end{aligned}$$

Solution of  $\textcircled{1}$  is given by

$$\bar{y}(\lambda, t) = A \cos cpt + B \sin cpt \dots \textcircled{2}$$

Putting  $t = 0$  on both sides

$$\bar{y}(\lambda, 0) = A \dots \textcircled{3}$$

Given that  $y(x, 0) = f(x)$

$$\Rightarrow \bar{y}(\lambda, 0) = \bar{f}(\lambda) \dots \textcircled{4}$$

From  $\textcircled{3}$  and  $\textcircled{4}$

$$A = \bar{f}(\lambda) \dots \textcircled{5}$$

Using  $\textcircled{5}$  in  $\textcircled{2}$

$$\bar{y}(\lambda, t) = \bar{f}(\lambda) \cos cpt + B \sin cpt \dots \textcircled{6}$$

$$\Rightarrow \frac{\partial y}{\partial t} = -cp \bar{f}(\lambda) \sin cpt + cpB \cos cpt$$

$$\Rightarrow \left. \frac{\partial y}{\partial t} \right|_{t=0} = cpB \dots \textcircled{7}$$

$$\text{Also given that } \left. \frac{\partial y}{\partial t} \right|_{t=0} = 0 \dots \textcircled{8}$$

$$\text{From } \textcircled{7} \text{ and } \textcircled{8}, \text{ we get } B = 0 \dots \textcircled{9}$$

Using  $\textcircled{9}$  in  $\textcircled{6}$ , we get

$$\bar{y}(\lambda, t) = \bar{f}(\lambda) \cos cpt$$

Taking inverse Fourier transform

$$y(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{y}(\lambda, t) e^{-i\lambda x} d\lambda$$

$$\Rightarrow y(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(\lambda) \cos cpt e^{-i\lambda x} d\lambda$$

### Exercise 2A

1. Find the Fourier transform of  $f(x) = \begin{cases} a - |x|, & |x| \leq a \\ 0, & |x| > a \end{cases}$

Hence prove that  $\int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$

2. Solve the integral equation  $\int_0^{\infty} f(x) \cos \lambda x dx = e^{-\lambda}, \lambda > 0$

3. Obtain Fourier sine integral of the function  $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$

4. Prove that Fourier integral of the function  $f(x) = \begin{cases} 1, & |x| \leq 1 \\ 0, & \text{otherwise} \end{cases}$  is given by

5. Find the Fourier sine and cosine transforms of  $xe^{-ax}$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda. \quad \text{Hence show that } \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

6. The temperature  $u(x, t)$  at any point of a semi infinite bar satisfies the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0, \quad \text{subject to conditions}$$

i.  $u(0, t) = 0, t > 0$

ii.  $u(x, 0) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$  Determine the expression for  $u(x, t)$

7. Determine the distribution of temperature in the semi infinite medium,  $x \geq 0$ , when the end at  $x = 0$  is maintained at zero temperature and initial distribution of temperature is  $f(x)$ .

### Answers

1.  $\frac{2(1 - \cos a\lambda)}{\lambda^2}$       2.  $f(x) = \frac{2}{\pi(1+x^2)}$       3.  $f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{(2 \sin \lambda - \sin 2\lambda)}{\lambda^2} \sin \lambda x d\lambda$

5.  $\frac{2a\lambda}{(a^2 + \lambda^2)^2}, \frac{a^2 - \lambda^2}{(a^2 + \lambda^2)^2}$       6.  $u(x, t) = \frac{2}{\pi} \int_0^{\infty} \frac{1 - \cos \lambda}{\lambda} e^{-\lambda^2 t} \sin \lambda x d\lambda$

7.  $u(x, t) = \frac{2}{\pi} \int_0^{\infty} \bar{f}_s(\lambda) e^{-c^2 \lambda^2 t} \sin \lambda x d\lambda$

# Z-TRANSFORMS

## 4.1 Introduction

Z- Transform plays an important role in discrete analysis and may be seen as discrete analogue of Laplace transform. Role of Z- Transforms in discrete analysis is the same as that of Laplace and Fourier transforms in continuous systems.

**Definition:** The Z-Transform of a sequence  $u_n$  defined for discrete values  $n = 0, 1, 2, 3, \dots$  and ( $u_n = 0$  for  $n < 0$ ) is defined as  $Z\{u_n\} = \sum_{n=0}^{\infty} u_n z^{-n}$ . Z- Transform of the sequence  $u_n$  i.e.  $Z\{u_n\}$  is a function of  $z$  and may be denoted by  $U(z)$

**Remark:**

- Z- Transform exists only when the infinite series  $\sum_{n=0}^{\infty} u_n z^{-n}$  is convergent.
- $Z\{u_n\} = \sum_{n=0}^{\infty} u_n z^{-n}$  is termed as one sided transform and for two sided Z - transform  $Z\{u_n\} = \sum_{n=-\infty}^{\infty} u_n z^{-n}$

### Results on Z- Transforms of standard sequences

1. 
$$Z\{a^n\} = \frac{z}{z - a}$$

$$\begin{aligned} Z\{a^n\} &= \sum_{n=0}^{\infty} a^n z^{-n} \\ &= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \dots + \frac{a^n}{z^n} + \dots \\ &= \frac{1}{1 - \frac{a}{z}}, \quad \left| \frac{a}{z} \right| < 1 \end{aligned}$$

$$\therefore Z\{a^n\} = \frac{z}{z - a}, \quad \left| \frac{a}{z} \right| < 1$$

2. 
$$Z\{1\} = \frac{z}{z - 1}$$

$$Z\{1\} = \frac{z}{z - 1} \quad \text{Putting } a = 1 \text{ in Result 1}$$

3. 
$$Z\{(-1)^n\} = \frac{z}{z + 1}$$

$$Z\{(-1)^n\} = \frac{z}{z + 1} \quad \text{Putting } a = -1 \text{ in Result 1}$$

4. 
$$Z\{k\} = \frac{kz}{z - 1}$$

$$\begin{aligned} Z\{k\} &= \sum_{n=0}^{\infty} k z^{-n} = k \sum_{n=0}^{\infty} z^{-n} \\ &= k \left[ 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots + \frac{1}{z^n} + \dots \right] \\ \therefore Z\{k\} &= \frac{kz}{z - 1} \end{aligned}$$

5. Recurrence formula for  $n^p$ :

$$Z\{n^p\} = -z \frac{d}{dz} Z\{n^{p-1}\}$$

$$Z\{n^p\} = \sum_{n=0}^{\infty} n^p z^{-n}, p \text{ is a positive integer} \quad \dots \textcircled{1}$$

$$Z\{n^{p-1}\} = \sum_{n=0}^{\infty} n^{p-1} z^{-n} \quad \dots \textcircled{2}$$

Differentiating  $\textcircled{2}$  w.r.t.  $z$ , we get

$$\begin{aligned} \frac{d}{dz} Z\{n^{p-1}\} &= \sum_{n=0}^{\infty} n^{p-1} (-n) z^{-n-1} \\ &= -z^{-1} \sum_{n=0}^{\infty} n^p z^{-n} \end{aligned}$$

$$\Rightarrow \frac{d}{dz} Z\{n^{p-1}\} = -z^{-1} Z\{n^p\} \quad \text{using } \textcircled{1}$$

$$\Rightarrow Z\{n^p\} = -z \frac{d}{dz} Z\{n^{p-1}\}$$

6. **Multiplication by  $n$ :**

$$Z\{nu_n\} = -z \frac{d}{dz} Z\{u_n\}$$

$$\begin{aligned} Z\{nu_n\} &= \sum_{n=0}^{\infty} nu_n z^{-n} \\ &= -z \sum_{n=0}^{\infty} u_n (-n) z^{-n-1} \\ &= -z \sum_{n=0}^{\infty} u_n \frac{d}{dz} z^{-n} \\ &= -z \sum_{n=0}^{\infty} \frac{d}{dz} (u_n z^{-n}) \\ &= -z \frac{d}{dz} \left( \sum_{n=0}^{\infty} u_n z^{-n} \right) \\ &= -z \frac{d}{dz} Z\{u_n\} \end{aligned}$$

7.

$$Z\{n\} = \frac{z}{(z-1)^2}$$

$$Z\{n\} = -z \frac{d}{dz} Z\{n^0\} \quad \text{using Recurrence Result 5 or 6}$$

$$= -z \frac{d}{dz} Z\{1\}$$

$$= -z \frac{d}{dz} \frac{z}{z-1} \quad \text{using result 2}$$

$$\Rightarrow Z\{n\} = \frac{z}{(z-1)^2}$$

8.

$$Z\{n^2\} = \frac{z^2 + z}{(z-1)^3}$$

$$Z\{n^2\} = -z \frac{d}{dz} Z\{n\} \quad \text{using Recurrence Result 5 or 6}$$

$$= -z \frac{d}{dz} \frac{z}{(z-1)^2} \quad \text{using Result 7}$$

$$\Rightarrow Z\{n^2\} = \frac{z^2 + z}{(z-1)^3}$$

9.

$$Z\{u(n)\} = \frac{z}{z-1} \quad u(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases} \text{ is Unit step sequence}$$

$$\begin{aligned} Z\{u(n)\} &= \sum_{n=0}^{\infty} u(n) z^{-n} = \sum_{n=0}^{\infty} 1 z^{-n} \\ &= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots + \frac{1}{z^n} + \dots \end{aligned}$$

$$\Rightarrow Z\{u(n)\} = \frac{z}{z-1}$$

$$10. \quad \boxed{Z\{\delta(n) = \begin{cases} 1, n=0 \\ 0, n \neq 0 \end{cases} = 1} \quad \delta(n) = \begin{cases} 1, n=0 \\ 0, n \neq 0 \end{cases} \text{ is Unit impulse sequence}$$

$$\begin{aligned} Z\{\delta(n)\} &= \sum_{n=0}^{\infty} \delta(n)z^{-n} \\ &= 1 + 0 + 0 + \dots \\ \Rightarrow Z\{\delta(n)\} &= 1 \end{aligned}$$

## 4.2 Properties of Z-Transforms

### 1. Linearity: $Z\{au_n + bv_n\} = aZ\{u_n\} + bZ\{v_n\}$

$$\begin{aligned} \text{Proof: } Z\{au_n + bv_n\} &= \sum_{n=0}^{\infty} (au_n + bv_n)z^{-n} \\ &= a \sum_{n=0}^{\infty} u_n z^{-n} + b \sum_{n=0}^{\infty} v_n z^{-n} \\ &= aZ\{u_n\} + bZ\{v_n\} \end{aligned}$$

### 2. Change of scale (or Damping rule):

If  $Z\{u_n\} \equiv U(z)$ , then  $Z\{a^{-n}u_n\} \equiv U(az)$  and  $Z\{a^n u_n\} \equiv U\left(\frac{z}{a}\right)$

$$\begin{aligned} \text{Proof: } Z\{a^{-n}u_n\} &= \sum_{n=0}^{\infty} a^{-n}u_n z^{-n} \\ &= \sum_{n=0}^{\infty} u_n (az)^{-n} \equiv U(az) \end{aligned}$$

Similarly  $Z\{a^n u_n\} \equiv U\left(\frac{z}{a}\right)$

### Results from application of Damping rule

$$\text{i. } \boxed{Z\{a^n n\} = \frac{az}{(z-a)^2}}$$

$$\begin{aligned} \text{Proof: } Z\{n\} &= \frac{z}{(z-1)^2} \equiv U(z) \text{ say} \\ \therefore Z\{a^n n\} &\equiv U\left(\frac{z}{a}\right) = \frac{\frac{z}{a}}{\left(\frac{z}{a}-1\right)^2} = \frac{az}{(z-a)^2} \end{aligned}$$

$$\text{ii. } \boxed{Z\{a^n n^2\} = \frac{az^2 + a^2 z}{(z-a)^3}}$$

$$\begin{aligned} \text{Proof: } Z\{n^2\} &= \frac{z^2+z}{(z-1)^3} \equiv U(z) \text{ say} \\ \therefore Z\{a^n n^2\} &\equiv U\left(\frac{z}{a}\right) = \frac{\left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right)}{\left(\left(\frac{z}{a}\right)-1\right)^3} = \frac{a(z^2+az)}{(z-a)^3} \end{aligned}$$

$$\text{iii. } \boxed{Z\{\cos n\theta\} = \frac{z(z-\cos \theta)}{z^2-2z\cos \theta+1}, \quad Z\{\sin n\theta\} = \frac{z \sin \theta}{z^2-2z\cos \theta+1}}$$

$$\text{Proof: We have } Z\{e^{-in\theta}\} = Z\{(e^{i\theta})^{-n}\} = Z\{(e^{i\theta})^{-n} \cdot 1\}$$

$$\text{Now } Z\{1\} = \frac{z}{z-1}$$

$$\begin{aligned} \therefore Z\{(e^{i\theta})^{-n} \cdot 1\} &= \frac{ze^{i\theta}}{ze^{i\theta}-1} \quad \therefore Z\{a^{-n}u_n\} \equiv U(az) \\ &= \frac{z}{z-e^{-i\theta}} \\ &= \frac{z(z-e^{i\theta})}{(z-e^{-i\theta})(z-e^{i\theta})} \end{aligned}$$

$$\begin{aligned}
&= \frac{z(z - \cos\theta - i\sin\theta)}{z^2 - z(e^{i\theta} + e^{-i\theta}) + 1} \quad \because e^{i\theta} = \cos\theta + i\sin\theta \\
&= \frac{z(z - \cos\theta - i\sin\theta)}{z^2 - 2z\cos\theta + 1} \quad \because \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \\
\therefore Z\{e^{-in\theta}\} &= \frac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1} - i \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1} \\
\Rightarrow Z\{\cos n\theta - i\sin n\theta\} &= \frac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1} - i \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1} \\
\therefore Z\{\cos n\theta\} &= \frac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1} \quad \dots \textcircled{3}
\end{aligned}$$

$$\text{and } Z\{\sin n\theta\} = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1} \quad \dots \textcircled{4}$$

$$\text{iv. } \boxed{Z\{a^n \cos n\theta\} = \frac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}, \quad Z\{a^n \sin n\theta\} = \frac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}}$$

By Damping rule, replacing  $z$  by  $\frac{z}{a}$  in  $\textcircled{3}$  and  $\textcircled{4}$ , we get

$$Z\{a^n \cos n\theta\} = \frac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2} \quad \text{and} \quad Z\{a^n \sin n\theta\} = \frac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$$

### 3. Right Shifting Property

For  $n \geq k$ ,  $Z\{u_{n-k}\} = z^{-k}Z\{u_n\}$ ,  $k$  is positive integer

$$\begin{aligned}
\text{Proof: } Z\{u_{n-k}\} &= \sum_{n=0}^{\infty} u_{n-k} z^{-n} \\
&= u_{-k} z^0 + u_{1-k} z^{-1} + \dots + u_{-1} z^{-k+1} + u_0 z^{-k} + u_1 z^{-(k+1)} + u_2 z^{-(k+2)} + \dots \\
&= 0 + u_0 z^{-k} + u_1 z^{-(k+1)} + u_2 z^{-(k+2)} + \dots \quad \because u_n = 0 \text{ for } n < 0 \\
&= \sum_{n=k}^{\infty} u_{n-k} z^{-n} \\
&= \sum_{m=0}^{\infty} u_m z^{-k-m} \\
&= z^{-k} \sum_{m=0}^{\infty} u_m z^{-m} \\
&= z^{-k} \sum_{n=0}^{\infty} u_n z^{-n} \\
&= z^{-k} Z\{u_n\}
\end{aligned}$$

### 4. Left Shifting Property

If  $k$  is a positive integer  $Z\{u_{n+k}\} = z^k \left[ Z\{u_n\} - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} - \dots - \frac{u_{k-1}}{z^{k-1}} \right]$

$$\begin{aligned}
\text{Proof: } Z\{u_{n+k}\} &= \sum_{n=0}^{\infty} u_{n+k} z^{-n} \\
&= z^k \sum_{n=0}^{\infty} u_{n+k} z^{-(n+k)} \\
&= z^k [u_k z^{-k} + u_{1+k} z^{-(1+k)} + u_{2+k} z^{-(2+k)} + \dots] \\
&= z^k [u_0 + u_1 z^{-1} + u_2 z^{-2} + \dots + u_{k-1} z^{-(k-1)} + u_k z^{-k} + \dots] \\
&\quad - z^k [u_0 + u_1 z^{-1} + u_2 z^{-2} + \dots + u_{k-1} z^{-(k-1)}] \\
&= z^k [\sum_{n=0}^{\infty} u_n z^{-n} - \sum_{n=0}^{k-1} u_n z^{-n}] \\
&= z^k [\sum_{n=0}^{\infty} u_n z^{-n} - \sum_{n=0}^{k-1} u_n z^{-n}] \\
&= z^k \left[ Z\{u_n\} - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} - \dots - \frac{u_{k-1}}{z^{k-1}} \right]
\end{aligned}$$

In particular for  $k = 1, 2, 3$

$$\begin{aligned}
Z\{u_{n+1}\} &= z[Z\{u_n\} - u_0] \\
Z\{u_{n+2}\} &= z^2 \left[ Z\{u_n\} - u_0 - \frac{u_1}{z} \right]
\end{aligned}$$

$$Z\{u_{n+3}\} = z^3 \left[ Z\{u_n\} - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} \right]$$

**5. Initial Value theorem:**

If  $Z\{u_n\} = U(z)$ , then  $u_0 = \lim_{z \rightarrow \infty} U(z)$

$$u_1 = \lim_{z \rightarrow \infty} z[U(z) - u_0]$$

$$u_2 = \lim_{z \rightarrow \infty} z^2 \left[ U(z) - u_0 - \frac{u_1}{z} \right]$$

$\vdots$

Proof: By definition  $U(z) = Z\{u_n\} = \sum_{n=0}^{\infty} u_n z^{-n}$

$$\Rightarrow U(z) = u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \dots \quad \dots \textcircled{5}$$

$$\begin{aligned} \therefore u_0 &= \lim_{z \rightarrow \infty} U(z) = \lim_{z \rightarrow \infty} \left[ u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \dots \right] \\ &= u_0 + 0 + 0 + 0 + \dots = u_0 \end{aligned}$$

Again from  $\textcircled{5}$ , we get

$$U(z) - u_0 = \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \dots$$

$$\Rightarrow z[U(z) - u_0] = u_1 + \frac{u_2}{z} + \frac{u_3}{z^2} + \dots$$

$$\Rightarrow \lim_{z \rightarrow \infty} z[U(z) - u_0] = \lim_{z \rightarrow \infty} \left[ u_1 + \frac{u_2}{z} + \frac{u_3}{z^2} + \dots \right] = u_1$$

$$\text{Similarly } u_2 = \lim_{z \rightarrow \infty} z^2 \left[ U(z) - u_0 - \frac{u_1}{z} \right]$$

**Note:** Initial value theorem may be used to determine the sequence  $u_n$  from the given function  $U(z)$

**6. Final Value theorem:**

If  $Z\{u_n\} = U(z)$ , then  $\lim_{n \rightarrow \infty} u_n = \lim_{z \rightarrow 1} (z - 1)U(z)$

Proof:  $Z\{u_{n+1} - u_n\} = \sum_{n=0}^{\infty} (u_{n+1} - u_n)z^{-n}$

$$\Rightarrow Z\{u_{n+1}\} - Z\{u_n\} = \sum_{n=0}^{\infty} (u_{n+1} - u_n)z^{-n}$$

$$\Rightarrow z[Z\{u_n\} - u_0] - Z\{u_n\} = \sum_{n=0}^{\infty} (u_{n+1} - u_n)z^{-n}$$

By using left shifting property for  $k = 1$

$$\Rightarrow (z - 1)Z\{u_n\} - u_0 = \sum_{n=0}^{\infty} (u_{n+1} - u_n)z^{-n}$$

$$\text{or } (z - 1)U(z) - u_0 = \sum_{n=0}^{\infty} (u_{n+1} - u_n)z^{-n} \quad \because Z\{u_n\} = U(z)$$

Taking limits  $z \rightarrow 1$  on both sides

$$\lim_{z \rightarrow 1} (z - 1)U(z) - u_0 = \sum_{n=0}^{\infty} (u_{n+1} - u_n)$$

$$\begin{aligned} \text{or } \lim_{z \rightarrow 1} (z - 1)U(z) - u_0 &= \lim_{n \rightarrow \infty} [(u_1 - u_0) + (u_2 - u_1) + \dots + (u_{n+1} - u_n)] \\ &= \lim_{n \rightarrow \infty} [u_{n+1}] - u_0 \end{aligned}$$

$$\Rightarrow \lim_{z \rightarrow 1} (z - 1)U(z) = u_{\infty}$$

$$\text{or } \lim_{z \rightarrow 1} (z - 1)U(z) = \lim_{n \rightarrow \infty} u_n$$

**Note:** Initial value and final value theorems determine the value of  $u_n$  for  $n = 0$  and for  $n \rightarrow \infty$  from the given function  $U(z)$ .

**7. Convolution theorem**

Convolution of two sequences  $u_n$  and  $v_n$  is defined as  $u_n * v_n = \sum_{m=0}^n u_m v_{n-m}$

Convolution theorem for Z-transforms states that

If  $U(z) = Z\{u_n\}$  and  $V(z) = Z\{v_n\}$ , then  $Z\{u_n * v_n\} = U(z).V(z)$

$$\begin{aligned} \text{Proof: } U(z).V(z) &= Z\{u_n\}.Z\{v_n\} \\ &= \left[ \sum_{n=0}^{\infty} u_n z^{-n} \right] \cdot \left[ \sum_{n=0}^{\infty} v_n z^{-n} \right] \end{aligned}$$

$$\begin{aligned}
&= \left[ u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \cdots + \frac{u_n}{z^n} + \cdots \right] \cdot \left[ v_0 + \frac{v_1}{z} + \frac{v_2}{z^2} + \cdots + \frac{v_n}{z^n} + \cdots \right] \\
&= (u_0 v_0) + (u_0 v_1 + u_1 v_0) z^{-1} + (u_0 v_2 + u_1 v_1 + u_2 v_0) z^{-2} + \cdots \\
&= \sum_{n=0}^{\infty} (u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + \cdots + u_n v_0) z^{-n} \\
&= \sum_{n=0}^{\infty} \left( \sum_{m=0}^n u_m v_{n-m} \right) z^{-n} \\
\Rightarrow U(z) \cdot V(z) &= Z \left\{ \sum_{m=0}^n u_m v_{n-m} \right\} \quad \because \sum_{n=0}^{\infty} u_n z^{-n} = Z \{u_n\} \\
\Rightarrow U(z) \cdot V(z) &= Z \{u_n * v_n\} \quad \because u_n * v_n = \sum_{m=0}^n u_m v_{n-m}
\end{aligned}$$

**Example1** Find the Z-transform of  $2n + 3 \sin \frac{n\pi}{4} - 5a^4$

**Solution:** By linearity property

$$\begin{aligned}
Z \left\{ 2n + 3 \sin \frac{n\pi}{4} - 5a^4 \right\} &= 2Z\{n\} + 3Z \left\{ \sin \frac{n\pi}{4} \right\} - 5Z\{a^4\} \\
&= 2Z\{n\} + 3Z \left\{ \sin \frac{n\pi}{4} \right\} - 5a^4 Z\{1\} \\
&= \frac{2z}{(z-1)^2} + \frac{3z \sin \frac{\pi}{4}}{z^2 - 2z \cos \frac{\pi}{4} + 1} - \frac{5a^4 z}{z-1} \\
\because Z\{n\} &= \frac{z}{(z-1)^2}, Z\{\sin n\theta\} = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}, Z\{1\} = \frac{z}{z-1} \\
\therefore Z \left\{ 2n + 3 \sin \frac{n\pi}{4} - 5a^4 \right\} &= \frac{2z}{(z-1)^2} + \frac{\frac{3z}{\sqrt{2}}}{z^2 - \sqrt{2}z + 1} - \frac{5a^4 z}{z-1}
\end{aligned}$$

**Example2** Find the Z-transform of the sequence  $\{4, 8, 16, 32, \dots\}$

**Solution:**  $u_n = 2^{n+2}, n = 0, 1, 2, 3, \dots$

$$\begin{aligned}
Z\{2^{n+2}\} &= Z\{2^2 2^n\} \\
&= 4 Z\{2^n\} = \frac{4z}{z-2}, \left| \frac{2}{z} \right| < 1 \because Z\{a^n\} = \frac{z}{z-a}, \left| \frac{a}{z} \right| < 1
\end{aligned}$$

**Example3** Find the Z-transform of  $(n+1)^2$

$$\begin{aligned}
\text{Solution: } Z\{(n+1)^2\} &= Z\{n^2 + 2n + 1\} \\
&= Z\{n^2\} + 2Z\{n\} + Z\{1\} \\
&= \frac{z^2 + z}{(z-1)^3} + \frac{2z}{(z-1)^2} + \frac{z}{z-1} \\
\because Z\{n^2\} &= \frac{z^2 + z}{(z-1)^3}, Z\{n\} = \frac{z}{(z-1)^2}, Z\{1\} = \frac{z}{z-1}
\end{aligned}$$

**Example4** Find the Z-transform of

$$\text{i. } n \cos n\theta \quad \text{ii. } \sin \left( \frac{n\pi}{2} + \frac{\pi}{4} \right) \quad \text{iii. } e^{-2n} \quad \text{iv. } a^{|n|}$$

$$\begin{aligned}
\text{Solution: i. } Z\{\cos n\theta\} &= \frac{z(z - \cos \theta)}{z^2 - 2z \cos \theta + 1} \\
\therefore Z\{n \cos n\theta\} &= -z \frac{d}{dz} \left[ \frac{z(z - \cos \theta)}{z^2 - 2z \cos \theta + 1} \right] \\
&\quad \because Z\{nu_n\} = -z \frac{d}{dz} Z\{u_n\} \\
&= -z \left[ \frac{-z^2 \cos \theta + 2z - \cos \theta}{(z^2 - 2z \cos \theta + 1)^2} \right] \\
&= \frac{z^3 \cos \theta - 2z^2 + z \cos \theta}{(z^2 - 2z \cos \theta + 1)^2} \\
\text{ii. } Z \left\{ \sin \left( \frac{n\pi}{2} + \frac{\pi}{4} \right) \right\} &= Z \left\{ \sin \frac{n\pi}{2} \cos \frac{\pi}{4} + \cos \frac{n\pi}{2} \sin \frac{\pi}{4} \right\} \\
&= \cos \frac{\pi}{4} Z \left\{ \sin \frac{n\pi}{2} \right\} + \sin \frac{\pi}{4} Z \left\{ \cos \frac{n\pi}{2} \right\}
\end{aligned}$$

$$= \frac{1}{\sqrt{2}} \left[ \frac{z \sin \frac{\pi}{2}}{z^2 - 2z \cos \frac{\pi}{2} + 1} + \frac{z \left( z - \cos \frac{\pi}{2} \right)}{z^2 - 2z \cos \frac{\pi}{2} + 1} \right]$$

$$\therefore Z\{\sin n\theta\} = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}, Z\{\cos n\theta\} = \frac{z(z - \cos \theta)}{z^2 - 2z \cos \theta + 1}$$

$$= \frac{1}{\sqrt{2}} \left[ \frac{z}{z^2 + 1} + \frac{z^2}{z^2 + 1} \right] = \frac{1}{\sqrt{2}} \left[ \frac{z + z^2}{z^2 + 1} \right]$$

$$\text{iii. } Z\{e^{-2n}\} = Z\{(e^{-2})^n\}$$

$$= \frac{z}{z - e^{-2}}$$

$$\therefore Z\{a^n\} = \frac{z}{z - a}$$

$$= \frac{ze^2}{ze^2 - 1}$$

$$\text{iv. } a^{|n|} = \begin{cases} a^{-n}, & n < 0 \\ a^n, & n \geq 0 \end{cases}$$

$$\text{Taking two sided } Z\text{-transform: } Z\{a^{|n|}\} = \sum_{n=-\infty}^{\infty} a^{|n|} z^{-n}$$

$$\therefore Z\{a^{|n|}\} = \sum_{-\infty}^{-1} a^{-n} z^{-n} + \sum_0^{\infty} a^n z^{-n}$$

$$= [\dots + a^3 z^3 + a^2 z^2 + az] + \left[ 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \dots \right]$$

$$= \frac{az}{1-az} + \frac{1}{1-\frac{a}{z}}, \quad |az| < 1 \text{ and } \left| \frac{a}{z} \right| < 1$$

$$= \frac{az}{1-az} + \frac{z}{z-a}, \quad |z| < \frac{1}{|a|} \text{ and } |a| < |z|$$

$$= \frac{z(1-az)}{(1-az)(z-a)}, \quad |a| < |z| < \frac{1}{|a|}$$

$$\Rightarrow Z\{a^{|n|}\} = \frac{z}{(z-a)}, \quad |a| < |z| < \frac{1}{|a|}$$

**Example5** Find the  $Z$ -transform of  $u_n = \begin{cases} 2^n, & n < 0 \\ 3^n, & n \geq 0 \end{cases}$

$$\text{Taking two sided } Z\text{-transform: } Z\{u_n\} = \sum_{n=-\infty}^{\infty} u_n z^{-n}$$

$$\therefore Z\{u_n\} = \sum_{-\infty}^{-1} 2^n z^{-n} + \sum_0^{\infty} 3^n z^{-n}$$

$$= \left[ \dots + \frac{z^3}{2^3} + \frac{z^2}{2^2} + \frac{z}{2} \right] + \left[ 1 + \frac{3}{z} + \frac{3^2}{z^2} + \frac{3^3}{z^3} + \dots \right]$$

$$= \frac{\frac{z}{2}}{1-\frac{z}{2}} + \frac{1}{1-\frac{3}{z}}, \quad \left| \frac{z}{2} \right| < 1 \text{ and } \left| \frac{3}{z} \right| < 1$$

$$= \frac{z}{2-z} + \frac{z}{z-3}, \quad |z| < 2 \text{ and } 3 < |z|$$

$$= \frac{z(z-3)+z(2-z)}{(2-z)(z-3)}, \quad 3 < |z| < 2$$

$$\Rightarrow Z\{u_n\} = \frac{z}{z^2 - 5z + 6}, \quad 3 < |z| < 2$$

$\therefore Z$ -transform does not exist for  $u_n = \begin{cases} 2^n, & n < 0 \\ 3^n, & n \geq 0 \end{cases}$  as the set  $3 < |z| < 2$  is infeasible.

**Example6** Find the  $Z$ -transform of

$$\text{i. } {}^n C_r \quad 0 \leq r \leq n \quad \text{ii. } {}^{n+r} C_r \quad \text{iii. } \frac{1}{(n+r)!} \quad \text{iv. } \frac{1}{(n-r)!}$$

$$\text{Solution: i. } Z\{{}^n C_r\} = \sum_{r=0}^n {}^n C_r z^{-r}$$

$$= 1 + {}^n C_1 z^{-1} + {}^n C_2 z^{-2} + \dots + {}^n C_n z^{-n}$$

$$= (1 + z^{-1})^n$$

$$\text{ii. } Z\{{}^{n+r} C_r\} = \sum_{r=0}^{\infty} {}^{n+r} C_r z^{-r}$$

$$= 1 + {}^{n+1} C_1 z^{-1} + {}^{n+2} C_2 z^{-2} + {}^{n+3} C_3 z^{-3} + \dots$$

$$= 1 + (n+1) z^{-1} + \frac{(n+2)(n+1)}{2!} z^{-2} + \frac{(n+3)(n+2)(n+1)}{3!} z^{-3} + \dots$$

$$\begin{aligned}
&= 1 + (-n-1)(-z^{-1}) + \frac{(-n-1)(-n-2)}{2!} (-z^{-1})^{-2} \\
&\quad + \frac{(-n-1)(-n-2)(-n-3)}{3!} (-z^{-1})^{-3} + \dots \\
&= (1 - z^{-1})^{-n-1}
\end{aligned}$$

**Example 7** Find the  $Z$ -transform of

i.  $\frac{1}{n!}$     ii.  $\frac{1}{(n+r)!}$     iii.  $\frac{1}{(n-r)!}$

i.  $Z\left\{\frac{1}{n!}\right\} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = 1 + \frac{1}{1!} z^{-1} + \frac{1}{2!} z^{-2} + \frac{1}{3!} z^{-3} + \dots$

$$= 1 + \frac{1}{1!} \frac{1}{z} + \frac{1}{2!} \frac{1}{z^2} + \frac{1}{3!} \frac{1}{z^3} + \dots = e^{\frac{1}{z}}$$

$$\Rightarrow Z\left\{\frac{1}{n!}\right\} = e^{\frac{1}{z}}$$

ii.  $Z\left\{\frac{1}{(n+r)!}\right\} = \sum_{n=0}^{\infty} \frac{1}{(n+r)!} z^{-n}$

Now  $Z\left\{\frac{1}{n!}\right\} = e^{\frac{1}{z}}$

Also from left shifting property  $Z\{u_{n+k}\} = z^k \left[ Z\{u_n\} - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} - \dots - \frac{u_{k-1}}{z^{k-1}} \right]$

$$\therefore Z\left\{\frac{1}{(n+r)!}\right\} = z^r \left[ e^{\frac{1}{z}} - 1 - \frac{1}{z} - \frac{1}{2! z^2} - \dots - \frac{1}{(r-1)! z^{r-1}} \right]$$

In particular  $\left\{\frac{1}{(n+1)!}\right\} = z^1 \left[ e^{\frac{1}{z}} - 1 \right]$

$$\left\{\frac{1}{(n+2)!}\right\} = z^2 \left[ e^{\frac{1}{z}} - 1 - \frac{1}{z} \right]$$

$\vdots$

iii.  $Z\left\{\frac{1}{(n-r)!}\right\} = \sum_{n=0}^{\infty} \frac{1}{(n-r)!} z^{-n}$

Now  $Z\left\{\frac{1}{n!}\right\} = e^{\frac{1}{z}}$

Also from right shifting property,  $Z\{u_{n-k}\} = z^{-k} Z\{u_n\}$ ,  $k$  is positive integer

$$\therefore Z\left\{\frac{1}{(n-r)!}\right\} = z^{-r} e^{\frac{1}{z}}$$

In particular  $\left\{\frac{1}{(n-1)!}\right\} = z^{-1} e^{\frac{1}{z}}$

$$\left\{\frac{1}{(n+2)!}\right\} = z^{-2} e^{\frac{1}{z}}$$

$\vdots$

**Example 8** Find  $Z\{u_{n+2}\}$  if  $Z\{u_n\} = \frac{z}{z-1} + \frac{z}{z^2+1}$

**Solution:** Given  $Z\{u_n\} = U(z) = \frac{z}{z-1} + \frac{z}{z^2+1}$

From left shifting property  $Z\{u_{n+2}\} = z^2 \left[ Z\{u_n\} - u_0 - \frac{u_1}{z} \right] \dots \textcircled{1}$

Now from initial value theorem  $u_0 = \lim_{z \rightarrow \infty} U(z)$

$$= \lim_{z \rightarrow \infty} \left[ \frac{z}{z-1} + \frac{z}{z^2+1} \right]$$

$$= \lim_{z \rightarrow \infty} \left[ \frac{1}{1 - \frac{1}{z}} + \frac{\frac{1}{z}}{1 + \frac{1}{z^2}} \right] = 1 + 0$$

$$\therefore u_0 = 1 \quad \dots \textcircled{2}$$

Also from initial value theorem  $u_1 = \lim_{z \rightarrow \infty} z[U(z) - u_0]$

$$= \lim_{z \rightarrow \infty} z \left[ \frac{z}{z-1} + \frac{z}{z^2+1} - 1 \right]$$

$$= \lim_{z \rightarrow \infty} z \left[ \frac{2z^2 - z + 1}{(z^2+1)(z-1)} \right] = 2$$

$$\therefore u_1 = 2 \quad \dots \textcircled{3}$$

Using ② and ③ in ①, we get  $Z\{u_{n+2}\} = z^2 \left[ \frac{z}{z-1} + \frac{z}{z^2+1} - 1 - \frac{2}{z} \right]$

$$\Rightarrow Z\{u_{n+2}\} = \frac{z(z^2 - z + 2)}{(z-1)(z^2+1)}$$

**Example9** Verify convolution theorem for  $u_n = n$  and  $v_n = 1$

**Solution:** Convolution theorem states that  $Z\{u_n * v_n\} = U(z).V(z)$

We know that  $u_n * v_n = \sum_{m=0}^n u_m v_{n-m}$

$$= \sum_{m=0}^n m \cdot 1$$

$$= 0 + 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$\Rightarrow Z\{u_n * v_n\} = \sum_{n=0}^{\infty} \frac{n(n+1)}{2} z^{-n} = \frac{1}{2} [\sum_{n=0}^{\infty} n^2 z^{-n} + \sum_{n=0}^{\infty} n z^{-n}]$$

$$= \frac{1}{2} [Z\{n^2\} + Z\{n\}]$$

$$= \frac{1}{2} \left[ \frac{z(z+1)}{(z-1)^3} + \frac{z}{(z-1)^2} \right] = \frac{1}{2} \left[ \frac{z(z+1) + z(z-1)}{(z-1)^3} \right] = \frac{1}{2} \left[ \frac{2z^2}{(z-1)^3} \right]$$

$$\therefore Z\{u_n * v_n\} = \frac{z^2}{(z-1)^3} \quad \dots \textcircled{1}$$

$$\text{Also } U(z) = Z\{n\} = \frac{z}{(z-1)^2} \quad \text{and } V(z) = Z\{1\} = \frac{z}{z-1}$$

$$\Rightarrow U(z).V(z) = \frac{z^2}{(z-1)^3} \quad \dots \textcircled{2}$$

$$\text{From ① and ② } Z\{u_n * v_n\} = U(z).V(z)$$

**Example10** If  $u_n = \delta(n) - \delta(n-1)$ ,  $v_n = 2\delta(n) + \delta(n-1)$ , Find the Z-transform of their convolution.

**Solution:**  $U(z) = Z\{\delta(n) - \delta(n-1)\}$ ,  $V(z) = Z\{2\delta(n) + \delta(n-1)\}$

$$\text{Now } Z\{\delta(n)\} = 1 \text{ and } Z\{\delta(n-1)\} = z^{-1} \quad \because Z\{u_{n-k}\} = z^{-k} Z\{u_n\}$$

$$\therefore U(z) = 1 - z^{-1} \text{ and } V(z) = 2 + z^{-1}$$

$$\text{Also } Z\{u_n * v_n\} = U(z).V(z)$$

$$\Rightarrow Z\{u_n * v_n\} = (1 - z^{-1})(2 + z^{-1}) = 2 - z^{-1} - z^{-2}$$

### 4.3 Inverse Z-Transform

Given a function  $U(z)$ , we can find the sequence  $u_n$  by one of the following methods

- Inspection (Direct inversion)
- Direct division
- Partial fractions

- Residues (Inverse integral)
- Power-Series
- Convolution theorem

#### 4.3.1 Inspection (Direct inversion) method

Sometimes by observing the coefficients in the given series  $U(z)$ , it is possible to find the sequence  $u_n$  as illustrated in the given examples.

**Example 11** Find  $u_n$  if  $U(z) = 1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{8}z^{-3} + \dots$

**Solution:** Given that  $U(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$  ... ①

Also by the definition of Z-transform  $U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$  ... ②

.Comparing ① and ②, we get  $u_n = \left(\frac{1}{2}\right)^n$

**Example 12** Find  $u_n$  if  $U(z) = \frac{z^3}{(z-1)^3}$

**Solution:**  $U(z) = \frac{z^3}{(z-1)^3} = \left(\frac{z-1}{z}\right)^{-3} = \left(1 - \frac{1}{z}\right)^{-3}$

$$\therefore U(z) = 1 + \frac{3}{z} + \frac{6}{z^2} + \frac{10}{z^3} + \dots$$

$$\because (1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \dots$$

$$\Rightarrow U(z) = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} z^{-n}$$

Comparing with  $U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$ , we get  $u_n = \frac{(n+1)(n+2)}{2}$

**Example 13** Find inverse Z-transform of  $3 + \frac{2z}{z-1} - \frac{z}{2z-1}$

**Solution:** Given that  $U(z) = 3 + \frac{2z}{z-1} - \frac{z}{2z-1}$

$$\therefore u_n = 3Z^{-1}[1] + 2Z^{-1}\left[\frac{z}{z-1}\right] - \frac{1}{2}Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right]$$

$$\Rightarrow u_n = 3\delta(n) + 2u(n) - \frac{1}{2}\left(\frac{1}{2}\right)^n = 3\delta(n) + 2u(n) - \left(\frac{1}{2}\right)^{n+1}$$

$\because Z^{-1}[1] = \delta(n)$ ,  $Z^{-1}\left[\frac{z}{z-1}\right] = u(n)$ ,  $Z^{-1}\left[\frac{z}{z-a}\right] = a^n$  where  $\delta(n)$  and  $u(n)$  are unit impulse and unit step sequences respectively.

**Example 14** Find inverse Z-transform of  $2z^{-2} - \frac{z^{-3}}{z-1} + \frac{2z^{-5}}{2z-1}$

**Solution:** Given that  $U(z) = 2z^{-2} - \frac{z^{-3}}{z-1} + \frac{2z^{-5}}{2z-1}$

$$\therefore u_n = 2Z^{-1}[z^{-2}.1] - Z^{-1}\left[z^{-4}.\frac{z}{z-1}\right] + Z^{-1}\left[z^{-6}.\frac{z}{z-\frac{1}{2}}\right]$$

$$\Rightarrow u_n = 2\delta(n-2) - u(n-4) + \left(\frac{1}{2}\right)^{n-6} u(n-6)$$

$\because$  From Right shifting property  $Z^{-1}[z^{-k}U(Z)] = u_{n-k}$

#### 4.3.2 Direct division method

Direct division is one of the simplest methods for finding inverse Z-transform and can be used for almost every type of expression given in fractional form.

**Example 15** Find the inverse Z-transform of  $\frac{z}{z^2-3z+2}$

**Solution:** Given that  $U(z) = \frac{z}{z^2-3z+2}$

By actual division, we get

$$\begin{array}{r} z^2 - 3z + 2 \overline{) z^{-1} + 3z^{-2} + 7z^{-3}} \\ \underline{z - 3 + 2z^{-1}} \phantom{+ 7z^{-3}} \\ 3 - 2z^{-1} \phantom{+ 7z^{-3}} \\ \underline{3 - 9z^{-1} + 6z^{-2}} \phantom{+ 7z^{-3}} \\ 7z^{-1} - 6z^{-2} \phantom{+ 7z^{-3}} \\ \underline{7z^{-1} - 21z^{-2} + 14z^{-3}} \phantom{+ 7z^{-3}} \\ 15z^{-2} - 14z^{-3} \phantom{+ 7z^{-3}} \\ \vdots \end{array}$$

$$\begin{aligned} \Rightarrow U(z) &= z^{-1} + 3z^{-2} + 7z^{-3} + \dots \\ &= \sum_{n=0}^{\infty} (2^n - 1)z^{-n} \\ \therefore u_n &= 2^n - 1 \end{aligned}$$

**Example16** Find the inverse  $Z$  -transform of  $\frac{4z^2+2z}{2z^2-3z+1}$

**Solution:** Given that  $U(z) = \frac{4z^2+2z}{2z^2-3z+1}$ , by actual division, we get

$$\begin{array}{r} 2z^2 - 3z + 1 \overline{) 2 + 4z^{-1} + 5z^{-2}} \\ \underline{4z^2 + 2z} \phantom{+ 5z^{-2}} \\ 4z^2 - 6z + 2 \phantom{+ 5z^{-2}} \\ \underline{8z - 2} \phantom{+ 5z^{-2}} \\ 8z - 12 + 4z^{-1} \phantom{+ 5z^{-2}} \\ \underline{10 - 4z^{-1}} \phantom{+ 5z^{-2}} \\ 10 - 15z^{-1} + 5z^{-2} \phantom{+ 5z^{-2}} \\ \underline{11z^{-1} - 5z^{-2}} \phantom{+ 5z^{-2}} \\ \vdots \end{array}$$

$$\begin{aligned} \Rightarrow U(z) &= 2 + 4z^{-1} + 5z^{-2} + \dots \\ &= \sum_{n=0}^{\infty} (6 - 2^{2-n})z^{-n} \\ \therefore u_n &= 6 - 2^{2-n} \\ \therefore u_n &= 6 - 4\left(\frac{1}{2}\right)^n \end{aligned}$$

### 4.3.3 Partial fractions method

Partial fractions method can be used only if order of expression in the numerator is less than or equal to that in the denominator. If order of expression in the numerator is greater, then the fraction may be brought to desired form by direct division. Partial fractions are formed of the expression  $\frac{U(z)}{z}$  as demonstrated in the examples below.

**Example17** Find the inverse  $Z$  -transform of  $\frac{z}{6z^2-5z+1}$

**Solution:** Given that  $U(z) = \frac{z}{6z^2-5z+1}$

$$\begin{aligned} \therefore \frac{U(z)}{z} &= \frac{1}{6z^2-5z+1} = \frac{1}{(2z-1)(3z-1)} \\ \Rightarrow \frac{U(z)}{z} &= \frac{2}{2z-1} - \frac{3}{3z-1} \\ \Rightarrow U(z) &= \frac{z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{3}} \\ \therefore u_n &= \left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n \quad \because Z\{a^n\} = \frac{z}{z-a} \text{ or } Z^{-1}\left[\frac{z}{z-a}\right] = a^n \end{aligned}$$

i.e.  $u_n = 2^{-n} - 3^{-n}$

**Example18** Find the inverse  $Z$  -transform of  $\frac{4z^2+2z}{2z^2-3z+1}$

**Solution:** Given that  $U(z) = \frac{4z^2-2z}{2z^2-3z+1} = \frac{2z(2z-1)}{(2z-1)(z-1)}$

$$\therefore \frac{U(z)}{z} = \frac{2(2z+1)}{(2z-1)(z-1)}$$

By partial fractions, we get

$$\begin{aligned} \frac{U(z)}{z} &= \frac{-8}{2z-1} + \frac{6}{z-1} \\ \Rightarrow U(z) &= \frac{-8z}{2z-1} + \frac{6z}{z-1} \\ \therefore u_n &= -4Z^{-1} \left[ \frac{z}{z-\frac{1}{2}} \right] + 6Z^{-1} \left[ \frac{z}{z-1} \right] \\ \Rightarrow u_n &= -4 \left( \frac{1}{2} \right)^n + 6(1)^n \quad \because Z^{-1} \left[ \frac{z}{z-a} \right] = a^n \\ \text{i.e. } u_n &= -4 \left( \frac{1}{2} \right)^n + 6 \end{aligned}$$

**Example19** Find the inverse  $Z$  -transform of  $\frac{1}{(1-z^{-1})(2-z^{-1})}$

**Solution:** Given that  $U(z) = \frac{1}{(1-z^{-1})(2-z^{-1})}$

Multiplying and dividing by  $z^2$ , we get

$$\begin{aligned} U(z) &= \frac{z^2}{z(1-z^{-1})z(2-z^{-1})} = \frac{z^2}{(z-1)(2z-1)} \\ \therefore \frac{U(z)}{z} &= \frac{z}{(z-1)(2z-1)} \end{aligned}$$

By partial fractions, we get

$$\begin{aligned} \frac{U(z)}{z} &= \frac{1}{(z-1)} - \frac{1}{(2z-1)} \\ \Rightarrow U(z) &= \frac{z}{(z-1)} - \frac{z}{(2z-1)} \\ \therefore u_n &= Z^{-1} \left[ \frac{z}{(z-1)} \right] - \frac{1}{2} Z^{-1} \left[ \frac{z}{z-\frac{1}{2}} \right] \\ \Rightarrow u_n &= (1)^n - \frac{1}{2} \left( \frac{1}{2} \right)^n \quad \because Z^{-1} \left[ \frac{z}{z-a} \right] = a^n \\ \text{i.e. } u_n &= 1 - \left( \frac{1}{2} \right)^{n+1} \end{aligned}$$

**Example20** Find the inverse  $Z$  -transform of  $\frac{4z^2-2z}{z^3-5z^2+8z-4}$

**Solution:** Given that  $U(z) = \frac{4z^2-2z}{z^3-5z^2+8z-4} = \frac{2z(2z-1)}{(z-1)(z-2)^2}$

$$\therefore \frac{U(z)}{z} = \frac{2(2z-1)}{(z-1)(z-2)^2}$$

By partial fractions, we get

$$\begin{aligned} \frac{U(z)}{z} &= \frac{2}{z-1} - \frac{2}{z-2} + \frac{6}{(z-2)^2} \\ \Rightarrow U(z) &= \frac{2z}{z-1} - \frac{2z}{z-2} + \frac{6z}{(z-2)^2} \end{aligned}$$

$$\begin{aligned}\therefore u_n &= 2Z^{-1} \left[ \frac{z}{z-1} \right] - 2Z^{-1} \left[ \frac{z}{z-2} \right] + 3Z^{-1} \left[ \frac{2z}{(z-2)^2} \right] \\ \Rightarrow u_n &= 2(1)^n - 2(2)^n + 3n(2)^n \quad \because Z^{-1} \left[ \frac{z}{z-a} \right] = a^n \text{ and } Z^{-1} \left[ \frac{az}{(z-a)^2} \right] = na^n \\ \text{i.e. } u_n &= 2 - 2^{n+1} + 3n \cdot 2^n\end{aligned}$$

#### 4.3.4 Method of residues (Inverse integral)

By using the theory of complex variables, it can be shown that the inverse Z-transform is given by  $u_n = \frac{1}{2\pi i} \oint_c U(z) z^{n-1} dz$  = sum of residues of  $U(z)$  where  $c$  is the closed contour which contains all the isolated singularities of  $U(z)$  in the region of convergence.

Method of residues is one of the most efficient methods and can be used to find the inverse Z-transform where partial fractions are tedious to find.

**Example21** Find the inverse z-transform of  $\frac{z}{z^2+7z+10}$

**Solution:**  $U(z) = \frac{z}{z^2+7z+10}$

Now  $u_n = \frac{1}{2\pi i} \oint_c U(z) z^{n-1} dz$

$$\begin{aligned}\Rightarrow u_n &= \frac{1}{2\pi i} \oint_c \frac{z}{z^2+7z+10} z^{n-1} dz \\ &= \frac{1}{2\pi i} \oint_c \frac{z^n}{z^2+7z+10} dz \\ &= \frac{1}{2\pi i} \oint_c \frac{z^n}{(z+2)(z+5)} dz\end{aligned}$$

There are two simple poles at  $z = -2$  and  $z = -5$

Residue at  $z = -2$  is given by  $\lim_{z \rightarrow -2} (z+2) \frac{z^n}{(z+2)(z+5)} = \frac{(-2)^n}{3}$

Residue at  $z = -5$  is given by  $\lim_{z \rightarrow -5} (z+5) \frac{z^n}{(z+2)(z+5)} = \frac{(-5)^n}{-3}$

$$\therefore u_n = \text{sum of residues} = \frac{(-2)^n}{3} + \frac{(-5)^n}{-3} = \frac{1}{3} \{ (-2)^n - (-5)^n \}$$

**Example22** Find the inverse z-transform of  $\frac{z^2+z}{(z-1)(z^2+1)}$

**Solution:**  $U(z) = \frac{z^2+z}{(z-1)(z^2+1)}$

Now  $u_n = \frac{1}{2\pi i} \oint_c U(z) z^{n-1} dz$

$$\begin{aligned}\Rightarrow u_n &= \frac{1}{2\pi i} \oint_c \frac{z^2+z}{(z-1)(z^2+1)} z^{n-1} dz \\ &= \frac{1}{2\pi i} \oint_c \frac{z^n(z+1)}{(z-1)(z+i)(z-i)} dz\end{aligned}$$

There are three simple poles at  $z = 1$ ,  $z = -i$  and  $z = i$

Residue at  $z = 1$  is given by  $\lim_{z \rightarrow 1} (z-1) \frac{z^n(z+1)}{(z-1)(z+i)(z-i)} = 1$

Residue at  $z = -i$  is given by  $\lim_{z \rightarrow -i} (z+i) \frac{z^n(z+1)}{(z-1)(z+i)(z-i)} = -\frac{1}{2}(-i)^n$

Residue at  $z = i$  is given by  $\lim_{z \rightarrow i} (z-i) \frac{z^n(z+1)}{(z-1)(z+i)(z-i)} = -\frac{1}{2}i^n$

$$\therefore u_n = \text{sum of residues} = 1 - \frac{1}{2}(-i)^n - \frac{1}{2}i^n = 1 - \frac{1}{2} \{ (-i)^n + i^n \}$$

**Example23** Find the inverse z-transform of  $\frac{z(z+1)}{(z-1)^3}$

**Solution:**  $U(z) = \frac{z(z+1)}{(z-1)^3}$

$$\begin{aligned}\text{Now } u_n &= \frac{1}{2\pi i} \oint_c U(z) z^{n-1} dz \\ \Rightarrow u_n &= \frac{1}{2\pi i} \oint_c \frac{z^n(z+1)}{(z-1)^3} dz\end{aligned}$$

Here  $z = 1$  is a pole of order 3

$$\begin{aligned}\text{Residue at } z = 1 \text{ is given by } & \frac{1}{2!} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} \left[ \frac{(z-1)^3 z^n (z+1)}{(z-1)^3} \right] \\ &= \frac{1}{2!} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} [z^n (z+1)] \\ &= \frac{1}{2!} \lim_{z \rightarrow 1} \frac{d}{dz} [(n+1)z^n + nz^{n-1}] \\ &= \frac{1}{2!} \lim_{z \rightarrow 1} [(n+1)nz^{n-1} + n(n-1)z^{n-2}] \\ &= [n^2 + n + n^2 - n] = n^2 \\ \therefore u_n &= \text{Sum of residues} = n^2\end{aligned}$$

#### 4.3.5 Power series method

In this method, we find the inverse Z - transform by expanding  $U(z)$  in power series.

**Example 24** Find  $u_n$  if  $U(z) = \log \frac{z}{z+1}$

**Solution:** Given  $U(z) = \log \frac{z}{z+1} = \log \left( \frac{z+1}{z} \right)^{-1} = -\log \frac{z+1}{z} = -\log \left( 1 + \frac{1}{z} \right)$

$$\begin{aligned}\therefore U(z) &= -\log(1+y) \quad \text{Putting } \frac{1}{z} = y \\ &= -y + \frac{y^2}{2} - \frac{y^3}{3} + \frac{y^4}{4} - \dots \\ &\quad \because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \\ \Rightarrow U(z) &= -\frac{1}{z} + \frac{1}{2z^2} - \frac{1}{3z^3} + \frac{1}{4z^4} - \dots \quad \because y = \frac{1}{z} \\ \Rightarrow U(z) &= 0 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{-n}\end{aligned}$$

Comparing with  $U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$ , we get

$$u_n = \begin{cases} 0 & \text{for } n = 0 \\ \frac{(-1)^n}{n}, & \text{otherwise} \end{cases}$$

#### 4.3.6 Convolution theorem method

Convolution theorem for Z-transforms states that:

$$\begin{aligned}\text{If } U(z) &= Z\{u_n\} \text{ and } V(z) = Z\{v_n\}, \text{ then } Z\{u_n * v_n\} = U(z) \cdot V(z) \\ \Rightarrow Z^{-1}[U(z) \cdot V(z)] &= u_n * v_n\end{aligned}$$

**Example25** Find the inverse z-transform of  $\frac{z^2}{(z-1)(2z-1)}$  using convolution theorem.

**Solution:** Let  $U(z) = Z\{u_n\} = \frac{z}{(z-1)}$  and  $V(z) = Z\{v_n\} = \frac{z}{(2z-1)} = \frac{1}{2} \left( \frac{z}{z-\frac{1}{2}} \right)$

$$\text{Clearly } u_n = (1)^n \text{ and } u_n = \frac{1}{2} \left( \frac{1}{2} \right)^n \quad \because Z^{-1} \left[ \frac{z}{z-a} \right] = a^n$$

Now by convolution theorem  $Z^{-1}[U(z) \cdot V(z)] = u_n * v_n$

$$\Rightarrow Z^{-1} \left[ \frac{z^2}{(z-1)(2z-1)} \right] = (1)^n * \left( \frac{1}{2} \right)^{n+1}$$

We know that  $u_n * v_n = \sum_{m=0}^n u_m v_{n-m}$

$$\begin{aligned}
 &= \sum_{m=0}^n (1)^m \left( \frac{1}{2} \right)^{n+1-m} \\
 &= \left( \frac{1}{2} \right)^{n+1} + \left( \frac{1}{2} \right)^n + \left( \frac{1}{2} \right)^{n-1} + \dots + \frac{1}{2} \\
 &= \frac{1}{2} \left[ 1 + \frac{1}{2} + \left( \frac{1}{2} \right)^2 + \dots + \left( \frac{1}{2} \right)^n \right] \\
 &= \frac{1}{2} \left[ \frac{1}{1-\frac{1}{2}} \left( 1 - \left( \frac{1}{2} \right)^{n+1} \right) \right] \\
 &\quad \because S_n = \frac{a}{1-r} (1 - r^n) \\
 &= \frac{1}{2} \left[ 2 \left( 1 - \left( \frac{1}{2} \right)^{n+1} \right) \right] \\
 &= 1 - \left( \frac{1}{2} \right)^{n+1}
 \end{aligned}$$

### Exercise 4A

- Find the Z-transform of  $u_n = \begin{cases} 2^n, & n < 0 \\ 3^n, & n \geq 0 \end{cases}$
- Find the Z-transform of  $u_n = \frac{1}{(n-p)!}$
- Find the inverse Z-transform of  $u_n = \frac{2z}{(z-1)(z^2+1)}$
- Solve the difference equation  $y_{x+2} + 4y_{x+1} + 3y_x = 3^x$ ,  $y_0 = 0, y_1 = 1$  using Z-transforms

### Answers

- $\frac{2z}{z^2-8z+15}, \quad 3 < |z| < 5$
- $z^{-p} e^{\frac{1}{z}}$
- $1 - \cos \frac{n\pi}{2} - i \sin \frac{n\pi}{2}$
- $y_x = \frac{1}{24} 3^x - \frac{5}{12} (-3)^x + \frac{3}{8} (-1)^x$

**MODULE-3**  
**PARTIAL DIFFERENTIAL EQUATIONS**

**Definition**

An equation involving one or more partial derivatives of a function of two or more variables is called a **Partial differential equation[PDE]**.

The *order* of a PDE is the order of the highest derivative and the *degree* of highest order derivative after clearing the equation of fractional powers.

A PDE is said to be linear if it is of first first degree in the dependent variable and its partial derivatives.

If each term of the PDE contains either the dependent variable or one of its partial derivatives, the PDE is said to be **homogeneous**. Otherwise it is said to be a **nonhomogeneous** PDE.

**Examples**

1.  $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$  [Order – 1, Degree – 1, Homogeneous PDE]

2.  $\frac{\partial^2 z}{\partial x \partial y} = xy$  [Order – 2, Degree – 1, NonHomogeneous PDE]

3.  $\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y} + 2z$  [Order – 2, Degree – 1, Homogeneous PDE]

4.  $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$  [Order – 2, Degree – 1, Homogeneous PDE]

5.  $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$  [Order – 3, Degree – 1, NonHomogeneous PDE]

6.  $\frac{\partial^2 u}{\partial x^2} = x + y$  [Order – 2, Degree – 1, NonHomogeneous PDE]

7.  $\frac{\partial^2 z}{\partial x^2} - 16z = 0$  [Order – 2, Degree – 1, Homogeneous PDE]

8.  $\frac{\partial^2 u}{\partial x \partial y} + 4 \frac{\partial u}{\partial y} = 0$  [Order – 2, Degree – 1, NonHomogeneous PDE]

**Formation of PDE by eliminating arbitrary constants and arbitrary functions**

**Note:**

If  $z = f(x, y)$  , then

$$z_x = \frac{\partial z}{\partial x} = p$$

$$z_y = \frac{\partial z}{\partial y} = q$$

$$z_{xx} = \frac{\partial^2 z}{\partial x^2} = r$$

$$z_{xy} = \frac{\partial^2 z}{\partial x \partial y} = s$$

$$z_{yy} = \frac{\partial^2 z}{\partial y^2} = t$$

### **Problems**

**Form the PDE by eliminating the arbitrary constant in the following**

1.  $z = (x + a)(y + b)$

**Sol:** Given  $z = (x + a)(y + b)$  --- (1)

Differentiate (1) w.r.t 'x' and 'y' partially

$$\frac{\partial z}{\partial x} = (y + b).1 \quad \Rightarrow \quad p = (y + b)$$

$$\text{Also, } \frac{\partial z}{\partial y} = (x + a).1 \quad \Rightarrow \quad q = (x + a)$$

$$(1) \Rightarrow z = q.p$$

$z = pq$  , is the required PDE

2.  $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

**Sol:** Given  $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$  --- (1)

Differentiate (1) w.r.t 'x' and 'y' partially

$$2 \frac{\partial z}{\partial x} = \frac{2x}{a^2}$$

$$\Rightarrow p = \frac{x}{a^2}$$

Multiply by 'x'

$$px = \frac{x^2}{a^2}$$

$$\text{Also, } 2 \frac{\partial z}{\partial y} = \frac{2y}{b^2}$$

$$\Rightarrow q = \frac{y}{b^2}$$

Multiply by 'y'

$$qy = \frac{y^2}{b^2}$$

(1)  $\Rightarrow 2z = px + qy$ , is the required PDE.

$$3. ax^2 + by^2 + z^2 = 1$$

$$\text{Sol: Given } ax^2 + by^2 + z^2 = 1$$

$$z^2 = 1 - ax^2 - by^2 \quad \text{--- (1)}$$

Differentiate (1) w.r.t 'x' and 'y' partially

$$2z \frac{\partial z}{\partial x} = 0 - 2ax - 0$$

$$\Rightarrow 2zp = -2ax$$

$$ax = -zp$$

Multiply by 'x'

$$ax^2 = -zpx$$

$$\text{Also, } 2z \frac{\partial z}{\partial y} = 0 - 0 - 2by$$

$$\Rightarrow 2zq = -2by$$

$$by = -zq$$

Multiply by 'y'

$$by^2 = -zqy$$

$$(1) \Rightarrow z^2 = 1 - (-zpx) - (-zqy)$$

$$z^2 - 1 = zpx + zqy$$

$$z^2 - 1 = z(px + qy), \text{ is the required PDE.}$$

$$4. z = a \log(x^2 + y^2) + b$$

$$\text{Sol: Given } z = a \log(x^2 + y^2) + b \quad \text{--- (1)}$$

Differentiate (1) w.r.t 'x' and 'y' partially

$$\frac{\partial z}{\partial x} = a \cdot \frac{1}{(x^2 + y^2)} \cdot 2x$$

$$\Rightarrow \mathbf{p} = \frac{2xa}{(x^2+y^2)}$$

$$\Rightarrow \frac{p}{x} = \frac{2a}{(x^2+y^2)} \quad \text{--- (2)}$$

$$\text{Also, } \frac{\partial z}{\partial y} = a \cdot \frac{1}{(x^2+y^2)} \cdot 2y$$

$$\Rightarrow \frac{q}{y} = \frac{2a}{(x^2+y^2)} \quad \text{--- (3)}$$

From (2) and (3) , we get

$$\frac{p}{x} = \frac{q}{y}$$

$$py = qx$$

$\mathbf{py - qx = 0}$  , is the required PDE.

$$\text{5. } z = xy + y\sqrt{x^2 - a^2} + b$$

$$\text{Sol: Given } z = xy + y\sqrt{x^2 - a^2} + b \quad \text{--- (1)}$$

Differentiate (1) w.r.t 'x' and 'y' partially

$$\frac{\partial z}{\partial x} = y + y \frac{1}{2\sqrt{x^2 - a^2}} \cdot 2x$$

$$\Rightarrow \mathbf{p} = y + \frac{xy}{\sqrt{x^2 - a^2}}$$

$$\Rightarrow \mathbf{p - y} = \frac{xy}{\sqrt{x^2 - a^2}} \quad \text{--- (2)}$$

$$\text{Also, } \frac{\partial z}{\partial y} = x \cdot 1 + \sqrt{x^2 - a^2} \cdot 1$$

$$\mathbf{q - x} = \sqrt{x^2 - a^2} \quad \text{--- (3)}$$

Using (3) in (2), we get

$$(2) \Rightarrow p - y = \frac{xy}{(q-x)}$$

$$(p - y)(q - x) = xy$$

$$pq - px - qy + xy = xy$$

$\mathbf{pq = px + qy}$  , is the required PDE.

6. Find the PDE of the family of spheres whose centre lie on the plane  $z = 0$  and have a constant radius 'r'.

Sol: The co-ordinates of the centre of the sphere can be taken as  $(a, b, 0)$  where  $a$  and  $b$  are arbitrary.  $r$  is the constant radius.

The equation of the sphere is given by

$$(x - a)^2 + (y - b)^2 + (z - 0)^2 = r^2$$

$$(x - a)^2 + (y - b)^2 + z^2 = r^2$$

$$z^2 = r^2 - (x - a)^2 - (y - b)^2 \quad \text{--- (1)}$$

Differentiate (1) w.r.t 'x' and 'y' partially

$$2z \frac{\partial z}{\partial x} = 0 - 2(x - a) - 0$$

$$\Rightarrow 2zp = -2(x - a)$$

$$\Rightarrow (x - a) = -zp$$

$$\text{Also } 2z \frac{\partial z}{\partial y} = 0 - 0 - 2(y - b)$$

$$\Rightarrow 2zq = -2(y - b)$$

$$\Rightarrow (y - b) = -zq$$

$$(1) \Rightarrow z^2 = r^2 - (-zp)^2 - (-zq)^2$$

$$z^2 = r^2 - z^2p^2 - z^2q^2$$

$$z^2 + z^2p^2 + z^2q^2 = r^2$$

$$z^2(1 + p^2 + q^2) = r^2, \text{ is the required PDE.}$$

$$7. \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\text{Sol: Given } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \quad \text{--- (1)}$$

Differentiate (1) w.r.t 'x' and 'y' partially

$$\frac{2z}{c^2} \frac{\partial z}{\partial x} = 0 - \frac{2x}{a^2} - 0$$

$$\frac{2z}{c^2} p = -\frac{2x}{a^2}$$

$$\frac{zp}{c^2} = -\frac{x}{a^2} \quad \text{--- (2)}$$

$$\text{Also, } \frac{2z}{c^2} \frac{\partial z}{\partial y} = 0 - 0 - \frac{2y}{b^2}$$

$$\frac{2z}{c^2} q = -\frac{2y}{b^2}$$

$$\frac{zq}{c^2} = -\frac{y}{b^2} \quad \text{--- (3)}$$

Differentiate (2) w.r.t 'x' partially

$$\frac{1}{c^2} \frac{\partial}{\partial x} \left[ z \cdot \frac{\partial z}{\partial x} \right] = -\frac{1}{a^2} \frac{\partial}{\partial x} [x] \quad \text{Since } p = \frac{\partial z}{\partial x}$$

$$\frac{1}{c^2} \left[ \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial x} + z \cdot \frac{\partial^2 z}{\partial x^2} \right] = -\frac{1}{a^2} \cdot 1$$

$$\frac{1}{c^2} [p^2 + z \cdot r] = -\frac{1}{a^2}$$

Multiply by 'x'

$$\frac{x}{c^2} [p^2 + zr] = -\frac{x}{a^2}$$

$$\frac{x}{c^2} [p^2 + zr] = \frac{zp}{c^2} \quad \text{From (2)}$$

$$x[p^2 + zr] = zp$$

$$xp^2 + xzr = zp$$

$$xp^2 + xzr - zp = 0$$

$$xp^2 + z(xr - p) = 0, \text{ is the required PDE.}$$

**8. Form the PDE of the family of spheres whose centre lies on  $x$  - axis with constant radius 3.**

**Sol:** Equation of sphere with centre on  $x$  - axis i.e.,  $C = (a, 0, 0)$  and radius  $r = 3$ .

The equation of the sphere is given by

$$(x - a)^2 + (y - 0)^2 + (z - 0)^2 = 3^2$$

$$(x - a)^2 + y^2 + z^2 = 9$$

$$z^2 = 9 - (x - a)^2 - y^2 \quad \text{--- (1)}$$

Differentiate (1) w.r.t 'x' and 'y' partially

$$2z \frac{\partial z}{\partial x} = 0 - 2(x - a) - 0$$

$$\Rightarrow 2zp = -2(x - a)$$

$$\Rightarrow (x - a) = -zp$$

$$\text{Also } 2z \frac{\partial z}{\partial y} = 0 - 0 - 2(y - b)$$

$$\Rightarrow 2zq = -2(y - b)$$

$$\Rightarrow (y - b) = -zq$$

$$(1) \Rightarrow z^2 = 9 - (-zp)^2 - (-zq)^2$$

$$z^2 = 9 - z^2p^2 - z^2q^2$$

$$z^2 + z^2p^2 + z^2q^2 = 9$$

$$z^2(1 + p^2 + q^2) = 9, \text{ is the required PDE.}$$

**Form the PDE by eliminating arbitrary functions in the following**

$$1. z = f(x^2 + y^2)$$

$$\text{Sol: Given } z = f(x^2 + y^2) \quad \text{--- (1)}$$

Differentiating (1) partially w.r.t 'x' and 'y', we have

$$\frac{\partial z}{\partial x} = p = f'(x^2 + y^2) \cdot 2x$$

$$\frac{\partial z}{\partial y} = q = f'(x^2 + y^2) \cdot 2y$$

$$\frac{p}{x} = 2f'(x^2 + y^2) \quad \text{--- (2)}$$

$$\frac{q}{y} = 2f'(x^2 + y^2) \quad \text{--- (3)}$$

From (2) and (3), we have

$$\frac{p}{x} = \frac{q}{y}$$

$$py = qx$$

$$py - qx = 0, \text{ is the required PDE.}$$

$$2. z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$$

$$\text{Sol: Given } z = y^2 + 2f\left(\frac{1}{x} + \log y\right) \quad \text{--- (1)}$$

Differentiating (1) partially w.r.t 'x' and 'y', we have

$$\frac{\partial z}{\partial x} = p = 0 + 2f'\left(\frac{1}{x} + \log y\right) \cdot \left(-\frac{1}{x^2}\right)$$

$$\frac{\partial z}{\partial y} = q = 2y + 2f'\left(\frac{1}{x} + \log y\right) \cdot \left(\frac{1}{y}\right)$$

$$p = -\frac{1}{x^2} \cdot 2f'\left(\frac{1}{x} + \log y\right)$$

$$q - 2y = \frac{1}{y} \cdot 2f'\left(\frac{1}{x} + \log y\right)$$

$$x^2p = -2f'\left(\frac{1}{x} + \log y\right) \quad \text{--- (2)}$$

$$(q - 2y)y = 2f'\left(\frac{1}{x} + \log y\right) \quad \text{--- (3)}$$

From (2) and (3), we have

$$x^2p = -(q - 2y)y$$

$$x^2p = -qy + 2y^2$$

$$x^2p + qy = 2y^2, \text{ is the required PDE.}$$

$$3. z = e^y f(x + y)$$

$$\text{Sol: Given } z = e^y f(x + y) \text{ --- (1)}$$

Differentiating (1) partially w.r.t 'x' and 'y', we have

$$\frac{\partial z}{\partial x} = p = e^y f'(x + y) \qquad \frac{\partial z}{\partial y} = q = e^y \cdot f'(x + y) + f(x + y) \cdot e^y$$

$$p = e^y f'(x + y) \text{ --- (2)}$$

$$q = e^y \cdot f'(x + y) + z$$

$$q - z = e^y \cdot f'(x + y) \text{ --- (3)}$$

From (2) and (3), we have

$$p = q - z$$

$p + z = q$ , is the required PDE.

$$4. z = e^{ax+by} f(ax - by)$$

$$\text{Sol: Given } z = e^{ax+by} f(ax - by) \text{ --- (1)}$$

Differentiating (1) partially w.r.t 'x', we have

$$\frac{\partial z}{\partial x} = e^{ax+by} f'(ax - by) \cdot a + f(ax - by) \cdot e^{ax+by} \cdot a$$

$$p = e^{ax+by} f'(ax - by) \cdot a + az$$

$$p - az = ae^{ax+by} f'(ax - by)$$

Multiply by 'b'

$$pb - abz = abe^{ax+by} f'(ax - by) \text{ --- (2)}$$

Differentiating (1) partially w.r.t 'y', we have

$$\frac{\partial z}{\partial y} = e^{ax+by} f'(ax - by) \cdot (-b) + f(ax - by) \cdot e^{ax+by} \cdot b$$

$$q = -be^{ax+by} f'(ax - by) + bz$$

$$q - bz = -be^{ax+by} f'(ax - by)$$

Multiply by 'a'

$$qa - abz = -abe^{ax+by} f'(ax - by) \text{ --- (3)}$$

From (2) and (3), we get

$$pb - abz = -(qa - abz)$$

$$pb - abz = -qa + abz$$

$$pb + qa = abz + abz$$

**$pb + qa = 2abz$** , is the required PDE.

$$3. z = e^{ax+by} f(ax - by)$$

**Sol:** Given  $z = e^{ax+by} f(ax - by)$  --- (1)

Differentiating (1) partially w.r.t 'x', we have

$$\frac{\partial z}{\partial x} = e^{ax+by} f'(ax - by).a + f(ax - by).e^{ax+by}.a$$

$$p = e^{ax+by} f'(ax - by).a + az$$

$$p - az = ae^{ax+by} f'(ax - by)$$

Multiply by 'b'

$$pb - abz = abe^{ax+by} f'(ax - by) \text{ --- (2)}$$

Differentiating (1) partially w.r.t 'y', we have

$$\frac{\partial z}{\partial y} = e^{ax+by} f'(ax - by).(-b) + f(ax - by).e^{ax+by}.b$$

$$q = -be^{ax+by} f'(ax - by) + bz$$

$$q - bz = -be^{ax+by} f'(ax - by)$$

Multiply by 'a'

$$qa - abz = -abe^{ax+by} f'(ax - by) \text{ --- (3)}$$

From (2) and (3), we get

$$pb - abz = -(qa - abz)$$

$$pb - abz = -qa + abz$$

$$pb + qa = abz + abz$$

**$pb + qa = 2abz$** , is the required PDE.

$$5. lx + my + nz = \varphi(x^2 + y^2 + z^2)$$

**Sol:** Given  $lx + my + nz = \varphi(x^2 + y^2 + z^2)$  --- (1)

Differentiating (1) partially w.r.t 'x', we have

$$l.1 + 0 + n \frac{\partial z}{\partial x} = \varphi'(x^2 + y^2 + z^2). \left( 2x + 0 + 2z \frac{\partial z}{\partial x} \right)$$

$$l + np = 2\varphi'(x^2 + y^2 + z^2). (x + zp)$$

$$\frac{(l+np)}{(x+zp)} = 2\varphi'(x^2 + y^2 + z^2) \text{ --- (2)}$$

Differentiating (1) partially w.r.t 'y', we have

$$0 + m.1 + n \frac{\partial z}{\partial y} = \varphi'(x^2 + y^2 + z^2). \left( 0 + 2y + 2z \frac{\partial z}{\partial y} \right)$$

$$m + nq = 2\varphi'(x^2 + y^2 + z^2). (y + zq)$$

$$\frac{(m+nq)}{(y+zq)} = 2\varphi'(x^2 + y^2 + z^2) \quad \text{--- (3)}$$

From (2) and (3), we get

$$\frac{(l+np)}{(x+zp)} = \frac{(m+nq)}{(y+zq)}$$

$$(l + np)(y + zq) = (x + zp)(m + nq)$$

$$ly + lzq + npy + npzq = mx + nqx + mzp + zpnq$$

$$ly - mx = nqx + mzp - lzq - npy$$

$$ly - mx = p(mz - ny) + q(nx - lz), \text{ is the required PDE.}$$

$$6. \varphi(x + y + z, x^2 + y^2 - z^2) = 0$$

$$\text{Sol: Given } \varphi(x + y + z, x^2 + y^2 - z^2) = 0$$

$$\text{Let } u = x + y + z$$

$$v = x^2 + y^2 - z^2$$

$$\frac{\partial u}{\partial x} = 1 + 0 + \frac{\partial z}{\partial x}$$

$$\frac{\partial v}{\partial x} = 2x + 0 - 2z \frac{\partial z}{\partial x}$$

$$\frac{\partial u}{\partial x} = 1 + p$$

$$\frac{\partial v}{\partial x} = 2x - 2zp$$

$$\frac{\partial u}{\partial y} = 0 + 1 + \frac{\partial z}{\partial y}$$

$$\frac{\partial v}{\partial y} = 0 + 2y - 2z \frac{\partial z}{\partial y}$$

$$\frac{\partial u}{\partial y} = 1 + q$$

$$\frac{\partial v}{\partial y} = 2y - 2zq$$

$$\text{Now } \varphi(u, v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\varphi(u, v) = 0$$

$$\begin{vmatrix} (1+p) & (1+q) \\ (2x-2zp) & (2y-2zq) \end{vmatrix} = 0$$

$$(1+p)(2y-2zq) - (1+q)(2x-2zp) = 0$$

$$2y - 2zq + 2py - 2zpq - 2x + 2zp - 2qx + 2zpq = 0 \quad \div \text{ by } 2$$

$$y - zq + py - x + zp - qx = 0$$

$$-zq + py + zp - qx = x - y$$

$$p(y + z) - q(x + z) = x - y, \text{ is the required PDE.}$$

$$7. \varphi(xy + z^2, x + y + z) = 0$$

$$\text{Sol: Given } \varphi(xy + z^2, x + y + z) = 0$$

$$\text{Let } u = xy + z^2$$

$$\frac{\partial u}{\partial x} = y \cdot 1 + 2z \frac{\partial z}{\partial x}$$

$$\frac{\partial u}{\partial x} = \mathbf{y} + \mathbf{2zp}$$

$$\frac{\partial u}{\partial y} = x \cdot 1 + 0 + 2z \frac{\partial z}{\partial y}$$

$$\frac{\partial u}{\partial y} = \mathbf{x} + \mathbf{2zq}$$

$$v = x + y + z$$

$$\frac{\partial v}{\partial x} = 1 + 0 + \frac{\partial z}{\partial x}$$

$$\frac{\partial v}{\partial x} = \mathbf{1} + \mathbf{p}$$

$$\frac{\partial v}{\partial y} = 1 + 0 + \frac{\partial z}{\partial y}$$

$$\frac{\partial v}{\partial y} = \mathbf{1} + \mathbf{q}$$

$$\text{Now } \varphi(u, v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\varphi(u, v) = 0$$

$$\begin{vmatrix} (y + 2zp) & (x + 2zq) \\ (1 + p) & (1 + q) \end{vmatrix} = 0$$

$$(y + 2zp)(1 + q) - (x + 2zq)(1 + p) = 0$$

$$y + yq + 2zp + 2zpq - x - 2zq - px - 2pzq = 0$$

$$y + yq + 2zp - x - 2zq - px = 0$$

$$yq + 2zp - 2zq - px = x - y$$

$$\mathbf{p(2z - x) + q(y - 2z) = x - y}$$
, is the required PDE.

$$\mathbf{8. f(x^2 + y^2, z - xy) = 0}$$

$$\text{Sol: Given } f(x^2 + y^2, z - xy) = 0$$

$$\text{Let } u = x^2 + y^2$$

$$\frac{\partial u}{\partial x} = 2x + 0$$

$$\frac{\partial u}{\partial x} = \mathbf{2x}$$

$$\frac{\partial u}{\partial y} = 0 + 2y$$

$$\frac{\partial u}{\partial y} = \mathbf{2y}$$

$$v = z - xy$$

$$\frac{\partial v}{\partial x} = \frac{\partial z}{\partial x} - y \cdot 1$$

$$\frac{\partial v}{\partial x} = \mathbf{p - y}$$

$$\frac{\partial v}{\partial y} = \frac{\partial z}{\partial y} - x \cdot 1$$

$$\frac{\partial v}{\partial y} = \mathbf{q - x}$$

$$\text{Now } f(u, v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$f(u, v) = 0$$

$$\begin{vmatrix} (2x) & (2y) \\ (p-y) & (q-x) \end{vmatrix} = 0$$

$$(2x)(q-x) - (2y)(p-y) = 0$$

$$2xq - 2x^2 - 2yp + 2y^2 = 0 \quad \div \text{ by } 2$$

$$xq - yp = x^2 - y^2, \text{ is the required PDE.}$$

$$9. f(x^2 + 2yz, y^2 + 2zx) = 0$$

$$\text{Sol: Given } f(x^2 + 2yz, y^2 + 2zx) = 0$$

$$\text{Let } u = x^2 + 2yz$$

$$v = y^2 + 2zx$$

$$\frac{\partial u}{\partial x} = 2x + 2y \frac{\partial z}{\partial x}$$

$$\frac{\partial v}{\partial x} = 0 + 2x \cdot \frac{\partial z}{\partial x} + 2z \cdot 1$$

$$\frac{\partial u}{\partial x} = 2x + 2yp$$

$$\frac{\partial v}{\partial x} = 2xp + 2z$$

$$\frac{\partial u}{\partial y} = 0 + 2y \cdot \frac{\partial z}{\partial y} + 2z \cdot 1$$

$$\frac{\partial v}{\partial y} = 2y + 2x \frac{\partial z}{\partial y}$$

$$\frac{\partial u}{\partial y} = 2yq + 2z$$

$$\frac{\partial v}{\partial y} = 2y + 2xq$$

$$\text{Now } f(u, v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$f(u, v) = 0$$

$$\begin{vmatrix} (2x + 2yp) & (2yq + 2z) \\ (2z + 2xp) & (2y + 2xq) \end{vmatrix} = 0$$

$$(2x + 2yp)(2y + 2xq) - (2yq + 2z)(2z + 2xp) = 0$$

$$4xy + 4x^2q + 4y^2p + 4xypq - 4zyq - 4xypq - 4z^2 - 4xzp = 0 \quad \div \text{ by } 4$$

$$xy + x^2q + y^2p - zyq - z^2 - xzp = 0$$

$$x^2q + y^2p - zyq - xzp = z^2 - xy$$

$$p(y^2 - xz) + q(x^2 - yz) = z^2 - xy, \text{ is the required PDE.}$$

$$10. f\left(\frac{xy}{z}, z\right) = 0$$

$$\text{Sol: Given } f\left(\frac{xy}{z}, z\right) = 0$$

$$\text{Let } u = \frac{xy}{z}$$

$$v = z$$

$$\frac{\partial u}{\partial x} = y \cdot \frac{(z \cdot 1 - x \cdot \frac{\partial z}{\partial x})}{z^2}$$

$$\frac{\partial v}{\partial x} = \frac{\partial z}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{y(z - xp)}{z^2}$$

$$\frac{\partial v}{\partial x} = p$$

$$\frac{\partial u}{\partial y} = x \cdot \frac{(z \cdot 1 - y \cdot \frac{\partial z}{\partial y})}{z^2}$$

$$\frac{\partial v}{\partial y} = \frac{\partial z}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{x(z - yq)}{z^2}$$

$$\frac{\partial v}{\partial y} = q$$

$$\text{Now } f(u, v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$f(u, v) = 0$$

$$\begin{vmatrix} \frac{y(z-xp)}{z^2} & \frac{x(z-yq)}{z^2} \\ p & q \end{vmatrix} = 0$$

$$\frac{y(z-xp)}{z^2} \cdot q - \frac{x(z-yq)}{z^2} \cdot p = 0$$

$$\frac{yq(z-xp)}{z^2} - \frac{xp(z-yq)}{z^2} = 0$$

$$\frac{yq(z-xp) - xp(z-yq)}{z^2} = 0$$

$$\frac{yqz - xppq - xzq + xypq}{z^2} = 0$$

$$yqz - xzq = 0 \quad ;$$

$$z(yq - xq) = 0$$

$$yq - xp = 0, \text{ is the required PDE.}$$

$$11. z = yf(x) + x\varphi(y)$$

$$\text{Sol: Given } z = yf(x) + x\varphi(y) \quad \text{--- (1)}$$

Differentiate (1) partially

$$\frac{\partial z}{\partial x} = y \cdot f'(x) + \varphi(y) \cdot 1$$

$$p = yf'(x) + \varphi(y) \quad \text{--- (2)}$$

$$\frac{\partial z}{\partial y} = f(x) \cdot 1 + x \cdot \varphi'(y)$$

$$q = f(x) + x\varphi'(y) \quad \text{--- (3)}$$

$$\frac{\partial^2 z}{\partial x^2} = y \cdot f''(x) + 0$$

$$\mathbf{r} = yf''(x) \quad \text{--- (4)}$$

$$\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = f'(x) \cdot 1 + \varphi'(y)$$

$$\frac{\partial^2 z}{\partial y \partial x} = f'(x) + \varphi'(y)$$

$$\mathbf{s} = f'(x) + \varphi'(y) \quad \text{--- (5)}$$

$$\frac{\partial^2 z}{\partial y^2} = 0 + x\varphi''(y)$$

$$\mathbf{t} = x\varphi''(y) \quad \text{--- (6)}$$

$$(1)^* x + (2)^* y$$

$$px + qy = xyf'(x) + x\varphi(y) + yf(x) + xy\varphi'(y)$$

$$px + qy = [x\varphi(y) + yf(x)] + xy[f'(x) + \varphi'(y)]$$

$$px + qy = z + xys \quad \text{From (1) and (5)}$$

$\mathbf{px + qy - xys = z}$ , is the required PDE.

$$\mathbf{12. \textcolor{red}{z = f(y + x) + g(y + 2x)}}$$

$$\mathbf{Sol:} \text{ Given } z = f(y + x) + g(y + 2x) \quad \text{--- (1)}$$

Differentiate (1) partially

$$\frac{\partial z}{\partial x} = f'(y + x) + 2g'(y + 2x)$$

$$\mathbf{p = f' + 2g'} \quad \text{--- (2)}$$

$$\frac{\partial z}{\partial y} = f'(y + x) + g'(y + 2x)$$

$$\mathbf{q = f' + g'} \quad \text{--- (3)}$$

$$\frac{\partial^2 z}{\partial x^2} = f''(y + x) + 4g''(y + 2x)$$

$$\mathbf{r = f'' + 4g''} \quad \text{--- (4)}$$

$$\frac{\partial^2 z}{\partial x \partial y} = f''(y + x) + 4g''(y + 2x)$$

$$\mathbf{s = f'' + 2g''} \quad \text{--- (5)}$$

$$\frac{\partial^2 z}{\partial y^2} = f''(y + x) + g''(y + 2x)$$

$$\mathbf{t = f'' + g''} \quad \text{--- (6)}$$

$$(4) - (5)$$

$$r - s = f'' + 4g'' - f'' - 2g''$$

$$r - s = 2g'' \quad \text{--- (7)}$$

$$(5) - (6)$$

$$s - t = f'' + 2g'' - f'' - g''$$

$$s - t = g'' \quad \text{--- (8)}$$

$$(7) \Rightarrow r - s = 2(s - t) \quad \text{From (8)}$$

$$r - s = 2s - 2t$$

$$r - s - 2s + 2t = 0$$

$$r - 3s + 2t = 0, \text{ is the required PDE.}$$

$$13. z = f_1(y - 2x) + f_2(2y - x)$$

$$\text{Sol: Given } z = f_1(y - 2x) + f_2(2y - x) \quad \text{--- (1)}$$

Differentiate (1) partially

$$\frac{\partial z}{\partial x} = -2f'_1(y - 2x) - f'_2(2y - x)$$

$$p = -2f'_1 - f'_2 \quad \text{--- (2)}$$

$$\frac{\partial z}{\partial y} = f'_1(y - 2x) + 2f'_2(2y - x)$$

$$q = f'_1 + 2f'_2 \quad \text{--- (3)}$$

$$\frac{\partial^2 z}{\partial x^2} = 4f''_1(y - 2x) + f''_2(2y - x)$$

$$r = 4f''_1 + f''_2 \quad \text{--- (4)}$$

$$\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = -2f''_1(y - 2x) - 2f''_2(2y - x)$$

$$s = -2f''_1 - 2f''_2 \quad \text{--- (5)}$$

$$\frac{\partial^2 z}{\partial y^2} = f''_1(y - 2x) + 4f''_2(2y - x)$$

$$t = f''_1 + 4f''_2 \quad \text{--- (6)}$$

Now

$$r + 2s = 4f''_1 + f''_2 - 4f''_1 - 4f''_2$$

$$r + 2s = -3f''_2 \quad \text{--- (7)}$$

Also

$$s + 2t = -2f_1'' - 2f_2'' + 2f_1'' + 8f_2''$$

$$s + 2t = 6f_2'' \quad \text{--- (8)}$$

$$(8) \div (7)$$

$$\frac{s+2t}{r+2s} = \frac{6f_2''}{-3f_2''}$$

$$\frac{s+2t}{r+2s} = -2$$

$$s + 2t = -2(r + 2s)$$

$$s + 2t = -2r - 4s$$

$$s + 2t + 2r + 4s = 0$$

**5s + 2t + 2r = 0** , is the required PDE.

$$14. z = f(x) + e^y g(x)$$

**Sol:** Given  $z = f(x) + e^y g(x)$  --- (1)

Differentiate (1) partially

$$\frac{\partial z}{\partial x} = f'(x) + e^y g'(x)$$

$$p = f'(x) + e^y g'(x) \quad \text{--- (2)}$$

$$\frac{\partial z}{\partial y} = e^y g(x)$$

$$q = e^y g(x) \quad \text{--- (3)}$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x) + e^y g''(x)$$

$$r = f''(x) + e^y g''(x) \quad \text{--- (4)}$$

$$\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = 0 + e^y g'(x)$$

$$s = e^y g'(x) \quad \text{--- (5)}$$

$$\frac{\partial^2 z}{\partial y^2} = e^y g(x)$$

$$t = e^y g(x) \quad \text{--- (6)}$$

From (3) and (6)

$$q = t$$

**t - q = 0** , is the required the PDE.

$$15. z = f(x + ct) + g(x - ct)$$

$$\text{Sol: Given } z = f(x + ct) + g(x - ct) \text{ --- (1)}$$

$$\frac{\partial z}{\partial x} = f'(x + ct) + g'(x - ct)$$

$$p = f' + g' \text{ --- (2)}$$

$$\frac{\partial z}{\partial t} = cf'(x + ct) - cg'(x - ct)$$

$$q = cf' - cg' \text{ --- (3)}$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x + ct) + g''(x - ct)$$

$$r = f'' + g'' \text{ --- (4)}$$

$$\frac{\partial}{\partial t} \left( \frac{\partial z}{\partial x} \right) = cf''(x + ct) - cg''(x - ct)$$

$$s = cf'' - cg'' \text{ --- (5)}$$

$$\frac{\partial^2 z}{\partial t^2} = c^2 f''(x + ct) + c^2 g''(x - ct)$$

$$t = c^2 f'' + c^2 g'' \text{ --- (6)}$$

Consider

$$t = c^2 f'' + c^2 g''$$

$$t = c^2 [f'' + g'']$$

$$t = c^2 r$$

$$t - c^2 r = 0, \text{ is the required PDE.}$$

### ***Solution of Non Homogeneous PDE by Direct Integration***

$$1. \text{ Solve } \frac{\partial^2 u}{\partial x^2} = x + y$$

$$\text{Sol: Given } \frac{\partial^2 u}{\partial x^2} = x + y$$

$$\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = x + y$$

Integrate w.r.t 'x'

$$\frac{\partial u}{\partial x} = \int x \, dx + y \int 1 \, dx$$

$$\frac{\partial u}{\partial x} = \frac{x^2}{2} + yx + f(y)$$

Again integrating w.r.t 'x'

$$u = \frac{1}{2} \int x^2 dx + y \int x dx + f(y) \int 1 dx$$

$$u = \frac{1}{2} \left[ \frac{x^3}{3} \right] + y \left[ \frac{x^2}{2} \right] + f(y).x + g(y)$$

$$u = \frac{x^3}{6} + \frac{x^2 y}{2} + x f(y) + g(y), \text{ is the required solution.}$$

$$2. \text{ Solve } \frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + a$$

$$\text{Sol: Given } \frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + a$$

$$\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{x}{y} + a$$

Integrate w.r.t 'x'

$$\frac{\partial z}{\partial y} = \frac{1}{y} \int x dx + a \int 1 dx$$

$$\frac{\partial z}{\partial y} = \frac{x^2}{2y} + ax + f(y)$$

Integrate w.r.t 'y'

$$z = \frac{x^2}{2} \int \frac{1}{y} dy + ax \int 1 dy + \int f(y) dy$$

$$z = \frac{x^2}{2} \log y + axy + F(y) + g(x), \text{ is the required solution.}$$

$$3. \text{ Solve } \frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$$

$$\text{Sol: Given } \frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$$

$$\frac{\partial}{\partial x} \left( \frac{\partial^2 z}{\partial x \partial y} \right) = \cos(2x + 3y)$$

Integrate w.r.t 'x'

$$\frac{\partial^2 z}{\partial x \partial y} = \int \cos(2x + 3y) dx$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\sin(2x+3y)}{2} + f(y)$$

$$\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{\sin(2x+3y)}{2} + f(y)$$

Integrating w.r.t ' $x$ '

$$\frac{\partial z}{\partial y} = \frac{1}{2} \int \sin(2x + 3y) dx + f(y) \int 1 dx$$

$$\frac{\partial z}{\partial y} = \frac{-\cos(2x+3y)}{2.2} + f(y).x + g(y)$$

Integrating w.r.t ' $y$ '

$$z = -\frac{1}{4} \int \cos(2x + 3y) dy + x \int f(y) dy + \int g(y) dy$$

$$z = -\frac{1}{4} \cdot \frac{\sin(2x+3y)}{3} + xF(y) + G(y) + h(x)$$

$$z = -\frac{\sin(2x+3y)}{12} + xF(y) + G(y) + h(x), \text{ is the required solution.}$$

4. Solve  $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$  for which  $\frac{\partial z}{\partial y} = -2\sin y$  when  $x = 0$  and  $z = 0$  if  $y$  is an odd multiple of  $\frac{\pi}{2}$ . {or  $z = 0$  if  $y = (2n + 1) \frac{\pi}{2}$ }

**Sol:** Given  $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$

$$\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \sin x \sin y$$

Integrate w.r.t ' $x$ '

$$\frac{\partial z}{\partial y} = \sin y \int \sin x dx$$

$$\frac{\partial z}{\partial y} = \sin y (-\cos x) + f(y) \quad \text{--- (1)}$$

Integrate w.r.t ' $y$ '

$$z = -\cos x \int \sin y dy + \int f(y) dy$$

$$z = -\cos x (-\cos y) + F(y) + g(x)$$

$$z = \cos x \cdot \cos y + F(y) + g(x) \quad \text{--- (2)}$$

By data  $\frac{\partial z}{\partial y} = -2\sin y$  when  $x = 0$

$$(1) \Rightarrow -2\sin y = \sin y (-\cos 0) + f(y)$$

Since  $\cos 0 = 1$

$$-2\sin y = -\sin y + f(y)$$

$$f(y) = -2\sin y + \sin y$$

$$f(y) = -\sin y$$

Now

$$F(y) = \int f(y)dy$$

$$F(y) = - \int \sin y \, dy$$

$$F(y) = -(-\cos y)$$

$$F(y) = \cos y$$

$$(2) \Rightarrow z = \cos x \cdot \cos y + \cos y + g(x)$$

$$\text{Again by data } z = 0 \text{ if } y = (2n+1)\frac{\pi}{2}$$

$$0 = \cos x \cdot \cos \left[ (2n+1)\frac{\pi}{2} \right] + \cos \left[ (2n+1)\frac{\pi}{2} \right] + g(x) \quad \cos \left[ (2n+1)\frac{\pi}{2} \right] = 0$$

$$g(x) = 0$$

Thus,

$$(2) \Rightarrow z = \cos x \cdot \cos y + \cos y + 0$$

$$z = \cos x \cdot \cos y + \cos y$$

$$z = \cos y (\cos x + 1), \text{ is the solution.}$$

5. Solve  $\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$  given that  $u = 0$  when  $t = 0$  and  $\frac{\partial u}{\partial t} = 0$  at  $x = 0$ . Also show that  $u \rightarrow \sin x$  as  $t \rightarrow \infty$ .

$$\text{Sol: Given } \frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$$

$$\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} \right) = e^{-t} \cos x$$

Integrate w.r.t 'x'

$$\frac{\partial u}{\partial t} = e^{-t} \int \cos x \, dx$$

$$\frac{\partial u}{\partial t} = e^{-t} \sin x + f(t) \quad \text{--- (1)}$$

Integrate w.r.t 't'

$$u = \sin x \int e^{-t} dt + \int f(t) dt + g(x)$$

$$u = \sin x \left[ \frac{e^{-t}}{-1} \right] + F(t) + g(x)$$

$$u = -e^{-t} \sin x + F(t) + g(x) \quad \text{--- (2)}$$

$$\text{By data } \frac{\partial u}{\partial t} = 0 \text{ at } x = 0$$

$$(1) \Rightarrow 0 = e^{-t} (\sin 0) + f(t)$$

$$\text{Since } \sin 0 = 0$$

$$f(t) = 0$$

Now

$$F(t) = \int f(t) dt$$

$$\mathbf{F(t) = 0}$$

$$(2) \Rightarrow u = -e^{-t} \sin x + 0 + g(x)$$

$$u = -e^{-t} \sin x + g(x)$$

Also given  $u = 0$  when  $t = 0$

$$0 = -e^{-0} \sin x + g(x)$$

Since  $e^{-0} = 1$

$$0 = -\sin x + g(x)$$

$$\mathbf{g(x) = \sin x}$$

Thus,

$$(2) \Rightarrow u = -e^{-t} \sin x + 0 + \sin x$$

$$\mathbf{u = \sin x(1 - e^{-t}) , is the solution.}$$

As  $t \rightarrow \infty$

$$u = \sin x(1 - e^{-\infty})$$

$$u = \sin x(1 - 0)$$

$$\mathbf{u = \sin x}$$

$$\therefore \mathbf{u \rightarrow \sin x \quad as \quad t \rightarrow \infty}$$

6. Solve  $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y}$  subject to  $z = 0$  when  $x = 1$  and  $\frac{\partial z}{\partial x} = \log x$  when  $y = 1$ .

**Sol:** Given  $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y}$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{x}{y}$$

$$\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{x}{y}$$

Integrate w.r.t 'y'

$$\frac{\partial z}{\partial x} = x \int \frac{1}{y} dy$$

$$\frac{\partial z}{\partial x} = x \log y + f(x) \quad \text{--- (1)}$$

Integrate w.r.t 'x'

$$z = \log y \int x dx + \int f(x) dx$$

$$z = \log y \left[ \frac{x^2}{2} \right] + F(x) + g(y) \quad \text{--- (2)}$$

By data  $\frac{\partial z}{\partial x} = \log x$  when  $y = 1$

$$(1) \Rightarrow \log x = x \log 1 + f(x)$$

Since  $\log 1 = 0$

$$f(x) = \log x$$

Now

$$F(x) = \int f(x) dx$$

$$F(x) = \int \log x \cdot 1 dx$$

$$F(x) = x \log x - \int \frac{1}{x} \cdot x dx$$

$$F(x) = x \log x - x$$

$$(2) \Rightarrow z = \frac{x^2 \log y}{2} + x \log x - x + g(y)$$

Also given  $z = 0$  when  $x = 1$

$$0 = \frac{1 \cdot \log y}{2} + 1 \cdot \log 1 - 1 + g(y)$$

Since  $\log 1 = 0$

$$0 = \frac{\log y}{2} - 1 + g(y)$$

$$g(y) = 1 - \frac{\log y}{2}$$

Thus,

$$z = \frac{x^2 \log y}{2} + x \log x - x + 1 - \frac{\log y}{2}$$

$$z = \frac{\log y}{2} (x^2 - 1) + x(\log x - 1) + 1, \text{ is the required solution.}$$

7. Solve  $\frac{\partial^2 z}{\partial x^2} = xy$  subject to the conditions that  $\frac{\partial z}{\partial x} = \log(1 + y)$  when  $x = 1$  and  $z = 0$  when  $x = 0$ .

**Sol:** Given  $\frac{\partial^2 z}{\partial x^2} = xy$

$$\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = xy$$

Integrate w.r.t 'x'

$$\frac{\partial z}{\partial x} = y \int x dx$$

$$\frac{\partial z}{\partial x} = y \cdot \frac{x^2}{2} + f(y) \quad \text{--- (1)}$$

Integrate w.r.t 'x'

$$z = \frac{y}{2} \int x^2 dx + f(y) \int 1 dx$$

$$z = \frac{y}{2} \left[ \frac{x^3}{3} \right] + xf(y) + g(y)$$

$$z = \frac{x^3 y}{6} + xf(y) + g(y) \quad \text{--- (2)}$$

By data  $\frac{\partial z}{\partial x} = \log(1 + y)$  when  $x = 1$

$$(1) \Rightarrow \log(1 + y) = \frac{y}{2} + f(y)$$

$$f(y) = \log(1 + y) - \frac{y}{2}$$

$$(2) \Rightarrow z = \frac{x^3 y}{6} + x \left[ \log(1 + y) - \frac{y}{2} \right] + g(y)$$

Also by data  $z = 0$  when  $x = 0$

$$0 = 0 + 0 + g(y)$$

$$g(y) = 0$$

Thus,

$$z = \frac{x^3 y}{6} + x \left[ \log(1 + y) - \frac{y}{2} \right] + 0$$

$$z = \frac{x^3 y}{6} + x \left[ \log(1 + y) - \frac{y}{2} \right], \text{ is the required solution.}$$

## Lagrange's Method of Undetermined Multipliers

Suppose we require to find the maximum and minimum values of  $f(x, y)$  where  $x, y, z$  are subject to a constraint equation

$$g(x, y, z) = 0$$

We define a function

$$F(x, y, z, \lambda) = f(x, y, z) + \lambda g(x, y, z) \dots (1)$$

Where  $\lambda$  is called Lagrange Multiplier which is independent of  $x, y$ , and  $z$ .

The necessary conditions for a maximum or minimum are

$$\frac{\partial F}{\partial x} = 0 \dots (2)$$

$$\frac{\partial F}{\partial y} = 0 \dots (3)$$

$$\frac{\partial F}{\partial z} = 0 \dots (4)$$

Solving the four equations for four unknowns  $\lambda, x, y, z$ , we obtain the point  $(x, y, z)$ . The point may be a maxima, minima or neither which is decided by the physical consideration.

This method is also applicable when we have more than one constraint equation connecting the variables.

**Example:**

**Find the minimum value of  $x^2 + y^2 + z^2$  subject to the condition  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$**

**Solution:**

Let the auxiliary function 'F' be

$$F(x, y, z, \lambda) = (x^2 + y^2 + z^2) + \lambda \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 \right)$$

Where  $\lambda$  is Lagrange Multiplier

|                                                                                                           |                                                                                                           |                                                                                                           |
|-----------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------|
| $\frac{\partial F}{\partial x} = 2x + \lambda \left( \frac{-1}{x^2} \right)$ $= 2x - \frac{\lambda}{x^2}$ | $\frac{\partial F}{\partial y} = 2y + \lambda \left( \frac{-1}{y^2} \right)$ $= 2y - \frac{\lambda}{y^2}$ | $\frac{\partial F}{\partial z} = 2z + \lambda \left( \frac{-1}{z^2} \right)$ $= 2z - \frac{\lambda}{z^2}$ |
|-----------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------|

For a minimum at  $(x, y, z)$  we have

|                                                           |                                                           |                                                           |
|-----------------------------------------------------------|-----------------------------------------------------------|-----------------------------------------------------------|
| $\frac{\partial F}{\partial x} = 0$                       | $\frac{\partial F}{\partial y} = 0$                       | $\frac{\partial F}{\partial z} = 0$                       |
| $2x - \frac{\lambda}{x^2} = 0$ $2x = \frac{\lambda}{x^2}$ | $2y - \frac{\lambda}{y^2} = 0$ $2y = \frac{\lambda}{y^2}$ | $2z - \frac{\lambda}{z^2} = 0$ $2z = \frac{\lambda}{z^2}$ |

|                                                                                       |                                                                                       |                                                                                       |
|---------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------|
| $x^3 = \frac{\lambda}{2}$ $x = \left(\frac{\lambda}{2}\right)^{\frac{1}{3}} \dots(1)$ | $y^3 = \frac{\lambda}{2}$ $y = \left(\frac{\lambda}{2}\right)^{\frac{1}{3}} \dots(2)$ | $z^3 = \frac{\lambda}{2}$ $z = \left(\frac{\lambda}{2}\right)^{\frac{1}{3}} \dots(3)$ |
|---------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------|

From (1), (2) and (3), we get

$$x = y = z$$

Given:  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

$$\therefore 3 \frac{1}{x} = 1$$

$$\therefore 3 = x$$

$$\therefore \Rightarrow y = 3 \text{ and } z = 3$$

$\therefore (3, 3, 3)$  is the point where minimum values occur.

The minimum value is  $3^2 + 3^2 + 3^2 = 9 + 9 + 9 = 27$ .

**Example:**

**A rectangular box open at the top, is to have a volume of 32cc. find the dimensions of the box that requires the least material for its construction.**

**Solution:**

Let  $x, y, z$  be the length, breadth and height of the box.

$$\text{Surface area} = xy + 2yz + 2zx$$

$$\text{Volume} = xyz = 32$$

Let the auxiliary function  $F$  be

$$F(x, y, z, \lambda) = (xy + 2yz + 2zx) + \lambda(xyz - 32)$$

Where  $\lambda$  is Lagrange Multiplier

|                                                       |                                                       |                                                        |
|-------------------------------------------------------|-------------------------------------------------------|--------------------------------------------------------|
| $\frac{\partial F}{\partial x} = y + 2z + \lambda yz$ | $\frac{\partial F}{\partial y} = x + 2z + \lambda xz$ | $\frac{\partial F}{\partial z} = 2x + 2y + \lambda xy$ |
|-------------------------------------------------------|-------------------------------------------------------|--------------------------------------------------------|

When  $F$  is extremum

|                                                              |                                                              |                                                                |
|--------------------------------------------------------------|--------------------------------------------------------------|----------------------------------------------------------------|
| $\frac{\partial F}{\partial x} = 0$                          | $\frac{\partial F}{\partial y} = 0$                          | $\frac{\partial F}{\partial z} = 0$                            |
| $y + 2z + \lambda yz = 0$ $\Rightarrow y + 2z = -\lambda yz$ | $x + 2z + \lambda xz = 0$ $\Rightarrow x + 2z = -\lambda yz$ | $2x + 2y + \lambda xy = 0$ $\Rightarrow 2x + 2y = -\lambda xy$ |

|                                                             |                                                             |                                                             |
|-------------------------------------------------------------|-------------------------------------------------------------|-------------------------------------------------------------|
| $\Rightarrow \frac{1}{z} + \frac{z}{y} = -\lambda \dots(1)$ | $\Rightarrow \frac{1}{z} + \frac{z}{x} = -\lambda \dots(2)$ | $\Rightarrow \frac{z}{y} + \frac{z}{x} = -\lambda \dots(3)$ |
|-------------------------------------------------------------|-------------------------------------------------------------|-------------------------------------------------------------|

|                                                                                                                                      |                                                                                                                                       |
|--------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------|
| <p>From (1) and (2), we get</p> $\frac{1}{z} + \frac{z}{y} = \frac{1}{z} + \frac{z}{x}$ $\frac{z}{y} = \frac{z}{x}$ $x = y \dots(4)$ | <p>From (2) and (3), we get</p> $\frac{1}{z} + \frac{z}{x} = \frac{z}{y} + \frac{z}{x}$ $\frac{1}{z} = \frac{z}{y}$ $y = 2z \dots(5)$ |
|--------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------|

From (4) and (5), we get

$$x = y = 2z$$

$$\begin{aligned} \text{Volume} = xyz &= 32 \Rightarrow (2z)(2z)z = 32 \Rightarrow 4z^3 = 32 \\ z^3 &= \frac{32}{4} = 8 \Rightarrow z = 2 \quad \text{i.e. } x = 4, y = 4, z = 2 \end{aligned}$$

$\therefore$  Cost minimum when  $x = 4, y = 4, z = 2$

### Example:

**A rectangular box open at the top is to have a given capacity K. Find the dimensions of the box requiring least material for its construction.**

### Solution:

Let  $x, y, z$  be the dimensions of the box.

$$\text{Surface area} = xy + 2yz + 2zx$$

$$\text{Volume} = xyz = K$$

Let the auxiliary function F be

$$F(x, y, z, \lambda) = (xy + 2yz + 2zx) + \lambda(xyz - k)$$

Where  $\lambda$  is Lagrange Multiplier

|                                                       |                                                       |                                                        |
|-------------------------------------------------------|-------------------------------------------------------|--------------------------------------------------------|
| $\frac{\partial F}{\partial x} = y + 2z + \lambda yz$ | $\frac{\partial F}{\partial y} = x + 2z + \lambda zx$ | $\frac{\partial F}{\partial z} = 2x + 2y + \lambda xy$ |
|-------------------------------------------------------|-------------------------------------------------------|--------------------------------------------------------|

When F is extremum.

|                                     |                                     |                                     |
|-------------------------------------|-------------------------------------|-------------------------------------|
| $\frac{\partial F}{\partial x} = 0$ | $\frac{\partial F}{\partial y} = 0$ | $\frac{\partial F}{\partial z} = 0$ |
|-------------------------------------|-------------------------------------|-------------------------------------|

|                                                                                                                        |                                                                                                                                |                                                                                                                                  |
|------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------|
| $y + 2z + \lambda yz = 0$<br>$\Rightarrow y + 2z = -\lambda yz$<br>$\Rightarrow \frac{1}{z} + \frac{2}{y} = -\dots(1)$ | $x + 2z + \lambda zx = 0$<br>$\Rightarrow x + 2z = -\lambda zx$<br>$\Rightarrow \frac{1}{z} + \frac{2}{x} = -\lambda \dots(2)$ | $2x + 2y + \lambda xy = 0$<br>$\Rightarrow 2x + 2y = -\lambda xy$<br>$\Rightarrow \frac{2}{y} + \frac{2}{x} = -\lambda \dots(3)$ |
|------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------|

|                                                                                                                                           |                                                                                                                                            |
|-------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------|
| <p>From (1) and (2), we get</p> $\frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x}$ $\frac{2}{y} = \frac{2}{x}$ $x = y \dots\dots(5)$ | <p>From (2) and (3), we get</p> $\frac{1}{z} + \frac{2}{x} = \frac{2}{y} + \frac{2}{x}$ $\frac{1}{z} = \frac{2}{y}$ $y = 2z \dots\dots(6)$ |
|-------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------|

From (4) and (5), we get

$$x = y = 2z$$

$$\therefore \text{Volume} = xyz = k \Rightarrow (2z)(2z)z = k \Rightarrow 4z^3 = k$$

$$4z^3 = k \Rightarrow z^3 = \frac{k}{4}$$

$$z = \sqrt[3]{\frac{k}{4}} ; \quad x = 2 \sqrt[3]{\frac{k}{4}} \quad ; \quad y = 2 \sqrt[3]{\frac{k}{4}}$$

$$\therefore \text{Value of minimum} = xy + 2yz + 2zx$$

$$\begin{aligned}
&= 4 \left(\sqrt[3]{\frac{k}{4}}\right)^3 + 4 \left(\sqrt[3]{\frac{k}{4}}\right)^3 + 4 \left(\sqrt[3]{\frac{k}{4}}\right)^3 \\
&= 12 \left(\sqrt[3]{\frac{k}{4}}\right)^3 \\
&= 3(2k)^{2/3}
\end{aligned}$$

**Example:**

Find the point on the plane  $ax + by + cz = p$  at which  $f = x^2 + y^2 + z^2$  has a stationary value and find the stationary value of  $f$ , using Lagrange's method of multipliers

**Solution:**

Let the auxiliary function  $F$  be

$$F(x, y, z, \lambda) = x^2 + y^2 + z^2 + \lambda(ax + by + cz - p)$$

Where  $\lambda$  is Lagrange Multiplier

|                                                  |                                                  |                                                  |
|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|
| $\frac{\partial F}{\partial x} = 2x + \lambda a$ | $\frac{\partial F}{\partial y} = 2y + \lambda b$ | $\frac{\partial F}{\partial z} = 2z + \lambda c$ |
|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|

When  $F$  is extremum.

|                                                                                                                   |                                                                                                                   |                                                                                                                   |
|-------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------|
| $\frac{\partial F}{\partial x} = 0$                                                                               | $\frac{\partial F}{\partial y} = 0$                                                                               | $\frac{\partial F}{\partial z} = 0$                                                                               |
| $2x + \lambda a = 0$<br>$\Rightarrow 2x = -\lambda a$<br>$\Rightarrow \frac{x}{a} = \frac{-\lambda}{2} \dots (1)$ | $2y + \lambda b = 0$<br>$\Rightarrow 2y = -\lambda b$<br>$\Rightarrow \frac{y}{b} = \frac{-\lambda}{2} \dots (2)$ | $2z + \lambda c = 0$<br>$\Rightarrow 2z = -\lambda c$<br>$\Rightarrow \frac{z}{c} = \frac{-\lambda}{2} \dots (3)$ |

From (1), (2) & (3), we get

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$$

$$\frac{ax}{a^2} = \frac{by}{b^2} = \frac{cz}{c^2}$$

$$\Rightarrow \frac{ax}{a^2} = \frac{by}{b^2} = \frac{cz}{c^2} \Rightarrow \frac{ax + by + cz}{a^2 + b^2 + c^2} = \frac{p}{a^2 + b^2 + c^2}$$

$$x = \frac{ap}{a^2 + b^2 + c^2}; \quad y = \frac{bp}{a^2 + b^2 + c^2}; \quad z = \frac{cp}{a^2 + b^2 + c^2}$$

Stationary value of  $f = x^2 + y^2 + z^2$

$$= \left( \frac{ap}{a^2 + b^2 + c^2} \right)^2 + \left( \frac{bp}{a^2 + b^2 + c^2} \right)^2 + \left( \frac{cp}{a^2 + b^2 + c^2} \right)^2$$

$$= \frac{a^2 p^2 + b^2 p^2 + c^2 p^2}{a^2 + b^2 + c^2}$$

$$\begin{aligned}
 &= \frac{p^2}{(a^2 + b^2 + c^2)^2} \\
 &= \frac{(a^2 + b^2 + c^2)p^2}{(a^2 + b^2 + c^2)^2} = \frac{p^2}{a^2 + b^2 + c^2}
 \end{aligned}$$

**Example:**

**Find the dimensions of the rectangular box without top of maximum capacity with surface area 432 square metre**

**Solution:**

Let  $x, y, z$  be the length, breadth and height of the box.

$$\text{Surface area} = xy + 2yz + 2zx = 432$$

$$\text{Volume} = xyz$$

Let the auxiliary function  $F$  be

$$F(x, y, z, \lambda) = xyz + \lambda (xy + 2yz + 2zx - 432)$$

|                                                         |                                                         |                                                          |
|---------------------------------------------------------|---------------------------------------------------------|----------------------------------------------------------|
| $\frac{\partial F}{\partial x} = yz + \lambda (y + 2z)$ | $\frac{\partial F}{\partial y} = xz + \lambda (x + 2z)$ | $\frac{\partial F}{\partial z} = xy + \lambda (2y + 2x)$ |
|---------------------------------------------------------|---------------------------------------------------------|----------------------------------------------------------|

To find the stationary value.

| $F_x = 0$                                                                                                                                                                                             | $F_y = 0$                                                                                                                                                                                             | $F_z = 0$                                                                                                                                                                                                |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $yz + \lambda (y + 2z) = 0$<br>$\Rightarrow yz = -\lambda (y + 2z)$<br>$\Rightarrow \frac{y + 2z}{yz} = \frac{-1}{\lambda}$<br>$\Rightarrow \frac{1}{z} + \frac{2}{y} = -\frac{1}{\lambda} \dots (1)$ | $xz + \lambda (x + 2z) = 0$<br>$\Rightarrow xz = -\lambda (x + 2z)$<br>$\Rightarrow \frac{x + 2z}{xz} = \frac{-1}{\lambda}$<br>$\Rightarrow \frac{1}{z} + \frac{2}{x} = -\frac{1}{\lambda} \dots (2)$ | $xy + \lambda (2y + 2x) = 0$<br>$\Rightarrow xy = -\lambda (2y + 2x)$<br>$\Rightarrow \frac{2y + 2x}{xy} = \frac{-1}{\lambda}$<br>$\Rightarrow \frac{2}{x} + \frac{2}{y} = -\frac{1}{\lambda} \dots (3)$ |

|                                                                                                                                                                        |                                                                                                                                                                         |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p><i>From (1) &amp; (2), we get</i></p> $\frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x}$ $\Rightarrow \frac{2}{y} = \frac{2}{x}$ $\Rightarrow x = y \dots (4)$ | <p><i>From (2) &amp; (3), we get</i></p> $\frac{1}{z} + \frac{2}{x} = \frac{2}{y} + \frac{2}{x}$ $\Rightarrow \frac{1}{z} = \frac{2}{y}$ $\Rightarrow y = 2z \dots (5)$ |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

From (4) & (5), we get  $x = y = 2z$

Surface area =  $xy + 2yz + 2zx = 432$

$$(2z)(2z) + 2(2z)z + 2z(2z) = 432$$

$$4z^2 + 4z^2 + 4z^2 = 432$$

$$12z^2 = 432$$

$$z^2 = 36 \therefore z = 6$$

$$\therefore x = 12, y = 12, z = 6 \quad \text{by (6)}$$

Thus, the dimension of the box is 12, 12, 6. Maximum

volume =  $12 \times 12 \times 6 = 864$  cubic metres.

**Example:**

The temperature  $u(x, y, z)$  at any point in space is  $u = 400xyz^2$ . Find the highest temperature on surface of the sphere  $x^2 + y^2 + z^2 = 1$ .

**Solution:**

$$u = f = 400xyz^2 \quad \dots (A)$$

$$\phi = x^2 + y^2 + z^2 - 1 = 0 \quad \dots (B)$$

Let the auxiliary function F be

$$F(x, y, z, \lambda) = 400xyz^2 + \lambda(x^2 + y^2 + z^2 - 1)$$

To

|                                                         |                                                         |                                                        |
|---------------------------------------------------------|---------------------------------------------------------|--------------------------------------------------------|
| $\frac{\partial F}{\partial x} = 400yz^2 + \lambda(2x)$ | $\frac{\partial F}{\partial y} = 400xz^2 + \lambda(2y)$ | $\frac{\partial F}{\partial z} = 800xyz + \lambda(2z)$ |
|---------------------------------------------------------|---------------------------------------------------------|--------------------------------------------------------|

find the stationary value.

| $F_x = 0$                                      | $F_y = 0$                                      | $F_z = 0$                          |
|------------------------------------------------|------------------------------------------------|------------------------------------|
| $400yz^2 + \lambda(2x) = 0$                    | $400xz^2 + \lambda(2y) = 0$                    | $800xyz + \lambda(2z) = 0$         |
| $400yz^2 = -\lambda(2x)$                       | $400xz^2 = -\lambda(2y)$                       | $800xyz = -\lambda(2z)$            |
| $\frac{200yz^2}{x} = -\lambda \quad \dots (1)$ | $\frac{200xz^2}{y} = -\lambda \quad \dots (2)$ | $400xy = -\lambda \quad \dots (3)$ |

From (1) & (2), we get  $yz = x^2 \dots (4)$

From (2) & (3), we get  $z^2 = 2y^2 \dots (5)$

From (4) & (5), we get

$$x^2 = y^2 = \frac{1}{2} z^2 \quad \dots (6)$$

$$(B) \Rightarrow \frac{1}{2} z^2 + \frac{1}{2} z^2 + z^2 - 1 = 0$$

$$\Rightarrow 2z^2 = 1$$

$$\Rightarrow z^2 = \frac{1}{2} \Rightarrow z = \pm \frac{1}{\sqrt{2}}$$

$$(6) \Rightarrow x^2 = \frac{1}{2} \left( \frac{1}{2} \right) = \frac{1}{4} \Rightarrow x = \pm \frac{1}{2}$$

$$(6) \Rightarrow y^2 = \frac{1}{2} \left( \frac{1}{2} \right) = \frac{1}{4} \Rightarrow y = \pm \frac{1}{2}$$

$\therefore u = 400xyz^2$ , we select  $x, y, z$  to be positive

$$\Rightarrow u = 400 \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)$$

$$\Rightarrow u = 50$$

∴ Maximum temperature is 50

**Example:**

**Find the maximum volume of the largest rectangular parallelepiped that can be inscribed in an ellipsoid  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$**

**Solution:**

Let a vertex of such parallelepiped be (x, y, z)

Then all the vertices will be ( ±x , ±y, ±z )

Then, the sides of the solid be 2x, 2y, 2z (lengths)

Hence, the volume  $V = (2x)(2y)(2z) = 8xyz$

Let  $f = 8xyz$

$$\phi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

Let the auxiliary function F be

$$F(x, y, z, \lambda) = 8xyz + \lambda \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right)$$

|                                                                |                                                                |                                                                |
|----------------------------------------------------------------|----------------------------------------------------------------|----------------------------------------------------------------|
| $\frac{\partial F}{\partial x} = 8yz + \lambda \frac{2x}{a^2}$ | $\frac{\partial F}{\partial y} = 8xz + \lambda \frac{2y}{b^2}$ | $\frac{\partial F}{\partial z} = 8xy + \lambda \frac{2z}{c^2}$ |
|----------------------------------------------------------------|----------------------------------------------------------------|----------------------------------------------------------------|

To find the stationary value.

| $F_x = 0$                                                                                                                                                                       | $F_y = 0$                                                                                                                                                                       | $F_z = 0$                                                                                                                                                                       |
|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $8yz + \lambda \frac{2x}{a^2} = 0$<br><br>$\Rightarrow 8yz = -\lambda \frac{2x}{a^2}$<br><br>$X^{ly} \frac{x}{2} \Rightarrow \frac{4xyz}{-\lambda} = \frac{x^2}{a^2} \dots (1)$ | $8xz + \lambda \frac{2y}{b^2} = 0$<br><br>$\Rightarrow 8xz = -\lambda \frac{2y}{b^2}$<br><br>$X^{ly} \frac{y}{2} \Rightarrow \frac{4xyz}{-\lambda} = \frac{y^2}{b^2} \dots (2)$ | $8xy + \lambda \frac{2z}{c^2} = 0$<br><br>$\Rightarrow 8xy = -\lambda \frac{2z}{c^2}$<br><br>$X^{ly} \frac{z}{2} \Rightarrow \frac{4xyz}{-\lambda} = \frac{z^2}{c^2} \dots (3)$ |

From (1), (2) & (3), we get

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} \dots (4)$$

Given  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\Rightarrow \frac{3x^2}{a^2} = 1 \text{ by (4)}$$

$$\Rightarrow x^2 = \frac{a^2}{3} \Rightarrow x = \frac{a}{\sqrt{3}}$$

$$\text{Similarly, } y = \frac{b}{\sqrt{3}}; \quad z = \frac{c}{\sqrt{3}}$$

The extremum point is  $(\frac{a}{\sqrt[3]{3}}, \frac{b}{\sqrt[3]{3}}, \frac{c}{\sqrt[3]{3}})$

Maximum volume  $V = 8(\frac{abc}{3\sqrt[3]{3}})$

**Example:**

**Find the minimum values of  $x^2yz^3$  subject to the condition  $2x + y + 3z = a$ .**

**Solution:**

Let  $f = x^2yz^3$

$\phi = 2x + y + 3z - a = 0$

Let the auxiliary function F be

$$F(x, y, z, \lambda) = x^2yz^3 + \lambda(2x + y + 3z - a)$$

|                                                      |                                                    |                                                       |
|------------------------------------------------------|----------------------------------------------------|-------------------------------------------------------|
| $\frac{\partial F}{\partial x} = 2xyz^3 + \lambda 2$ | $\frac{\partial F}{\partial y} = x^2z^3 + \lambda$ | $\frac{\partial F}{\partial z} = 3x^2yz^2 + 3\lambda$ |
|------------------------------------------------------|----------------------------------------------------|-------------------------------------------------------|

To find the stationary value.

| $F_x = 0$                    | $F_y = 0$                     | $F_z = 0$                      |
|------------------------------|-------------------------------|--------------------------------|
| $2xyz^3 + \lambda 2 = 0$     | $x^2z^3 + \lambda = 0$        | $3x^2yz^2 + 3\lambda = 0$      |
| $xyz^3 = -\lambda \dots (1)$ | $x^2z^3 = -\lambda \dots (2)$ | $x^2yz^2 = -\lambda \dots (3)$ |

|                                                                      |                                                                        |
|----------------------------------------------------------------------|------------------------------------------------------------------------|
| <p>From (1) &amp; (2), we get</p> $xyz^3 = x^2z^3$ $x = y \dots (4)$ | <p>From (2) &amp; (3), we get</p> $x^2z^3 = x^2yz^2$ $z = y \dots (5)$ |
|----------------------------------------------------------------------|------------------------------------------------------------------------|

From (4) & (5), we get

$$x = y = z \dots (6)$$

Given:  $2x + y + 3z = a$

$$\Rightarrow 2z + z + 3z = a$$

$$\Rightarrow 6z = a$$

$$\Rightarrow z = \frac{a}{6} \quad (6) \Rightarrow x = y = z = \frac{a}{6}$$

$\therefore$  The stationary point is  $(\frac{a}{6}, \frac{a}{6}, \frac{a}{6})$

Hence, Minimum value of  $f = x^2yz^3$

$$= \binom{a}{6}^2 \binom{a}{6} \binom{a}{6}^3 = \binom{a}{6}^6$$

**Example:**

**Find the maximum value of  $x^m y^n z^p$ , when  $x + y + z = a$ . Solution:**

Let  $f = x^m y^n z^p$

$$\phi = x + y + z - a = 0$$

Let the auxiliary function F be

$$F(x, y, z, \lambda) = x^m y^n z^p + \lambda (x + y + z - a)$$

|                                                             |                                                              |                                                              |
|-------------------------------------------------------------|--------------------------------------------------------------|--------------------------------------------------------------|
| $\frac{\partial F}{\partial x} = mx^{m-1}y^n z^p + \lambda$ | $\frac{\partial F}{\partial y} = nx^m y^{n-1} z^p + \lambda$ | $\frac{\partial F}{\partial z} = px^m y^n z^{p-1} + \lambda$ |
|-------------------------------------------------------------|--------------------------------------------------------------|--------------------------------------------------------------|

To find the stationary value.

| $F_x = 0$                                                                                                                    | $F_y = 0$                                                                                                                      | $F_z = 0$                                                                                                                      |
|------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------|
| $mx^{m-1}y^n z^p + \lambda = 0$<br>$\Rightarrow mx^{m-1}y^n z^p = -\lambda$<br>$\frac{mx^m y^n z^p}{x} = -\lambda \dots (1)$ | $nx^m y^{n-1} z^p + \lambda = 0$<br>$\Rightarrow nx^m y^{n-1} z^p = -\lambda$<br>$\frac{nx^m y^n z^p}{y} = -\lambda \dots (2)$ | $px^m y^n z^{p-1} + \lambda = 0$<br>$\Rightarrow px^m y^n z^{p-1} = -\lambda$<br>$\frac{px^m y^n z^p}{z} = -\lambda \dots (3)$ |

From (1), (2) & (3), we get

$$\frac{mx^m y^n z^p}{x} = \frac{nx^m y^n z^p}{y} = \frac{px^m y^n z^p}{z}$$

$$\div x^m y^n z^p \Rightarrow \frac{m}{x} = \frac{n}{y} = \frac{p}{z} \dots (4)$$

$$x = \frac{m}{p} z \dots (5) \quad y = \frac{n}{p} z \dots (6)$$

Given:  $x + y + z = a$

$$\Rightarrow \frac{m}{p} z + \frac{n}{p} z + z = a$$

$$\Rightarrow \left[ \frac{m}{p} + \frac{n}{p} + 1 \right] z = a$$

$$\Rightarrow \left[ \frac{m+n+p}{p} \right] z = a$$

$$(5) \Rightarrow x = \frac{m}{p} \frac{ap}{m+n+p}$$

$$= \frac{am}{m+n+p}$$

$$(5) \Rightarrow y = \frac{n}{p} \frac{ap}{m+n+p}$$

\_\_\_\_\_

p m+n+p

$$\Rightarrow z = \frac{ap}{m+n+p}$$

$$= \frac{an}{m+n+p}$$

∴ The stationary point is  $\left( \frac{am}{m+n+p}, \frac{an}{m+n+p}, \frac{ap}{m+n+p} \right)$

∴ The Maximum value of  $f = \left( \frac{am}{m+n+p} \right)^m \left( \frac{an}{m+n+p} \right)^n \left( \frac{ap}{m+n+p} \right)^p$

$$\text{Max. Value of } f = \frac{a^{m+n+p} m^m n^n p^p}{(m+n+p)^{m+n+p}}$$

## NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

### Introduction:

Partial differential equations arise in the study of many branches of applied mathematics, e.g., in fluid dynamics, heat transfer, boundary layer flow, elasticity, quantum mechanics, and electromagnetic theory. Only a few of these equations can be solved by analytical methods which are also complicated by requiring use of advanced mathematical techniques. In most of the cases, it is easier to develop approximate solutions by numerical methods. Of all the numerical methods available for the solution of partial differential equations, the method of finite differences is most commonly used. In this method, the derivatives appearing in the equation and the boundary conditions are replaced by their finite difference approximations. Then the given equation is changed to a system of linear equations which are solved by iterative procedures. This process is slow but produces good results in many boundary value problems. An added advantage of this method is that the computation can be carried by electronic computers. To accelerate the solution, sometimes the method of relaxation proves quite effective.

### Classification of Second Order Equations

The general linear partial differential equation of the second order in two independent variables is of the form

$$A(X,Y) \frac{\partial^2 u}{\partial x^2} + B(X,Y) \frac{\partial^2 u}{\partial x \partial y} + C(X,Y) \frac{\partial^2 u}{\partial y^2} + (X,Y,U, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0$$

Such a partial differential equation is said to be

- (i) **elliptic** if  $B^2 - 4AC < 0$ , (ii) **parabolic** if  $B^2 - 4AC = 0$ , and  
(iii) **hyperbolic** if  $B^2 - 4AC > 0$

Now let us examine the nature of three PDEs which are under our discussion.

| Sl.no | P.D.E                                                       | A     | B | C  | $B^2 - 4AC$ | Nature of the PDE |
|-------|-------------------------------------------------------------|-------|---|----|-------------|-------------------|
| 1     | One dimensional wave equation:<br>$c^2 u_{xx} - u_{tt} = 0$ | $c^2$ | 0 | -1 | $4c^2 > 0$  | Hyperbolic        |
| 2     | One dimensional heat equation:                              | $c^2$ | 0 | 0  | 0           | Parabolic         |

|   |                                                                 |   |   |   |          |          |
|---|-----------------------------------------------------------------|---|---|---|----------|----------|
|   | $c^2 u_{xx} - u_t = 0$                                          |   |   |   |          |          |
| 3 | Two dimensional<br>Laplace's equation:<br>$u_{xx} + u_{yy} = 0$ | 1 | 0 | 1 | $-4 < 0$ | Elliptic |

## Finite difference approximation to ordinary and partial derivatives

Let  $y = y(x)$  and its derivatives be single valued continuous functions of  $x$ . we have Taylors expansion,

$$y(x + h) = y(x) + \frac{h}{1!} y'(x) + \frac{(h)^2}{2!} y''(x) + \frac{(h)^3}{3!} y'''(x) + \dots \quad (1)$$

$$y(x - h) = y(x) - \frac{h}{1!} y'(x) + \frac{(h)^2}{2!} y''(x) - \frac{(h)^3}{3!} y'''(x) + \dots \quad (2)$$

Assuming  $h$  to be small we shall neglect terms containing  $h^2, h^3, \dots$  in (1) and (2) so that we have

$$y'(x) = \frac{y(x+h) - y(x)}{h} \quad \text{from (1)}$$

$$y'(x) = \frac{y(x) - y(x-h)}{h} \quad \text{from (2)}$$

$$\text{Also we have by (1) + (2) we have } y'(x) = \frac{y(x+h) - y(x-h)}{2h}$$

These three expressions for  $y'(x)$  are the finite difference approximation in terms of forward, backward and central difference respectively.

Also (1) + (2) by neglecting terms containing  $h^3, h^4, \dots$  will give us,

$$y(x + h) - y(x - h) = 2y(x) + h^2 y''(x)$$

$$\text{Or } y''(x) = \frac{y(x+h) - 2y(x) + y(x-h)}{h^2}$$

This is the finite difference approximation for  $y''(x)$

## Extension to the Partial derivatives

Let  $u = u(x, y)$  be a function of two independent variables. The finite difference approximation for the first order partial derivatives:  $u_x, u_y$  and second order partial derivatives:  $u_{xx}, u_{yy}$  are as follows.

$$u_x = \frac{\partial u}{\partial x} = \frac{u(x+h, y) - u(x, y)}{h} \quad \dots\dots (1)$$

$$u_x = \frac{\partial u}{\partial x} = \frac{u(x, y) - u(x-h, y)}{h} \quad \dots\dots (2)$$

$$u_x = \frac{\partial u}{\partial x} = \frac{u(x+h, y) - u(x-h, y)}{2h} \quad \dots\dots (3)$$

$$u_y = \frac{\partial u}{\partial y} = \frac{u(x,y+k)-u(x,y)}{k} \quad \dots\dots (4)$$

$$u_y = \frac{\partial u}{\partial y} = \frac{u(x,y)-u(x,y-k)}{k} \quad \dots\dots (5)$$

$$u_y = \frac{\partial u}{\partial y} = \frac{u(x,y+k)-u(x,y-k)}{2k} \quad \dots\dots (6)$$

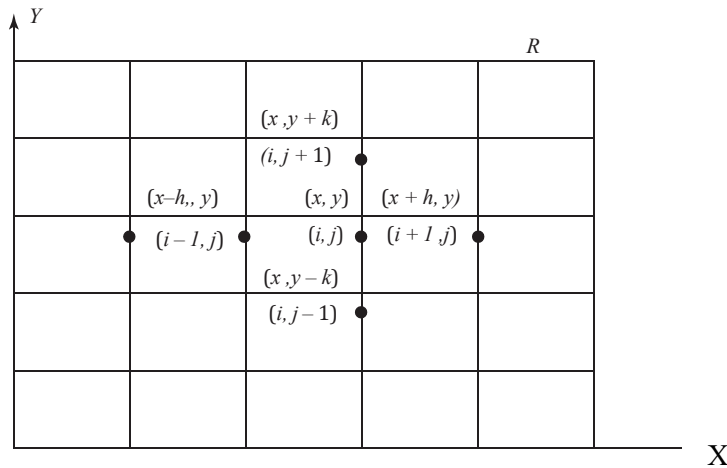
$$u_{xx} = \partial^2 u / \partial x^2 = \frac{u(x+h,y)-2u(x,y)+u(x-h,y)}{h^2} \quad \dots\dots (7)$$

$$u_{yy} = \partial^2 u / \partial y^2 = \frac{u(x,y+k)-2u(x,y)+u(x,y-k)}{k^2} \quad \dots\dots (8)$$

## NUMERICAL SOLUTION OF A PDE

Consider a rectangular region  $R$  in the  $x, y$  plane. Let us divide this region into a network of rectangular network of sides  $h$  and  $k$  as shown in Figure. In other words, we draw lines  $x=ih$ ,  $y=jk$ ;  $i, j = 1, 2, 3, \dots$  being parallel to the  $Y$ -axis and  $X$ -axis respectively resulting into a network of rectangles.

The points of intersection of the dividing lines are called *mesh points, nodal points, or grid points*



We write  $u(x, y) = u(ih, jk)$  and finite difference approximation for the partial derivatives given by (1) to (8) are put in the following modified form.

$$u_x = \frac{\partial u}{\partial x} = \frac{1}{h} [u_{i+1,j} - u_{i,j}]$$

$$u_x = \frac{\partial u}{\partial x} = \frac{1}{h} [u_{i,j} - u_{i-1,j}]$$

$$u_x = \frac{\partial u}{\partial x} = \frac{1}{2h} [u_{i+1,j} - u_{i-1,j}]$$

$$u_y = \frac{\partial u}{\partial y} = \frac{1}{k} [u_{i,j+1} - u_{i,j}]$$

$$u_y = \frac{\partial u}{\partial y} = \frac{1}{k} [u_{i,j} - u_{i,j-1}]$$

$$u_y = \frac{\partial u}{\partial y} = \frac{1}{2k} [u_{i,j+1} - u_{i,j-1}]$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{1}{h^2} [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]$$

$$u_{yy} = \frac{\partial^2 u}{\partial y^2} = \frac{1}{k^2} [u_{i,j+1} - 2u_{i,j} + u_{i,j-1}]$$

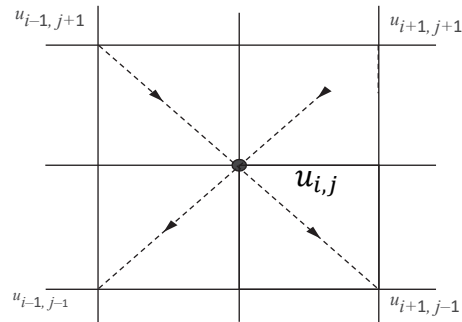
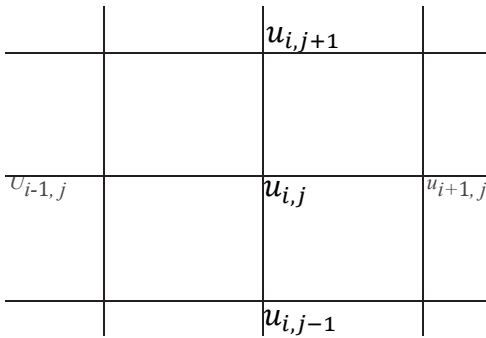
The substitution of these finite difference approximations into the given PDE converts the PDE into finite difference equations. we can solve this equation under given set of conditions.

This will enable us to determine  $u_{i,j}$  explicitly at all the interior mesh points. These values constitute a numerical solution to the given PDE.

## NUMERICAL SOLUTION OF THE LAPLACE EQUATION IN TWO DIMENSIONS

Laplace equation in two dimensions is  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$  ..... (I)

We consider a rectangular region R for which  $u(x, y)$  is known at the boundary. Let us suppose that the region is such that it can be divided into network of square mesh of side h.



Now we will substitute the finite difference approximation for the partial derivatives present in equation (I)

Hence we have

$$\frac{1}{h^2} [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] + \frac{1}{k^2} [u_{i,j+1} - 2u_{i,j} + u_{i,j-1}] = 0$$

That is ,  $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} = 4u_{i,j}$

$$\text{Or } u_{i,j} = \frac{1}{4} [u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}] \quad \text{..... (2)}$$

This is called the **Standard five-point formula**

Since the Laplace equation remains invariant when the coordinates are rotated through an angle  $45^\circ$  we can also have the formula in the form

$$u_{i,j} = \frac{1}{4} [u_{i+1,j-1} + u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j-1}] \quad \text{..... (3)}$$

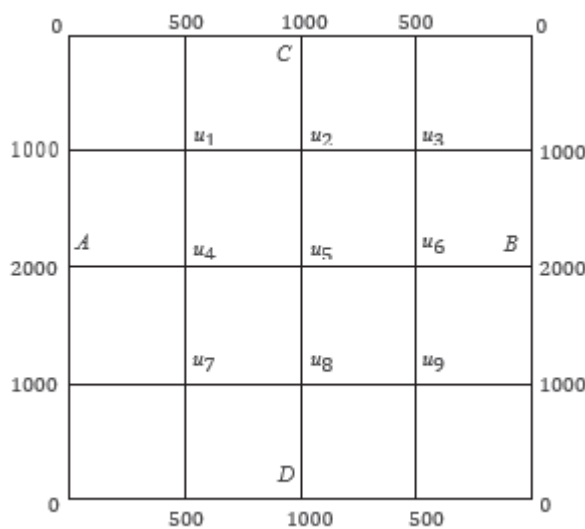
It is observed that the value of  $u_{i,j}$  is the average of its values at the four neighbouring diagonal mesh points. It is also called **diagonal five-point formula**.

The accuracy of the values of  $u_{i,j}$  can be improved by successive application of the formula (2). We regard the obtained values of  $u_{i,j}$  as the initial approximation denoted by  $u^{(0)}_{i,j}$ , then the first iteration is carried out by using the latest iterative values available on hand where we have,

$$u^{(1)}_{i,j} = \frac{1}{4} \left[ u^{(0)}_{i+1,j} + u^{(0)}_{i-1,j} + u^{(0)}_{i,j+1} + u^{(0)}_{i,j-1} \right]$$

The procedure will be repeated till we get the values to the desired degree of accuracy, this process is called **liebmann's iteration process**

**Example :1** Solve Laplace's equation  $u_{xx} + u_{yy} = 0$  for the following square mesh with boundary values as shown in the fig.



Here  $u_5$  is located at the center of the region and hence by the standard five point formula,

$$u_5 = \frac{1}{4} [2000 + 2000 + 1000 + 1000] = 1500$$

Next we shall compute  $u_1, u_3, u_7, u_9$  by the diagonal five point formula

$$u_1 = \frac{1}{4} [0 + 1500 + 1000 + 2000] = 1125$$

$$u_3 = \frac{1}{4} [1000 + 2000 + 1500 + 0] = 1125$$

Also  $u_7 = 1125 = u_9$

Further we compute  $u_2, u_4, u_6, u_8$  by standard five point formula

$$u_2 = \frac{1}{4} [u_1 + u_3 + u_5 + 1000] = \frac{1}{4} [1125 + 1125 + 1500 + 1000] = 1187.5$$

$$u_4 = \frac{1}{4} [2000 + u_5 + u_1 + u_7] = \frac{1}{4} [2000 + 1500 + 1125 + 1125] = 1437.5$$

$$u_6 = \frac{1}{4} [2000 + u_9 + u_3 + u_5] = \frac{1}{4} [2000 + 1125 + 1125 + 1500] = 1437.5$$

$$u_8 = \frac{1}{4} [u_9 + u_7 + u_5 + 1000] = \frac{1}{4} [1125 + 1125 + 1500 + 1000] = 1187.5$$

These are considered as initial approximation values for Liebmanns iteration. We compute  $u_i$  ( $i=1$  to 9) in the serial order by using the latest iterative values on hand by applying S. F.

#### First iteration

$$u_1^{(1)} = \frac{1}{4} [1000 + u_2 + 500 + u_4] = \frac{1}{4} [1000 + 1187.5 + 500 + 1437.5] = 1031.25$$

$$u_2^{(1)} = \frac{1}{4} [1000 + u_5 + u_1^{(1)} + u_3] = \frac{1}{4} [1000 + 1500 + 1031.25 + 1125] = 1164.0625$$

$$u_3^{(1)} = \frac{1}{4} [1000 + u_2^{(1)} + 500 + u_6] = \frac{1}{4} [1000 + 1164.0625 + 500 + 1437.5] = 1025.3906$$

$$u_4^{(1)} = \frac{1}{4} [u_1^{(1)} + u_7 + 2000 + u_5] = \frac{1}{4} [1031.25 + 1125 + 2000 + 1500] = 1414.0625$$

$$u_5^{(1)} = \frac{1}{4} [u_4^{(1)} + u_6 + u_2^{(1)} + u_8] = \frac{1}{4} [1414.0625 + 1437.5 + 1164.0625 + 1187.5] \\ = 1300.7813$$

$$u_6^{(1)} = \frac{1}{4} [u_5^{(1)} + 2000 + u_3^{(1)} + u_9] = \frac{1}{4} [1300.7813 + 2000 + 1025.3906 + 1125] \\ = 1362.7930$$

$$u_7^{(1)} = \frac{1}{4} [u_4^{(1)} + 500 + 1000 + u_8] = \frac{1}{4} [1414.0625 + 500 + 1000 + 1187.5] \\ = 1025.3906$$

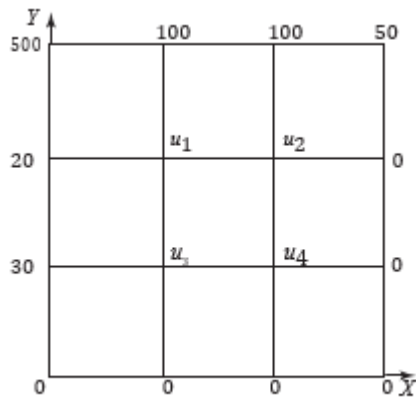
$$u_8^{(1)} = \frac{1}{4} [u_5^{(1)} + 1000 + u_7^{(1)} + u_9] = \frac{1}{4} [1300.7813 + 1000 + 1025.3906 + 1125] \\ = 1112.7930$$

$$u_9^{(1)} = \frac{1}{4} [u_8^{(1)} + 1000 + 500 + u_6^{(1)}] = \frac{1}{4} [1112.7930 + 1000 + 500 + 1362.7930] \\ = 993.8965$$

Thus the required first iterative values to the nearest integer are as follows

$$u_1 = 1031 \quad u_2 = 1164 \quad u_3 = 1025 \quad u_4 = 1414 \quad u_5 = 1301 \quad u_6 = 1363 \\ u_7 = 1025 \quad u_8 = 1113 \quad u_9 = 994$$

**Example :2** Solve Laplace's equation  $u_{xx} + u_{yy} = 0$  for the following square mesh with boundary values as shown in the fig



$$u_1 = \frac{1}{4} [20 + u_2 + 100 + u_3] ; \quad u_2 = \frac{1}{4} [u_1 + 0 + 100 + u_4]$$

$$u_3 = \frac{1}{4} [30 + u_4 + u_1 + 0] \quad ; \quad u_4 = \frac{1}{4} [u_3 + 0 + u_2 + 0]$$

Now we have a system of equations to be solved

$$4u_1 - u_2 - u_3 = 120 \quad \dots\dots\dots (1)$$

$$-u_1 + 4u_2 - u_4 = 100 \quad \dots\dots\dots (2)$$

$$-u_1 + 4u_3 - u_4 = 30 \quad \dots\dots\dots (3)$$

$$-u_2 - u_3 + 4u_4 = 0 \quad \dots\dots\dots (4)$$

Let us eliminate  $u_1$  from (1) and (2); (2) and (3)

$$15u_2 - u_3 - 4u_4 = 520 \quad \dots\dots\dots (5)$$

$$4u_2 - 4u_3 = 70 \quad \dots\dots\dots (6)$$

Now we will eliminate  $u_4$  from (4) and (5)

$$14u_2 - 2u_3 = 520 \quad \dots\dots\dots(7)$$

Now solve (6) and (7)

$$2u_2 - 2u_3 = 35$$

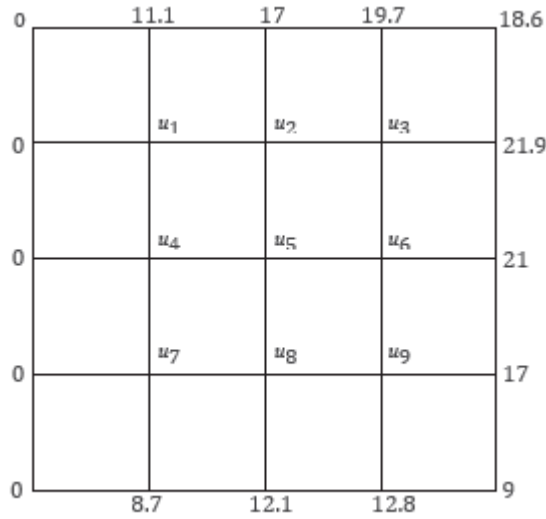
$$14u_2 - 2u_3 = 520$$

$$\therefore u_2 = 40.4 , u_3 = 22.9$$

Hence from (1)  $u_1 = 45.825$  and from (4)  $u_4 = 15.825$

Thus these are the required values at the interior mesh points.

**Example :3** Solve Laplace's equation  $u_{xx} + u_{yy} = 0$  for the following square mesh with boundary values as shown in the fig.



Here  $u_5$  is located at the center of the region and hence by the standard five point formula,

$$u_5 = \frac{1}{4} [0 + 21 + 17 + 12.1] = 12.525$$

Next we shall compute  $u_1, u_3, u_7, u_9$  by the diagonal five point formula

$$u_7 = \frac{1}{4} [0 + 12.525 + 0 + 12.1] = 6.15625$$

$$u_9 = \frac{1}{4} [12.1 + 21 + 12.525 + 9] = 13.65625$$

$$u_1 = \frac{1}{4} [0 + 17 + 0 + 12.525] = 7.38125$$

$$u_3 = \frac{1}{4} [12.525 + 18.6 + 17 + 21] = 17.28125$$

Further we compute  $u_2, u_4, u_6, u_8$  by standard five point formula

$$u_2 = \frac{1}{4} [u_1 + u_3 + u_5 + 17] = \frac{1}{4} [7.38125 + 17.28125 + 12.525 + 17] = 13.546875$$

$$u_4 = \frac{1}{4} [0 + u_5 + u_1 + u_7] = \frac{1}{4} [0 + 12.525 + 7.38125 + 6.15625] = 6.515625$$

$$u_6 = \frac{1}{4} [21 + u_9 + u_3 + u_5] = \frac{1}{4} [12.525 + 21 + 17.28125 + 13.65625] = 16.115625$$

$$u_8 = \frac{1}{4} [u_9 + u_7 + u_5 + 12.1] = \frac{1}{4} [6.15625 + 13.65625 + 12.525 + 12.1] = 11.109375$$

Thus the required mesh points corrected to two decimal points are

$$u_1 = 7.38 \quad u_2 = 13.55 \quad u_3 = 17.28 \quad u_4 = 6.52 \quad u_5 = 12.53 \quad u_6 = 16.12$$

$$u_7 = 6.16 \quad u_8 = 11.11 \quad u_9 = 13.66$$

**Example :4** Solve  $\nabla^2 u = 0$  in the square region bounded by the coordinate axes and the lines  $x = 4, y = 4$  with the boundary conditions given by the analytical expressions,

- (i)  $u(0, y) = 0$  for  $0 \leq y \leq 4$
- (ii)  $u(4, y) = 12 + y$  for  $0 \leq y \leq 4$

- (iii)  $u(x, 0) = 3x$  for  $0 \leq x \leq 4$   
 (iv)  $u(x, 4) = x^2$  for  $0 \leq x \leq 4$

Also employ Liebmann's iteration process to compute the second iterative of  $u(x, y)$  correct to two decimal places.

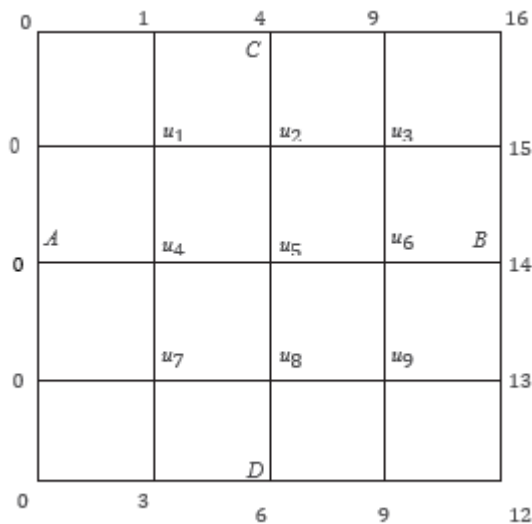
We have  $\nabla^2 u = 0$  represented by  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$  in two dimensions.

We shall divide the square region in to 16 squares of side one unit.

We shall derive the values of  $u(x, y)$  on the boundary from the given expressions.

- (i)  $u(0, y) = 0$  for  $0 \leq y \leq 4$   
 $\Rightarrow u(0, 1) = u(0, 2) = u(0, 3) = u(0, 4) = 0$
- (ii)  $u(4, y) = 12 + y$  for  $0 \leq y \leq 4$   
 $\Rightarrow u(4, 1) = 13, u(4, 2) = 14, u(4, 3) = 15, u(4, 4) = 16$
- (iii)  $u(x, 0) = 3x$  for  $0 \leq x \leq 4$   
 $\Rightarrow u(1, 0) = 3, u(2, 0) = 6, u(3, 0) = 9, u(4, 0) = 12$
- (iv)  $u(x, 4) = x^2$  for  $0 \leq x \leq 4$   
 $\Rightarrow u(1, 4) = 1, u(2, 4) = 4, u(3, 4) = 9, u(4, 4) = 16$

We shall represent these values on the square region and let  $u_1, u_2, u_3, u_4, \dots$  be the interior mesh points of the region



Here  $u_5$  is located at the center of the region and hence by the standard five point formula,

$$u_5 = \frac{1}{4} [0 + 14 + 4 + 6] = 6$$

Next we shall compute  $u_1, u_3, u_7, u_9$  by the diagonal five point formula

$$u_7 = \frac{1}{4} [0 + 6 + 0 + 6] = 3$$

$$u_9 = \frac{1}{4} [6 + 14 + 6 + 12] = 9.5$$

$$u_1 = \frac{1}{4} [0 + 4 + 0 + 6] = 2.5$$

$$u_3 = \frac{1}{4} [6 + 16 + 4 + 14] = 10$$

Further we compute  $u_2, u_4, u_6, u_8$  by standard five point formula

$$u_2 = \frac{1}{4} [u_1 + u_3 + u_5 + 4] = \frac{1}{4} [2.5 + 10 + 4 + 6] = 5.625$$

$$u_4 = \frac{1}{4} [0 + u_5 + u_1 + u_7] = \frac{1}{4} [0 + 6 + 2.5 + 3] = 2.875$$

$$u_6 = \frac{1}{4} [14 + u_9 + u_3 + u_5] = \frac{1}{4} [6 + 14 + 10 + 9.5] = 9.875$$

$$u_8 = \frac{1}{4} [u_9 + u_7 + u_5 + 6] = \frac{1}{4} [3 + 9.5 + 6 + 6] = 6.125$$

### FIRST ITERATION

$$u_1^{(1)} = \frac{1}{4} [0 + u_2 + 1 + u_4] = \frac{1}{4} [0 + 5.625 + 1 + 2.875] = 2.375$$

$$u_2^{(1)} = \frac{1}{4} [4 + u_5 + u_1^{(1)} + u_3] = \frac{1}{4} [2.375 + 10 + 4 + 6] = 5.59375$$

$$u_3^{(1)} = \frac{1}{4} [9 + u_2^{(1)} + 15 + u_6] = \frac{1}{4} [5.59375 + 15 + 9 + 9.875] = 9.8671875$$

$$u_4^{(1)} = \frac{1}{4} [u_1^{(1)} + u_7 + 0 + u_5] = \frac{1}{4} [0 + 6 + 2.375 + 3] = 2.84375$$

$$u_5^{(1)} = \frac{1}{4} [u_4^{(1)} + u_6 + u_2^{(1)} + u_8] = \frac{1}{4} [2.84375 + 9.875 + 5.59375 + 6.125] = 6.109375$$

$$u_6^{(1)} = \frac{1}{4} [u_5^{(1)} + 14 + u_3^{(1)} + u_9] = \frac{1}{4} [6.109375 + 14 + 9.8671875 + 9.5] = 9.8691406$$

$$u_7^{(1)} = \frac{1}{4} [u_4^{(1)} + 3 + 0 + u_8] = \frac{1}{4} [0 + 6.125 + 2.84375 + 3] = 2.9921875$$

$$u_8^{(1)} = \frac{1}{4} [u_5^{(1)} + 6 + u_7^{(1)} + u_9] = \frac{1}{4} [2.9921875 + 9.5 + 6.109375 + 6] = 6.1503906$$

$$u_9^{(1)} = \frac{1}{4} [u_8^{(1)} + 13 + 9 + u_6^{(1)}] = \frac{1}{4} [6.1503906 + 13 + 9.8691406 + 9] = 9.5048828$$

Thus the first iterative values corrected to three decimal points are

$$u_1 = 2.375 \quad u_2 = 5.594 \quad u_3 = 9.867 \quad u_4 = 2.844 \quad u_5 = 6.109 \quad u_6 = 9.869$$

$$u_7 = 2.992 \quad u_8 = 6.150 \quad u_9 = 9.505$$

### SECOND ITERATION

$$u_1^{(2)} = \frac{1}{4} [0 + u_2 + 1 + u_4] = \frac{1}{4} [0 + 5.594 + 1 + 2.844] = 2.36$$

$$u_2^{(2)} = \frac{1}{4} [4 + u_5 + u_1^{(2)} + u_3] = \frac{1}{4} [2.36 + 9.867 + 4 + 6.109] = 5.584$$

$$u_3^{(2)} = \frac{1}{4} [9 + u_2^{(2)} + 15 + u_6] = \frac{1}{4} [5.584 + 15 + 9 + 9.869] = 9.863$$

$$u_4^{(2)} = \frac{1}{4} [u_1^{(2)} + u_7 + 0 + u_5] = \frac{1}{4} [0 + 6.109 + 2.36 + 2.992] = 2.865$$

$$u_5^{(2)} = \frac{1}{4} [u_4^{(2)} + u_6 + u_2^{(2)} + u_8] = \frac{1}{4} [2.865 + 9.869 + 5.584 + 6.150] = 6.117$$

$$u_6^{(2)} = \frac{1}{4} [u_5^{(2)} + 14 + u_3^{(2)} + u_9] = \frac{1}{4} [6.117 + 14 + 9.863 + 9.505] = 9.871$$

$$u_7^{(2)} = \frac{1}{4} [u_4^{(2)} + 3 + 0 + u_8] = \frac{1}{4} [0 + 6.150 + 2.865 + 3] = 3.004$$

$$u_8^{(2)} = \frac{1}{4} [u_5^{(2)} + 6 + u_7^{(2)} + u_9] = \frac{1}{4} [3.004 + 9.505 + 6.117 + 6] = 6.156$$

$$u_9^{(2)} = \frac{1}{4} [u_8^{(2)} + 13 + 9 + u_6^{(2)}] = \frac{1}{4} [6.156 + 13 + 9.871 + 9] = 9.507$$

Thus the required second iterative values corrected to three decimal points are

$$u_1 = 2.36 \quad u_2 = 5.58 \quad u_3 = 9.86 \quad u_4 = 2.87 \quad u_5 = 6.12 \quad u_6 = 9.87$$

$$u_7 = 3 \quad u_8 = 6.16 \quad u_9 = 9.51$$

### Numerical Solution of the one dimensional heat equation

1. Find the numerical solution of the parabolic equation  $\frac{\partial^2 u}{\partial x^2} = 2 \frac{\partial u}{\partial t}$  when  $u(0, t) = 0 = u(4, t)$  and  $u(x, 0) = x(4 - x)$  by taking  $h = 1$ . Find the values upto  $t = 5$ .

The standard form of the one dimensional heat equation is  $u_t = c^2 u_{xx}$  and the given equation is of the form  $u_t = \frac{1}{2} u_{xx}$

$$\therefore c^2 = \frac{1}{2}$$

Since  $h = 1$ ,  $k = \frac{h^2}{2c^2} = 1$

The values of  $x$  in  $0 \leq x \leq 4$  with  $h = 1$  are 0,1,2,3,4 and the values of  $t$  with  $k = 1$  are 0,1,2,3,4,5

We have the following initial table wherein the values in the first and last column are zero since  $u(0, t) = 0, u(4, t) = 0$

| t \ x |   | $x_0$         | $x_1$     | $x_2$     | $x_3$     | $x_4$     |
|-------|---|---------------|-----------|-----------|-----------|-----------|
|       |   | 0             | 1         | 2         | 3         | 4         |
| $t_0$ | 0 | $u_{0,0} = 0$ | $u_{1,0}$ | $u_{2,0}$ | $u_{3,0}$ | $u_{4,0}$ |
| $t_1$ | 1 | $u_{0,1} = 0$ | $u_{1,1}$ | $u_{2,1}$ | $u_{3,1}$ | $u_{4,1}$ |

|       |   |               |           |           |           |           |
|-------|---|---------------|-----------|-----------|-----------|-----------|
| $t_2$ | 2 | $u_{0,2} = 0$ | $u_{1,2}$ | $u_{2,2}$ | $u_{3,2}$ | $u_{4,2}$ |
| $t_3$ | 3 | $u_{0,3} = 0$ | $u_{1,3}$ | $u_{2,3}$ | $u_{3,3}$ | $u_{4,3}$ |
| $t_4$ | 4 | $u_{0,4} = 0$ | $u_{1,4}$ | $u_{2,4}$ | $u_{3,4}$ | $u_{4,4}$ |
| $t_5$ | 5 | $u_{0,5} = 0$ | $u_{1,5}$ | $u_{2,5}$ | $u_{3,5}$ | $u_{4,5}$ |

Consider the initial conditions  $u(x, 0) = x(4 - x)$

$$\therefore u_{1,0} = u(1,0) = 3 ; u_{2,0} = 4 ; u_{3,0} = 3$$

(First row in the table is completed)

Consider the relation

$$u_{i,j+1} = \frac{1}{2} [u_{i-1,j} + u_{i+1,j}] \text{-----(1)}$$

$$u_{i,1} = \frac{1}{2} [u_{i-1,0} + u_{i+1,0}]$$

$$u_{1,1} = \frac{1}{2} [u_{0,0} + u_{2,0}] = \frac{1}{2} (0 + 4) = 2$$

$$u_{2,1} = \frac{1}{2} [u_{1,0} + u_{3,0}] = \frac{1}{2} (3 + 3) = 3$$

$$u_{3,1} = \frac{1}{2} [u_{2,0} + u_{4,0}] = \frac{1}{2} (4 + 0) = 2$$

(Second row in the table is completed)

Also from (1)

$$u_{i,2} = \frac{1}{2} [u_{i-1,1} + u_{i+1,1}]$$

$$u_{1,2} = \frac{1}{2} [u_{0,1} + u_{2,1}] = \frac{1}{2} (0 + 3) = 1.5$$

$$u_{2,2} = \frac{1}{2} [u_{1,1} + u_{3,1}] = \frac{1}{2} (2 + 2) = 2$$

$$u_{3,2} = \frac{1}{2} [u_{2,1} + u_{4,1}] = \frac{1}{2} (3 + 0) = 1.5$$

(Third row in the table is completed)

Also from (1)

$$u_{i,3} = \frac{1}{2}[u_{i-1,2} + u_{i+1,2}]$$

$$u_{1,3} = \frac{1}{2}[u_{0,2} + u_{2,2}] = \frac{1}{2}(0 + 2) = 1$$

$$u_{2,3} = \frac{1}{2}[u_{1,2} + u_{3,2}] = \frac{1}{2}(1.5 + 1.5) = 1.5$$

$$u_{3,3} = \frac{1}{2}[u_{2,2} + u_{4,2}] = \frac{1}{2}(2 + 0) = 1$$

$$u_{i,4} = \frac{1}{2}[u_{i-1,3} + u_{i+1,3}]$$

*(Fourth row in the table is completed)*

Also from (1)

$$u_{1,4} = \frac{1}{2}[u_{0,3} + u_{2,3}] = \frac{1}{2}(0 + 1.5) = 0.75$$

$$u_{2,4} = \frac{1}{2}[u_{1,3} + u_{3,3}] = \frac{1}{2}(1 + 1) = 1$$

$$u_{3,4} = \frac{1}{2}[u_{2,3} + u_{4,3}] = \frac{1}{2}(1.5 + 0) = 0.75$$

$$u_{i,5} = \frac{1}{2}[u_{i-1,4} + u_{i+1,4}]$$

*(Fifth row in the table is completed)*

Also from (1)

$$u_{1,5} = \frac{1}{2}[u_{0,4} + u_{2,4}] = \frac{1}{2}(0 + 1) = 0.5$$

$$u_{2,5} = \frac{1}{2}[u_{1,4} + u_{3,4}] = \frac{1}{2}(0.75 + 0.75) = 0.75$$

$$u_{3,5} = \frac{1}{2}[u_{2,4} + u_{4,4}] = \frac{1}{2}(1 + 0) = 0.5$$

Thus the required values of  $u_{i,j}$  are tabulated

| x \ t | 0 | 1    | 2    | 3    | 4 |
|-------|---|------|------|------|---|
| 0     | 0 | 3    | 4    | 3    | 0 |
| 1     | 0 | 2    | 3    | 2    | 0 |
| 2     | 0 | 1.5  | 2    | 1.5  | 0 |
| 3     | 0 | 1    | 1.5  | 1    | 0 |
| 4     | 0 | 0.75 | 1    | 0.75 | 0 |
| 5     | 0 | 0.5  | 0.75 | 0.5  | 0 |

2. Solve  $u_t = u_{xx}$  subject to the conditions  $u(0, t) = 0$ ,  $u(1, t) = 0$ ,  $u(x, 0) = \sin(\pi x)$  for  $0 \leq t \leq 0.1$  by taking  $h = 0.2$ . Write down the following values from the table.  
a)  $u(0.2, 0.04)$  b)  $u(0.4, 0.08)$  c)  $u(0.6, 0.06)$

Comparing the standard form of the heat equation

$$c^2 u_{xx} = u_t \quad \text{with the given equation } u_{xx} = u_t \quad \text{we have } c^2 = 1$$

$$\text{Since } h = 0.2, \quad k = \frac{h^2}{2c^2} = \frac{0.04}{2} = 0.02$$

The values of  $x$  in  $0 \leq x \leq 1$  with  $h = 0.2$  are 0, 0.2, 0.4, 0.6, 0.8 and 1.

Also the values of  $t$  in the given range  $0 \leq t \leq 0.1$  are 0, 0.02, 0.04, 0.06, 0.08 and 0.1

We have the following initial table, where in the values in the first and last column are zero since  $u(0, t) = 0$  and  $u(1, t) = 0$

| t \ x |      | $x_0$         | $x_1$     | $x_2$     | $x_3$     | $x_4$     | $x_5$         |
|-------|------|---------------|-----------|-----------|-----------|-----------|---------------|
|       |      | 0             | 0.2       | 0.4       | 0.6       | 0.8       | 1             |
| $t_0$ | 0    | $u_{0,0} = 0$ | $u_{1,0}$ | $u_{2,0}$ | $u_{3,0}$ | $u_{4,0}$ | $u_{5,0} = 0$ |
| $t_1$ | 0.02 | $u_{0,1} = 0$ | $u_{1,1}$ | $u_{2,1}$ | $u_{3,1}$ | $u_{4,1}$ | $u_{5,1} = 0$ |
| $t_2$ | 0.04 | $u_{0,2} = 0$ | $u_{1,2}$ | $u_{2,2}$ | $u_{3,2}$ | $u_{4,2}$ | $u_{5,2} = 0$ |
| $t_3$ | 0.06 | $u_{0,3} = 0$ | $u_{1,3}$ | $u_{2,3}$ | $u_{3,3}$ | $u_{4,3}$ | $u_{5,3} = 0$ |
| $t_4$ | 0.08 | $u_{0,4} = 0$ | $u_{1,4}$ | $u_{2,4}$ | $u_{3,4}$ | $u_{4,4}$ | $u_{5,4} = 0$ |
| $t_5$ | 0.1  | $u_{0,5} = 0$ | $u_{1,5}$ | $u_{2,5}$ | $u_{3,5}$ | $u_{4,5}$ | $u_{5,5} = 0$ |

Consider  $u(x, 0) = \sin(\pi x)$

$$u_{1,0} = u(0.2, 0) = 3 = \sin\left(\frac{\pi}{5}\right) = 0.59$$

$$u_{2,0} = 4 = u(0.4,0) = \sin\left(\frac{2\pi}{5}\right) = 0.95$$

$$u_{3,0} = u(0.6,0) = \sin\left(\frac{3\pi}{5}\right) = 0.95$$

$$u_{4,0} = u(0.8,0) = \sin\left(\frac{4\pi}{5}\right) = 0.59$$

(First row in the table is completed)

Consider the relation

$$u_{i,j+1} = \frac{1}{2}[u_{i-1,j} + u_{i+1,j}] \text{-----(1)}$$

$$u_{i,1} = \frac{1}{2}[u_{i-1,0} + u_{i+1,0}]$$

$$u_{1,1} = \frac{1}{2}[u_{0,0} + u_{2,0}] = \frac{1}{2}(0 + 0.95) = 0.475$$

$$u_{2,1} = \frac{1}{2}[u_{1,0} + u_{3,0}] = \frac{1}{2}(0.59 + 0.95) = 0.77$$

$$u_{3,1} = \frac{1}{2}[u_{2,0} + u_{4,0}] = \frac{1}{2}(0.95 + 0.59) = 0.77$$

$$u_{4,1} = \frac{1}{2}[u_{3,0} + u_{5,0}] = \frac{1}{2}(0.95 + 0.59) = 0.475$$

(Second row in the table is completed)

Also from (1)

$$u_{i,2} = \frac{1}{2}[u_{i-1,1} + u_{i+1,1}]$$

$$u_{1,2} = \frac{1}{2}[u_{0,1} + u_{2,1}] = \frac{1}{2}(0 + 0.77) = 0.385$$

$$u_{2,2} = \frac{1}{2}[u_{1,1} + u_{3,1}] = \frac{1}{2}(0.475 + 0.77) = 0.6225$$

$$u_{3,2} = \frac{1}{2}[u_{2,1} + u_{4,1}] = \frac{1}{2}(0.77 + 0.77) = 0.6225$$

$$u_{4,2} = \frac{1}{2}[u_{3,1} + u_{5,1}] = \frac{1}{2}(0.77 + 0) = 0.385$$

*(Third row in the table is completed)*

Also from (1)

$$u_{i,3} = \frac{1}{2}[u_{i-1,2} + u_{i+1,2}]$$

$$u_{1,3} = \frac{1}{2}[u_{0,2} + u_{2,2}] = \frac{1}{2}(0 + 0.6225) = 0.3113$$

$$u_{2,3} = \frac{1}{2}[u_{1,2} + u_{3,2}] = \frac{1}{2}(0.385 + 0.6225) = 0.504$$

$$u_{3,3} = \frac{1}{2}[u_{2,2} + u_{4,2}] = \frac{1}{2}(0.6225 + 0.385) = 0.504$$

$$u_{4,3} = \frac{1}{2}[u_{3,2} + u_{5,2}] = \frac{1}{2}(0.6225 + 0) = 0.3113$$

*(Fourth row in the table is completed)*

Also from (1)

$$u_{i,4} = \frac{1}{2}[u_{i-1,3} + u_{i+1,3}]$$

$$u_{1,4} = \frac{1}{2}[u_{0,3} + u_{2,3}] = \frac{1}{2}(0 + 0.504) = 0.252$$

$$u_{2,4} = \frac{1}{2}[u_{1,3} + u_{3,3}] = \frac{1}{2}(0.3113 + 0.504) = 0.408$$

$$u_{3,4} = \frac{1}{2}[u_{2,3} + u_{4,3}] = \frac{1}{2}(0.504 + 0.3113) = 0.408$$

$$u_{4,4} = \frac{1}{2}[u_{3,3} + u_{5,3}] = \frac{1}{2}(0.504 + 0) = 0.252$$

*(Fifth row in the table is completed)*

Also from (1)

$$u_{i,5} = \frac{1}{2} [u_{i-1,4} + u_{i+1,4}]$$

$$u_{1,5} = \frac{1}{2} [u_{0,4} + u_{2,4}] = \frac{1}{2} (0 + 0.408) = 0.204$$

$$u_{2,5} = \frac{1}{2} [u_{1,4} + u_{3,4}] = \frac{1}{2} (0.252 + 0.408) = 0.33$$

$$u_{3,5} = \frac{1}{2} [u_{2,4} + u_{4,4}] = \frac{1}{2} (0.408 + 0.252) = 0.33$$

$$u_{4,5} = \frac{1}{2} [u_{3,4} + u_{5,4}] = \frac{1}{2} (0.408 + 0) = 0.204$$

Thus the required values of  $u_{i,j}$  are tabulated

| $\begin{array}{c} \text{t} \\ \text{---} \\ \text{x} \end{array}$ | 0 | 0.2    | 0.4    | 0.6    | 0.8    | 1 |
|-------------------------------------------------------------------|---|--------|--------|--------|--------|---|
| 0                                                                 | 0 | 0.59   | 0.95   | 0.95   | 0.59   | 0 |
| 0.02                                                              | 0 | 0.475  | 0.77   | 0.77   | 0.475  | 0 |
| 0.04                                                              | 0 | 0.385  | 0.6225 | 0.6225 | 0.385  | 0 |
| 0.06                                                              | 0 | 0.3113 | 0.504  | 0.504  | 0.3113 | 0 |
| 0.08                                                              | 0 | 0.252  | 0.408  | 0.408  | 0.252  | 0 |
| 0.1                                                               | 0 | 0.204  | 0.33   | 0.33   | 0.204  | 0 |

Also from the table,

$$\text{a) } u(0.2, 0.04) = 0.385 \quad \text{b) } u(0.4, 0.08) = 0.408 \quad \text{c) } u(0.6, 0.06) = 0.504$$

### Numerical solution of the one dimensional wave equation

We seek numerical solution of the wave equation

$$c^2 u_{xx} = u_{tt} \quad \text{--- (1)}$$

subject to the boundary conditions

$$u(0, t) = 0 \quad \text{--- (2)}$$

$$u(l, t) = 0 \quad \text{--- (3)}$$

and the initial conditions

$$u(x, 0) = f(x) \quad \text{--- (4)}$$

$$u_t(x, 0) = 0 \quad \text{--- (5)}$$

We shall substitute the finite difference approximation for the partial derivatives present in (1)

$$c^2 \cdot \frac{1}{h^2} [u_{i-1,j} - 2u_{i,j} + u_{i+1,j}] = \frac{1}{k^2} [u_{i,j-1} - 2u_{i,j} + u_{i,j+1}]$$

$$c^2 \frac{k^2}{h^2} [u_{i-1,j} - 2u_{i,j} + u_{i+1,j}] = [u_{i,j-1} - 2u_{i,j} + u_{i,j+1}]$$

Taking  $\frac{k}{h} = \lambda$ , we have

$$c^2 \lambda^2 [u_{i-1,j} - 2u_{i,j} + u_{i+1,j}] = [u_{i,j-1} - 2u_{i,j} + u_{i,j+1}]$$

$$u_{i,j+1} = 2(1 - c^2 \lambda^2) u_{i,j} + c^2 \lambda^2 (u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \quad \text{--- (6)}$$

For convenience let us choose  $\lambda$  such that  $1 - c^2 \lambda^2 = 0$ .

$$1 - c^2 \left( \frac{k^2}{h^2} \right) = 0 \quad \text{or} \quad k^2 = \frac{h^2}{c^2} \Rightarrow k = \frac{h}{c}$$

$$(6) \Rightarrow u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \quad \text{--- (7)}$$

This is called the **explicit formula** for the solution of the wave equation.

Further we express the initial condition

$$u_t(x, 0) = 0$$

$$\frac{1}{2k} [u_{i,1} - u_{i,-1}] = 0 \quad \left\{ \text{Taking } j = 0 \text{ in } u_t = \frac{1}{2k} [u_{i,j+1} - u_{i,j-1}] \right\}$$

$$\text{or } u_{i,1} = u_{i,-1} \quad \text{--- (8)}$$

Putting  $j = 0$  in (7), we get

$$u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1}$$

$$u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,1} \quad \{ \text{from (8)} \}$$

$$u_{i,1} + u_{i,1} = u_{i-1,0} + u_{i+1,0}$$

$$2u_{i,1} = u_{i-1,0} + u_{i+1,0}$$

$$u_{i,1} = \frac{1}{2} [u_{i-1,0} + u_{i+1,0}] \quad \text{--- (9)}$$

Further  $u(0, t) = 0$  and  $u(l, t) = 0$  means  $u_{0,j} = 0$  and  $u_{l,j} = 0$  which implies that the values along the first and last column are zero.

$u(x, 0) = f(x)$  means  $u_{i,0} = f(x)$  gives the values of  $u$  along the first row.

Finally  $u_t(x, 0) = 0$  modified into the form (9) giving  $u_{i,1}$ , will give us the values of  $u$  along the second row. These values will help us to obtain the rest of the value of  $u$  at the mesh points by the explicit formula (7).

Thus we are able to determine  $u(x, t)$  at all interior mesh points.

### Working procedure for problems

- Based on the given step sizes  $h$  and  $k$ , we form the rectangular network of sides  $h$  and  $k$ .
- Given  $h$  only we can get  $k = \frac{h}{c}$ . The points of division of  $x$  are  $x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \dots, x_n = x_0 + nh$  and that of  $t$  are  $t_0, t_1 = t_0 + k, t_2 = t_0 + 2k, \dots, t_n = t_0 + nk$ .
- We form the basic informative table:

| $x \backslash t$ |         | $x_0$     | $x_1$     | $x_2$     | $\dots$ | $x_n$     |
|------------------|---------|-----------|-----------|-----------|---------|-----------|
|                  |         | 0         | 1         | 2         | $\dots$ | $n$       |
| $t_0$            | 0       | $u_{0,0}$ | $u_{1,0}$ | $u_{2,0}$ | $\dots$ | $u_{n,0}$ |
| $t_1$            | 1       | $u_{0,1}$ | $u_{1,1}$ | $u_{2,1}$ | $\dots$ | $u_{n,1}$ |
| $t_2$            | 2       | $u_{0,2}$ | $u_{1,2}$ | $u_{2,2}$ | $\dots$ | $u_{n,2}$ |
| $\dots$          | $\dots$ | $\dots$   | $\dots$   | $\dots$   | $\dots$ | $\dots$   |
| $t_n$            | $n$     | $u_{0,n}$ | $u_{1,n}$ | $u_{2,n}$ | $\dots$ | $u_{n,n}$ |

- We instantly write the value of  $u_{i,j}$  with reference to the conditions  $u(0, t) = 0$  and  $u(l, t) = 0$   
[Values along the first and last column are zero]
- The values along the first row are obtained by direct computation using the condition  $u(x, 0) = f(x)$
- The values along the second row are obtained by the relation  $u_{i,1} = \frac{1}{2}[u_{i-1,0} + u_{i+1,0}]$
- Rest of the values are found from the relation  $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$

**Note:** If the values of both  $h$  and  $k$  are small, we use the below

formula

$$u_{i,j+1} = 2(1 - c^2 \lambda^2)u_{i,j} + c^2 \lambda^2(u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \quad \text{--- (10)}$$

Putting  $j = 0$

$$(10) \Rightarrow u_{i,1} = 2(1 - c^2 \lambda^2)u_{i,0} + c^2 \lambda^2(u_{i-1,0} + u_{i+1,0}) - u_{i,-1} \quad \text{--- (11)}$$

Further we express the initial condition

$$u_t(x, 0) = 0$$

$$\frac{1}{2k} [u_{i,1} - u_{i,-1}] = 0 \quad \left\{ \text{Taking } j = 0 \text{ in } u_t = \frac{1}{2k} [u_{i,j+1} - u_{i,j-1}] \right\}$$

$$\text{or } u_{i,1} = u_{i,-1}$$

$$(11) \Rightarrow u_{i,1} = 2(1 - c^2 \lambda^2) u_{i,0} + c^2 \lambda^2 (u_{i-1,0} + u_{i+1,0}) - u_{i,1}$$

$$2u_{i,1} = 2(1 - c^2 \lambda^2) u_{i,0} + c^2 \lambda^2 (u_{i-1,0} + u_{i+1,0})$$

Putting  $j = 1$

$$(10) \Rightarrow u_{i,2} = 2(1 - c^2 \lambda^2) u_{i,1} + c^2 \lambda^2 (u_{i-1,1} + u_{i+1,1}) - u_{i,0}$$

### Problems

1. Solve the wave equation  $\frac{\partial^2 u}{\partial t^2} = 4 \frac{\partial^2 u}{\partial x^2}$  subject to  $u(0, t) = 0$ ,

$u(4, t) = 0$ ,  $u_t(x, 0) = 0$  and  $u(x, 0) = x(4 - x)$  by taking  $h = 1$ ,

$k = 0.5$  upto four steps.

**Sol:** Step size of  $x : h = 1$  ; Step size of  $t : k = 0.5$

Since  $0 \leq x \leq 4$ , the points of division are 0,1,2,3,4. Since  $k = 0.5$ , the values corresponding to  $t$  upto four steps are 0,0.5,1,1.5 and 2.

We have the following initial table. The values in the first and last column are zero since  $u(0, t) = 0 = u(4, t)$

| $\begin{matrix} x \\ t \end{matrix}$ |     | $x_0$         | $x_1$     | $x_2$     | $x_3$     | $x_4$         |
|--------------------------------------|-----|---------------|-----------|-----------|-----------|---------------|
|                                      |     | 0             | 1         | 2         | 3         | 4             |
| $t_0$                                | 0   | $u_{0,0} = 0$ | $u_{1,0}$ | $u_{2,0}$ | $u_{3,0}$ | $u_{4,0} = 0$ |
| $t_1$                                | 0.5 | $u_{0,1} = 0$ | $u_{1,1}$ | $u_{2,1}$ | $u_{3,1}$ | $u_{4,1} = 0$ |
| $t_2$                                | 1   | $u_{0,2} = 0$ | $u_{1,2}$ | $u_{2,2}$ | $u_{3,2}$ | $u_{4,2} = 0$ |

|       |     |               |           |           |           |               |
|-------|-----|---------------|-----------|-----------|-----------|---------------|
| $t_3$ | 1.5 | $u_{0,3} = 0$ | $u_{1,3}$ | $u_{2,3}$ | $u_{3,3}$ | $u_{4,3} = 0$ |
| $t_4$ | 2   | $u_{0,4} = 0$ | $u_{1,4}$ | $u_{2,4}$ | $u_{3,4}$ | $u_{4,4} = 0$ |

**To find first row elements**

Consider  $u(x, 0) = x(4 - x)$

$$\therefore u_{1,0} = u(1,0) = 3 ; u_{2,0} = u(2,0) = 4 ; u_{3,0} = u(3,0) = 3$$

**To find second row elements**

Consider  $u_{i,1} = \frac{1}{2}[u_{i-1,0} + u_{i+1,0}]$

$$u_{1,1} = \frac{1}{2}[u_{0,0} + u_{2,0}] = \frac{1}{2}[0 + 4] = 2$$

$$u_{2,1} = \frac{1}{2}[u_{1,0} + u_{3,0}] = \frac{1}{2}[3 + 3] = 3$$

$$u_{3,1} = \frac{1}{2}[u_{2,0} + u_{4,0}] = \frac{1}{2}[4 + 0] = 2$$

**To find remaining row values we use**

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

**Third row elements**

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 3 - 3 = 0 \quad \{By \text{ taking } i = 1, j = 1\}$$

$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 2 + 2 - 4 = 0 \quad \{By \text{ taking } i = 2, j = 1\}$$

$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 0 + 3 - 3 = 0 \quad \{By \text{ taking } i = 3, j = 1\}$$

**Fourth row elements**

$$u_{1,3} = u_{0,2} + u_{2,2} - u_{1,1} = 0 + 0 - 2 = -2 \quad \{By \text{ taking } i = 1, j = 2\}$$

$$u_{2,3} = u_{1,2} + u_{3,2} - u_{2,1} = 0 + 0 - 3 = -3 \quad \{By \text{ taking } i = 2, j = 2\}$$

$$u_{3,3} = u_{2,2} + u_{4,2} - u_{3,1} = 0 + 0 - 2 = -2 \quad \{By \text{ taking } i = 3, j = 2\}$$

**Fifth row elements**

$$u_{1,4} = u_{0,3} + u_{2,3} - u_{1,2} = 0 - 3 - 0 = -3 \quad \{By \text{ taking } i = 1, j = 3\}$$

$$u_{2,4} = u_{1,3} + u_{3,3} - u_{2,2} = -2 - 2 - 0 = -4 \quad \{By \text{ taking } i = 2, j = 3\}$$

$$u_{3,4} = u_{2,3} + u_{4,3} - u_{3,2} = -3 + 0 - 0 = -3 \quad \{By \text{ taking } i = 3, j = 3\}$$

Thus the required values of  $u_{i,j}$  are tabulated

| $x \backslash t$ | 0 | 1  | 2  | 3  | 4 |
|------------------|---|----|----|----|---|
| 0                | 0 | 3  | 4  | 3  | 0 |
| 0.5              | 0 | 2  | 3  | 2  | 0 |
| 1                | 0 | 0  | 0  | 0  | 0 |
| 1.5              | 0 | -2 | -3 | -2 | 0 |
| 2                | 0 | -3 | -4 | -3 | 0 |

2. Solve numerically  $u_{xx} = 0.0625u_{tt}$ , subject to the conditions

$u(0, t) = 0 = u(5, t)$ ,  $u(x, 0) = x^2(x - 5)$  and  $u_t(x, 0) = 0$  by taking  $h = 1$  for  $0 \leq t \leq 1$ .

**Sol:** The wave equation in the standard form is  $c^2 u_{xx} = u_{tt}$  and

hence the given equation can be written as  $\left(\frac{1}{0.0625}\right) u_{xx} = u_{tt}$

i.e.,  $16u_{xx} = u_{tt}$ , where  $c^2 = 16$  or  $c = 4$ .

Since  $h = 1$ , we have  $k = \frac{h}{c} = \frac{1}{4} = 0.25$

Step size of  $x$ :  $h = 1$  where  $0 \leq x \leq 5$

Step size of  $t$ :  $k = 0.25$  where  $0 \leq t \leq 1$  as required.

$\therefore$  Values of  $x$  are 0,1,2,3,4,5

Values of  $t$  are 0,0.25,0.5,0.75,1

We have the following initial table. The values in the first and last column are zero since  $u(0, t) = 0 = u(5, t)$

| $x \backslash t$ |      | $x_0$         | $x_1$     | $x_2$     | $x_3$     | $x_4$     | $x_5$         |
|------------------|------|---------------|-----------|-----------|-----------|-----------|---------------|
|                  |      | 0             | 1         | 2         | 3         | 4         | 5             |
| $t_0$            | 0    | $u_{0,0} = 0$ | $u_{1,0}$ | $u_{2,0}$ | $u_{3,0}$ | $u_{4,0}$ | $u_{5,0} = 0$ |
| $t_1$            | 0.25 | $u_{0,1} = 0$ | $u_{1,1}$ | $u_{2,1}$ | $u_{3,1}$ | $u_{4,1}$ | $u_{5,1} = 0$ |
| $t_2$            | 0.5  | $u_{0,2} = 0$ | $u_{1,2}$ | $u_{2,2}$ | $u_{3,2}$ | $u_{4,2}$ | $u_{5,2} = 0$ |
| $t_3$            | 0.75 | $u_{0,3} = 0$ | $u_{1,3}$ | $u_{2,3}$ | $u_{3,3}$ | $u_{4,3}$ | $u_{5,3} = 0$ |

|       |   |               |           |           |           |           |               |
|-------|---|---------------|-----------|-----------|-----------|-----------|---------------|
| $t_4$ | 1 | $u_{0,4} = 0$ | $u_{1,4}$ | $u_{2,4}$ | $u_{3,4}$ | $u_{4,4}$ | $u_{5,4} = 0$ |
|-------|---|---------------|-----------|-----------|-----------|-----------|---------------|

**To find first row elements**

Consider  $u(x, 0) = x^2(x - 5)$

$$\therefore u_{1,0} = u(1,0) = -4 ; u_{2,0} = u(2,0) = -12 ; u_{3,0} = u(3,0) = -18 ;$$

$$u_{4,0} = u(4,0) = -16$$

**To find second row elements**

Consider  $u_{i,1} = \frac{1}{2}[u_{i-1,0} + u_{i+1,0}]$

$$u_{1,1} = \frac{1}{2}[u_{0,0} + u_{2,0}] = \frac{1}{2}[0 - 12] = -6$$

$$u_{2,1} = \frac{1}{2}[u_{1,0} + u_{3,0}] = \frac{1}{2}[-4 - 18] = -11$$

$$u_{3,1} = \frac{1}{2}[u_{2,0} + u_{4,0}] = \frac{1}{2}[-12 - 16] = -14$$

$$u_{4,1} = \frac{1}{2}[u_{3,0} + u_{5,0}] = \frac{1}{2}[-18 + 0] = -9$$

**To find remaining row values we use**

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

**Third row elements**

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 - 11 + 4 = -7 \quad \{\text{By taking } i = 1, j = 1\}$$

$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = -6 - 14 + 12 = -8 \quad \{\text{By taking } i = 2, j = 1\}$$

$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = -11 - 9 + 18 = -2 \quad \{\text{By taking } i = 3, j = 1\}$$

$$u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0} = -14 + 0 + 16 = 2 \quad \{\text{By taking } i = 4, j = 1\}$$

**Fourth row elements**

$$u_{1,3} = u_{0,2} + u_{2,2} - u_{1,1} = 0 - 8 + 6 = -2 \quad \{\text{By taking } i = 1, j = 2\}$$

$$u_{2,3} = u_{1,2} + u_{3,2} - u_{2,1} = -7 - 2 + 11 = 2 \quad \{\text{By taking } i = 2, j = 2\}$$

$$u_{3,3} = u_{2,2} + u_{4,2} - u_{3,1} = -8 + 2 + 14 = 8 \quad \{\text{By taking } i = 3, j = 2\}$$

$$u_{4,3} = u_{3,2} + u_{5,2} - u_{4,1} = -2 + 0 + 9 = 7 \quad \{\text{By taking } i = 4, j = 2\}$$

**Fifth row elements**

$$u_{1,4} = u_{0,3} + u_{2,3} - u_{1,2} = 0 + 2 + 7 = 9 \quad \{\text{By taking } i = 1, j = 3\}$$

$$u_{2,4} = u_{1,3} + u_{3,3} - u_{2,2} = -2 + 8 + 8 = 14 \quad \{\text{By taking } i = 2, j = 3\}$$

$$u_{3,4} = u_{2,3} + u_{4,3} - u_{3,2} = 2 + 7 + 2 = 11 \quad \{\text{By taking } i = 3, j = 3\}$$

$$u_{4,4} = u_{3,3} + u_{5,3} - u_{4,2} = 8 + 0 - 2 = 6 \quad \{\text{By taking } i = 4, j = 3\}$$

Thus the required values of  $u_{i,j}$  are tabulated

| $x \backslash t$ | 0 | 1  | 2   | 3   | 4   | 5 |
|------------------|---|----|-----|-----|-----|---|
| 0                | 0 | -4 | -12 | -18 | -16 | 0 |
| 0.25             | 0 | -6 | -11 | -14 | -9  | 0 |
| 0.5              | 0 | -7 | -8  | -2  | 2   | 0 |
| 0.75             | 0 | -2 | 2   | 8   | 7   | 0 |
| 1                | 0 | 9  | 14  | 11  | 6   | 0 |

3. Solve  $25u_{xx} = u_{tt}$ , at the pivotal points given  $u(0, t) = 0 = u(5, t)$ ,

$$u_t(x, 0) = 0 \text{ and } u(x, 0) = \begin{cases} 20x, & 0 \leq x \leq 1 \\ 5(5 - x), & 1 \leq x \leq 5 \end{cases} \text{ by taking } h = 1.$$

Compute  $u(x, t)$  for  $0 \leq t \leq 1$ .

**Sol:** The wave equation in the standard form is  $c^2 u_{xx} = u_{tt}$  and

hence the given equation can be written as  $25u_{xx} = u_{tt}$ , where  $c^2 = 25$  or  $c = 5$ .

Since  $h = 1$ , we have  $k = \frac{h}{c} = \frac{1}{5} = 0.2$

Step size of  $x$ :  $h = 1$  where  $0 \leq x \leq 5$

Step size of  $t$ :  $k = 0.2$  where  $0 \leq t \leq 1$  as required.

$\therefore$  Values of  $x$  are 0,1,2,3,4,5

Values of  $t$  are 0,0.2,0.4,0.6,0.8,1

We have the following initial table. The values in the first and last column are zero since  $u(0, t) = 0 = u(5, t)$

| $x$   |     | $x_0$         | $x_1$     | $x_2$     | $x_3$     | $x_4$     | $x_5$         |
|-------|-----|---------------|-----------|-----------|-----------|-----------|---------------|
| $t$   |     | 0             | 1         | 2         | 3         | 4         | 5             |
| $t_0$ | 0   | $u_{0,0} = 0$ | $u_{1,0}$ | $u_{2,0}$ | $u_{3,0}$ | $u_{4,0}$ | $u_{5,0} = 0$ |
| $t_1$ | 0.2 | $u_{0,1} = 0$ | $u_{1,1}$ | $u_{2,1}$ | $u_{3,1}$ | $u_{4,1}$ | $u_{5,1} = 0$ |
| $t_2$ | 0.4 | $u_{0,2} = 0$ | $u_{1,2}$ | $u_{2,2}$ | $u_{3,2}$ | $u_{4,2}$ | $u_{5,2} = 0$ |
| $t_3$ | 0.6 | $u_{0,3} = 0$ | $u_{1,3}$ | $u_{2,3}$ | $u_{3,3}$ | $u_{4,3}$ | $u_{5,3} = 0$ |
| $t_4$ | 0.8 | $u_{0,4} = 0$ | $u_{1,4}$ | $u_{2,4}$ | $u_{3,4}$ | $u_{4,4}$ | $u_{5,4} = 0$ |
| $t_5$ | 1   | $u_{0,5} = 0$ | $u_{1,5}$ | $u_{2,5}$ | $u_{3,5}$ | $u_{4,5}$ | $u_{5,5} = 0$ |

### To find first row elements

We have  $u(x, 0) = \begin{cases} 20x, & 0 \leq x \leq 1 \\ 5(5 - x), & 1 \leq x \leq 5 \end{cases}$

$\therefore u_{1,0} = u(1,0) = 20 ; u_{2,0} = u(2,0) = 15 ; u_{3,0} = u(3,0) = 10 ;$

$u_{4,0} = u(4,0) = 5$

### To find second row elements

Consider  $u_{i,1} = \frac{1}{2}[u_{i-1,0} + u_{i+1,0}]$

$u_{1,1} = \frac{1}{2}[u_{0,0} + u_{2,0}] = \frac{1}{2}[0 + 15] = 7.5$

$u_{2,1} = \frac{1}{2}[u_{1,0} + u_{3,0}] = \frac{1}{2}[20 + 10] = 15$

$u_{3,1} = \frac{1}{2}[u_{2,0} + u_{4,0}] = \frac{1}{2}[15 + 5] = 10$

$u_{4,1} = \frac{1}{2}[u_{3,0} + u_{5,0}] = \frac{1}{2}[10 + 0] = 5$

### To find remaining row values we use

$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$

### Third row elements

$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 15 - 20 = -5 \quad \{By \text{ taking } i = 1, j = 1\}$

$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 7.5 + 10 - 15 = 2.5 \quad \{By \text{ taking } i = 2, j = 1\}$

$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 15 + 5 - 10 = 10 \quad \{By \text{ taking } i = 3, j = 1\}$

$u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0} = 10 + 0 - 5 = 5 \quad \{By \text{ taking } i = 4, j = 1\}$

### Fourth row elements

$$u_{1,3} = u_{0,2} + u_{2,2} - u_{1,1} = 0 + 2.5 - 7.5 = -5 \quad \{\text{By taking } i = 1, j = 2\}$$

$$u_{2,3} = u_{1,2} + u_{3,2} - u_{2,1} = -5 + 10 - 15 = -10 \quad \{\text{By taking } i = 2, j = 2\}$$

$$u_{3,3} = u_{2,2} + u_{4,2} - u_{3,1} = 2.5 + 5 - 10 = -2.5 \quad \{\text{By taking } i = 3, j = 2\}$$

$$u_{4,3} = u_{3,2} + u_{5,2} - u_{4,1} = 10 + 0 - 5 = 5 \quad \{\text{By taking } i = 4, j = 2\}$$

#### **Fifth row elements**

$$u_{1,4} = u_{0,3} + u_{2,3} - u_{1,2} = 0 - 10 + 5 = -5 \quad \{\text{By taking } i = 1, j = 3\}$$

$$u_{2,4} = u_{1,3} + u_{3,3} - u_{2,2} = -5 - 2.5 - 2.5 = -10 \quad \{\text{By taking } i = 2, j = 3\}$$

$$u_{3,4} = u_{2,3} + u_{4,3} - u_{3,2} = -10 + 5 - 10 = -15 \quad \{\text{By taking } i = 3, j = 3\}$$

$$u_{4,4} = u_{3,3} + u_{5,3} - u_{4,2} = -2.5 + 0 - 5 = -7.5 \quad \{\text{By taking } i = 4, j = 3\}$$

#### **Last row elements**

$$u_{1,5} = u_{0,4} + u_{2,4} - u_{1,3} = 0 - 10 + 5 = -5 \quad \{\text{By taking } i = 1, j = 4\}$$

$$u_{2,5} = u_{1,4} + u_{3,4} - u_{2,3} = -5 - 15 + 10 = -10 \quad \{\text{By taking } i = 2, j = 4\}$$

$$u_{3,5} = u_{2,4} + u_{4,4} - u_{3,3} = -10 - 7.5 + 2.5 = -15 \quad \{\text{By taking } i = 3, j = 4\}$$

$$u_{4,5} = u_{3,4} + u_{5,4} - u_{4,3} = -15 + 0 - 5 = -20 \quad \{\text{By taking } i = 4, j = 4\}$$

Thus the required values of  $u_{i,j}$  are tabulated

| $x \backslash t$ | 0 | 1   | 2   | 3    | 4    | 5 |
|------------------|---|-----|-----|------|------|---|
| 0                | 0 | 20  | 15  | 10   | 5    | 0 |
| 0.2              | 0 | 7.5 | 15  | 10   | 5    | 0 |
| 0.4              | 0 | -5  | 2.5 | 10   | 5    | 0 |
| 0.6              | 0 | -5  | -10 | -2.5 | 5    | 0 |
| 0.8              | 0 | -5  | -10 | -15  | -7.5 | 0 |
| 1                | 0 | -5  | -10 | -15  | -20  | 0 |

4. Solve  $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ , at the pivotal points given  $u(x, 0) = 0$ ;  $u(0, t) = 0$ ;  
 $u_t(x, 0) = 0$  and  $u(1, t) = 100\sin(\pi t)$  in the range  $0 \leq t \leq 1$  by  
 taking  $h = \frac{1}{4}$ .

**Sol:** The wave equation in the standard form is  $c^2 u_{xx} = u_{tt}$  and

hence the given equation can be written as  $u_{xx} = u_{tt}$ , where  $c^2 = 1$  or  $c = 1$ .

By data  $h = \frac{1}{4}$ , we have  $k = \frac{h}{c} = \frac{\frac{1}{4}}{1} = \frac{1}{4}$ .

The points of division of  $x$  (in  $0 \leq x \leq 1$ ) as well as  $t$  are  $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ .

The condition  $u(x, 0) = 0$  means that the value of  $u$  along the first row are zero and  $u(0, t) = 0$  means that the value of  $u$  along the first column are zero. The initial table showing the values given and the values to be computed as below

| $x \backslash t$ |               | $x_0$         | $x_1$         | $x_2$         | $x_3$         | $x_4$         |
|------------------|---------------|---------------|---------------|---------------|---------------|---------------|
|                  |               | 0             | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | 1             |
| $t_0$            | 0             | $u_{0,0} = 0$ | $u_{1,0} = 0$ | $u_{2,0} = 0$ | $u_{3,0} = 0$ | $u_{4,0} = 0$ |
| $t_1$            | $\frac{1}{4}$ | $u_{0,1} = 0$ | $u_{1,1}$     | $u_{2,1}$     | $u_{3,1}$     | $u_{4,1}$     |
| $t_2$            | $\frac{1}{2}$ | $u_{0,2} = 0$ | $u_{1,2}$     | $u_{2,2}$     | $u_{3,2}$     | $u_{4,2}$     |
| $t_3$            | $\frac{3}{4}$ | $u_{0,3} = 0$ | $u_{1,3}$     | $u_{2,3}$     | $u_{3,3}$     | $u_{4,3}$     |
| $t_4$            | 1             | $u_{0,4} = 0$ | $u_{1,4}$     | $u_{2,4}$     | $u_{3,4}$     | $u_{4,4}$     |

We have seen the condition  $u_t(x, 0) = 0$  will lead us the formula

**To find second row elements**

Consider  $u_{i,1} = \frac{1}{2}[u_{i-1,0} + u_{i+1,0}]$

$$u_{1,1} = \frac{1}{2}[u_{0,0} + u_{2,0}] = \frac{1}{2}[0 + 0] = 0$$

$$u_{2,1} = \frac{1}{2}[u_{1,0} + u_{3,0}] = \frac{1}{2}[0 + 0] = 0$$

$$u_{3,1} = \frac{1}{2}[u_{2,0} + u_{4,0}] = \frac{1}{2}[0 + 0] = 0$$

$u_{4,1} = \frac{1}{2}[u_{3,0} + u_{5,0}]$  and we do not have  $u_{5,0}$  on hand.

Hence  $u_{4,1} = u(x_4, t_1) = u\left(1, \frac{1}{4}\right) = 100\sin\left(\frac{\pi}{4}\right) = 70.7$

**To find remaining row values we use**

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

**Third row elements**

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 \quad \{\text{By taking } i = 1, j = 1\}$$

$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 0 \quad \{\text{By taking } i = 2, j = 1\}$$

$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 70.7 \quad \{\text{By taking } i = 3, j = 1\}$$

$u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0}$  {By taking  $i = 4, j = 1$ } is in applicable.

Hence  $u_{4,2} = u(x_4, t_2) = u\left(1, \frac{1}{2}\right) = 100\sin\left(\frac{\pi}{2}\right) = 100$ .

**Fourth row elements**

$$u_{1,3} = u_{0,2} + u_{2,2} - u_{1,1} = 0$$

$$u_{2,3} = u_{1,2} + u_{3,2} - u_{2,1} = 70.7$$

$$u_{3,3} = u_{2,2} + u_{4,2} - u_{3,1} = 100$$

$$u_{4,3} = u(x_4, t_3) = u\left(1, \frac{3}{4}\right) = 100\sin\left(\frac{3\pi}{4}\right) = 70.7.$$

**Last row elements**

$$u_{1,4} = u_{0,3} + u_{2,3} - u_{1,2} = 70.7$$

$$u_{2,4} = u_{1,3} + u_{3,3} - u_{2,2} = 100$$

$$u_{3,4} = u_{2,3} + u_{4,3} - u_{3,2} = 70.7$$

$$u_{4,4} = u(x_4, t_4) = u(1, 1) = 100\sin(\pi) = 0$$

Thus the required values of  $u_{i,j}$  are tabulated

|               |   |               |               |               |      |
|---------------|---|---------------|---------------|---------------|------|
| $x$<br>$t$    | 0 | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | 1    |
| 0             | 0 | 0             | 0             | 0             | 0    |
| $\frac{1}{4}$ | 0 | 0             | 0             | 0             | 70.7 |
| $\frac{1}{2}$ | 0 | 0             | 0             | 70.7          | 100  |
| $\frac{3}{4}$ | 0 | 0             | 70.7          | 100           | 70.7 |
| 1             | 0 | 70.7          | 100           | 70.7          | 0    |

5. Solve the wave equation  $4 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ , given  $u(x, 0) = u(5, t) = 0$ ,

$t \geq 0$   $u(x, 0) = x(5 - x)$ ,  $\frac{\partial}{\partial t} u(x, 0) = 0$ ,  $0 < x < 5$ . Find  $u$  at  $t = 2$

given  $h = 1, k = 0.5$ .

**Sol:** The wave equation in the standard form is  $c^2 u_{xx} = u_{tt}$  and

hence the given equation can be written as  $4u_{xx} = u_{tt}$

Here  $c^2 = 4$  or  $c = 2$  and  $k = \frac{h}{c}$

Since  $h = 1$ , we have  $k = \frac{h}{c} = \frac{1}{2} = 0.5$

Step size of  $x$ :  $h = 1$  where  $0 \leq x \leq 5$

Step size of  $t$ :  $k = 0.5$  where  $0 \leq t \leq 2$  as required.

$\therefore$  Values of  $x$  are 0,1,2,3,4,5

Values of  $t$  are 0,0.5,1,1.5,2

We have the following initial table. The values in the first and last column are zero since  $u(0, t) = 0 = u(5, t)$ .

| $x$   |     | $x_0$         | $x_1$     | $x_2$     | $x_3$     | $x_4$     | $x_5$         |
|-------|-----|---------------|-----------|-----------|-----------|-----------|---------------|
| $t$   |     | 0             | 1         | 2         | 3         | 4         | 5             |
| $t_0$ | 0   | $u_{0,0} = 0$ | $u_{1,0}$ | $u_{2,0}$ | $u_{3,0}$ | $u_{4,0}$ | $u_{5,0} = 0$ |
| $t_1$ | 0.5 | $u_{0,1} = 0$ | $u_{1,1}$ | $u_{2,1}$ | $u_{3,1}$ | $u_{4,1}$ | $u_{5,1} = 0$ |
| $t_2$ | 1   | $u_{0,2} = 0$ | $u_{1,2}$ | $u_{2,2}$ | $u_{3,2}$ | $u_{4,2}$ | $u_{5,2} = 0$ |
| $t_3$ | 1.5 | $u_{0,3} = 0$ | $u_{1,3}$ | $u_{2,3}$ | $u_{3,3}$ | $u_{4,3}$ | $u_{5,3} = 0$ |
| $t_4$ | 2   | $u_{0,4} = 0$ | $u_{1,4}$ | $u_{2,4}$ | $u_{3,4}$ | $u_{4,4}$ | $u_{5,4} = 0$ |

**To find first row elements**

Consider  $u(x, 0) = x(5 - x)$

$$\therefore u_{1,0} = u(1,0) = 4 ; u_{2,0} = u(2,0) = 6 ; u_{3,0} = u(3,0) = 6 ;$$

$$u_{4,0} = u(4,0) = 4$$

**To find second row elements**

Consider  $u_{i,1} = \frac{1}{2}[u_{i-1,0} + u_{i+1,0}]$

$$u_{1,1} = \frac{1}{2}[u_{0,0} + u_{2,0}] = \frac{1}{2}[0 + 6] = 3$$

$$u_{2,1} = \frac{1}{2}[u_{1,0} + u_{3,0}] = \frac{1}{2}[4 + 6] = 5$$

$$u_{3,1} = \frac{1}{2}[u_{2,0} + u_{4,0}] = \frac{1}{2}[6 + 4] = 5$$

$$u_{4,1} = \frac{1}{2}[u_{3,0} + u_{5,0}] = \frac{1}{2}[6 + 0] = 3$$

**To find remaining row values we use**

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

**Third row elements**

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 5 - 4 = 1$$

$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 3 + 5 - 6 = 2$$

$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 5 + 3 - 6 = 2$$

$$u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0} = 5 + 0 - 4 = 1$$

**Fourth row elements**

$$u_{1,3} = u_{0,2} + u_{2,2} - u_{1,1} = 0 + 2 - 3 = -1$$

$$u_{2,3} = u_{1,2} + u_{3,2} - u_{2,1} = 1 + 2 - 5 = -2$$

$$u_{3,3} = u_{2,2} + u_{4,2} - u_{3,1} = 2 + 1 - 5 = -2$$

$$u_{4,3} = u_{3,2} + u_{5,2} - u_{4,1} = 2 + 0 - 3 = -1$$

**Fifth row elements**

$$u_{1,4} = u_{0,3} + u_{2,3} - u_{1,2} = 0 - 2 - 1 = -3$$

$$u_{2,4} = u_{1,3} + u_{3,3} - u_{2,2} = -1 - 2 - 2 = -5$$

$$u_{3,4} = u_{2,3} + u_{4,3} - u_{3,2} = -2 - 1 - 2 = -5$$

$$u_{4,4} = u_{3,3} + u_{5,3} - u_{4,2} = -2 + 0 - 1 = -3$$

Thus the required values of  $t = 2$  are  $-3, -5, -5, -3$ .

6. Solve  $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ , subject to the boundary conditions,

$$u(0, t) = 0 = u(1, t); t \geq 0 \text{ and the initial conditions,}$$

$$u(x, 0) = \sin(\pi x), \frac{\partial u}{\partial t}(x, 0) = 0, 0 < x < 1 \text{ by taking}$$

taking  $h = \frac{1}{4}$  and  $k = \frac{1}{5}$ . Carryout second level solutions in the time scale.

**Sol:** The wave equation in the standard form is  $c^2 u_{xx} = u_{tt}$  and

hence the given equation can be written as  $u_{xx} = u_{tt}$ , where

$$c^2 = 1 \text{ or } c = 1.$$

By data  $h = \frac{1}{4}$ , we have  $k = \frac{1}{5}$ .

The points of division of  $x$  (in  $0 < x < 1$ ) and  $t$  are  $0, \frac{1}{5}, \frac{2}{5}$ .

We have the following initial table. The values in the first and last column are zero since  $u(0, t) = 0 = u(1, t)$ .

| $x \backslash t$ |               | $x_0$         | $x_1$         | $x_2$         | $x_3$         | $x_4$         |
|------------------|---------------|---------------|---------------|---------------|---------------|---------------|
|                  |               | 0             | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | 1             |
| $t_0$            | 0             | $u_{0,0} = 0$ | $u_{1,0}$     | $u_{2,0}$     | $u_{3,0}$     | $u_{4,0} = 0$ |
| $t_1$            | $\frac{1}{5}$ | $u_{0,1} = 0$ | $u_{1,1}$     | $u_{2,1}$     | $u_{3,1}$     | $u_{4,1} = 0$ |
| $t_2$            | $\frac{2}{5}$ | $u_{0,2} = 0$ | $u_{1,2}$     | $u_{2,2}$     | $u_{3,2}$     | $u_{4,2} = 0$ |

**To find first row elements**

Consider  $u(x, 0) = \sin(\pi x)$

$$\therefore u_{1,0} = u(x_1, t_0) = u\left(\frac{1}{4}, 0\right) = \sin\left(\frac{\pi}{4}\right) = 0.7071$$

$$u_{2,0} = u(x_2, t_0) = u\left(\frac{1}{2}, 0\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

$$u_{3,0} = u(x_3, t_0) = u\left(\frac{3}{4}, 0\right) = \sin\left(\frac{3\pi}{4}\right) = 0.7071$$

Since **both  $h$  and  $k$  are small**, we use the below formula

$$\text{Now, } c^2 \lambda^2 = c^2 \frac{k^2}{h^2} \Rightarrow c^2 \lambda^2 = 1 \cdot \frac{\frac{1}{25}}{\frac{1}{16}}$$

$$c^2 \lambda^2 = \frac{16}{25}$$

**To find second row elements**

$$\text{Consider } 2u_{i,1} = 2(1 - c^2 \lambda^2)u_{i,0} + c^2 \lambda^2(u_{i-1,0} + u_{i+1,0})$$

$$2u_{i,1} = 2\left(1 - \frac{16}{25}\right)u_{i,0} + \frac{16}{25}(u_{i-1,0} + u_{i+1,0})$$

$$2u_{i,1} = \frac{18}{25}u_{i,0} + \frac{16}{25}(u_{i-1,0} + u_{i+1,0})$$

$$u_{i,1} = \frac{9}{25}u_{i,0} + \frac{8}{25}(u_{i-1,0} + u_{i+1,0})$$

$$u_{1,1} = \frac{9}{25}u_{1,0} + \frac{8}{25}(u_{0,0} + u_{2,0}) = \frac{9}{25}(0.7071) + \frac{8}{25}(0 + 1) = 0.5746$$

$$u_{2,1} = \frac{9}{25}u_{2,0} + \frac{8}{25}(u_{1,0} + u_{3,0}) = \frac{9}{25}(1) + \frac{8}{25}(0.7071 + 0.7071) = 0.8125$$

$$u_{3,1} = \frac{9}{25}u_{3,0} + \frac{8}{25}(u_{2,0} + u_{4,0}) = \frac{9}{25}(0.7071) + \frac{8}{25}(1 + 0) = 0.5746$$

**To find third row elements**

$$\text{Consider } u_{i,2} = 2(1 - c^2 \lambda^2)u_{i,1} + c^2 \lambda^2(u_{i-1,1} + u_{i+1,1}) - u_{i,0}$$

$$u_{i,2} = 2\left(1 - \frac{16}{25}\right)u_{i,1} + \frac{16}{25}(u_{i-1,1} + u_{i+1,1}) - u_{i,0}$$

$$u_{i,2} = \frac{18}{25}u_{i,1} + \frac{16}{25}(u_{i-1,1} + u_{i+1,1}) - u_{i,0}$$

$$u_{1,2} = \frac{18}{25}u_{1,1} + \frac{16}{25}(u_{0,1} + u_{2,1}) - u_{1,0} = \frac{18}{25}(0.5746) + \frac{16}{25}(0 + 0.8125) - 0.7071 = 0.2266$$

$$u_{2,2} = \frac{18}{25}u_{2,1} + \frac{16}{25}(u_{1,1} + u_{3,1}) - u_{2,0} = \frac{18}{25}(0.8125) + \frac{16}{25}(0.5746 + 0.5746) - 1 = 0.3205$$

$$u_{3,2} = \frac{18}{25}u_{3,1} + \frac{16}{25}(u_{2,1} + u_{4,1}) - u_{3,0} = \frac{18}{25}(0.5746) + \frac{16}{25}(0.8125 + 0) - 0.7071 = 0.2266$$

| $t \backslash x$ |               | $x_0$         | $x_1$         | $x_2$         | $x_3$         | $x_4$         |
|------------------|---------------|---------------|---------------|---------------|---------------|---------------|
|                  |               | 0             | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | 1             |
| $t_0$            | 0             | $u_{0,0} = 0$ | 0.7071        | 1             | 0.7071        | $u_{4,0} = 0$ |
| $t_1$            | $\frac{1}{5}$ | $u_{0,1} = 0$ | 0.5746        | 0.8125        | 0.5746        | $u_{4,1} = 0$ |
| $t_2$            | $\frac{2}{5}$ | $u_{0,2} = 0$ | 0.2266        | 0.3205        | 0.2266        | $u_{4,2} = 0$ |

7. Solve  $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$  with the boundary conditions  $u(x, 0) = 0 = u(1, t)$ ,

$t \geq 0$  and initial conditions  $u(x, 0) = \sin \pi x$ ,  $\frac{\partial u}{\partial t}(x, 0) = 0$ ,  $0 < x < 1$

taking  $h = k = 0.2$  at  $t = 1$ .

**Sol:** The wave equation in the standard form is  $c^2 u_{xx} = u_{tt}$  and

hence the given equation can be written as  $u_{xx} = u_{tt}$

Here  $c^2 = 1$  or  $c = 1$  and  $k = \frac{h}{c}$

Step size of  $x$ :  $h = 0.2$  where  $0 < x < 1$

Step size of  $t$ :  $k = 0.2$  where  $t = 1$  as required.

$\therefore$  Values of  $x$  and  $t$  are 0, 0.2, 0.4, 0.6, 0.8, 1

The values in the first and last column are zero since

$$u(0, t) = 0 = u(1, t).$$

$$u_{0,0} = 0 = u_{0,1} = u_{0,2} = u_{0,3} = u_{0,4} = u_{0,5}$$

$$u_{5,0} = 0 = u_{5,1} = u_{5,2} = u_{5,3} = u_{5,4} = u_{5,5}$$

**To find first row elements**

Consider  $u(x, 0) = \sin \pi x$

$$\therefore u_{1,0} = u(1, 0) = \sin\left(\frac{\pi}{5}\right) = 0.5878;$$

$$u_{2,0} = u(2, 0) = \sin\left(\frac{2\pi}{5}\right) = 0.9511;$$

$$u_{3,0} = u(3,0) = \sin\left(\frac{3\pi}{5}\right) = 0.9511 ;$$

$$u_{4,0} = u(4,0) = \sin\left(\frac{4\pi}{5}\right) = 0.5878 .$$

**To find second row elements**

$$\text{Consider } u_{i,1} = \frac{1}{2}[u_{i-1,0} + u_{i+1,0}]$$

$$u_{1,1} = \frac{1}{2}[u_{0,0} + u_{2,0}] = \frac{1}{2}[0 + 0.9511] = 0.4755$$

$$u_{2,1} = \frac{1}{2}[u_{1,0} + u_{3,0}] = \frac{1}{2}[0.5878 + 0.9511] = 0.7695$$

$$u_{3,1} = \frac{1}{2}[u_{2,0} + u_{4,0}] = \frac{1}{2}[0.9511 + 0.5878] = 0.7695$$

$$u_{4,1} = \frac{1}{2}[u_{3,0} + u_{5,0}] = \frac{1}{2}[0.9511 + 0] = 0.4755$$

**To find remaining row values we use**

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

**Third row elements**

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 0.7695 - 0.5878 = 0.1817$$

$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 0.4755 + 0.7695 - 0.9511 = 0.2939$$

$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 0.7695 + 0.4755 - 0.9511 = 0.2939$$

$$u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0} = 0.7695 - 0.5878 = 0.1817$$

**Fourth row elements**

$$u_{1,3} = u_{0,2} + u_{2,2} - u_{1,1} = 0 + 0.2939 - 0.4755 = -0.1816$$

$$u_{2,3} = u_{1,2} + u_{3,2} - u_{2,1} = 0.1817 + 0.2939 - 0.7695 = -0.2939$$

$$u_{3,3} = u_{2,2} + u_{4,2} - u_{3,1} = 0.2939 + 0.1817 - 0.7695 = -0.2939$$

$$u_{4,3} = u_{3,2} + u_{5,2} - u_{4,1} = 0.2939 + 0 - 0.4755 = -0.1816$$

**Fifth row elements**

$$u_{1,4} = u_{0,3} + u_{2,3} - u_{1,2} = 0 - 0.2939 - 0.1817 = -0.4756$$

$$u_{2,4} = u_{1,3} + u_{3,3} - u_{2,2} = -0.1816 - 0.2938 - 0.2939 = -0.7693$$

$$u_{3,4} = u_{2,3} + u_{4,3} - u_{3,2} = -0.2938 - 0.1816 - 0.2939 = -0.7693$$

$$u_{4,4} = u_{3,3} + u_{5,3} - u_{4,2} = -0.2939 + 0 - 0.1817 = -0.4756$$

### Last row elements

$$u_{1,5} = u_{0,4} + u_{2,4} - u_{1,3} = 0 - 0.7693 + 0.1816 = -0.5877$$

$$u_{2,5} = u_{1,4} + u_{3,4} - u_{2,3} = -0.4756 - 0.7693 + 0.2938 = -0.9511$$

$$u_{3,5} = u_{2,4} + u_{4,4} - u_{3,3} = -0.7693 - 0.4756 + 0.2938 = -0.9511$$

$$u_{4,5} = u_{3,4} + u_{5,4} - u_{4,3} = -0.7693 + 0 + 0.1816 = -0.5877$$

Thus the required values of  $t = 1$  are  $-0.5877, -0.9511, -0.9511, -0.5877$ .

Using Crank-Nicholson Method solve  $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$  subject to the condition  $u(0, t) = 0, u(1, t) = 2t, u(x, 0) = 0$  taking  $h = 0.25$  for two time steps.

$$u(0, t) = 0 \quad x = 0, t = 0$$

$$u(1, t) = 2t \quad x = 1, t = 2t$$

$$u(x, 0) = 0 \quad x = x, t = 0$$

By Crank-Nicholson Method,

$$\frac{\partial^2 u}{\partial x^2} = a \frac{\partial u}{\partial t}$$

But given that,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Therefore,  $a=1$

$$k = ah^2$$

$$k = h^2$$

$$k = (0.25)^2 = 0.0625$$

| t \ x  | 0 | 0.25        | 0.5         | 0.75        | 1                 |
|--------|---|-------------|-------------|-------------|-------------------|
| 0      | 0 | 0           | 0           | 0           | $2(0)=0$          |
| 0.0625 | 0 | a<br>0.0023 | b<br>0.0089 | c<br>0.034  | $2(0.625)=0.1250$ |
| 0.125  | 0 | d<br>0.0119 | e<br>0.0387 | f<br>0.1068 | $2(0.125)=0.250$  |

$$a = \frac{1}{4}[b] \text{ -----(1)}$$

$$b = \frac{1}{4}[a + c] \text{ -----(2)}$$

$$c = \frac{1}{4}[b + 0.125] \text{ -----(3)}$$

By Eqn (1)  $4a - b = 0$

By Eqn (2)  $-a + 4b - c = 0$

By Eqn (3)  $0 - b + 4c = 0.125$

Solving the above equations, we get,

$$a = 0.0023$$

$$b = 0.0089$$

$$c = 0.034$$

$$s d = \frac{1}{4} [b + e]$$

$$d = \frac{1}{4} [0.0089 + e] \text{-----(4)}$$

$$e = \frac{1}{4} [a + c + d + f]$$

$$e = \frac{1}{4} [0.0362 + d + f] \text{-----(5)}$$

$$f = \frac{1}{4} (b + 0.125 + 0.125 + e)$$

$$f = \frac{1}{4} (0.3884 + e) \text{-----(6)}$$

By Eqn (4)  $4d - e = 0.00893$

By Eqn (5)  $-d + 4e - f = 0.0362$

By Eqn (6)  $-e + 4f = 0.3884$

Solving the above equations, we get,

$$d = 0.0119$$

$$e = 0.0387$$

$$f = 0.1068$$

**Solve by Crank-Nicholson Method**  $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0, 0 \leq x \leq 1$  subject to the condition  $u(0, t) = 0, u(1, t) = 0, u(x, 0) = 100x(1 - x)$  taking  $h = 0.25$  for one steps.

$$u(0, t) = 0 \quad x = 0, t = 0$$

$$u(1, t) = 0 \quad x = 1, t = 0$$

$$u(x, 0) = 100x(1 - x) \quad x = x, t = 100x(1 - x)$$

By Crank-Nicholson Method,

$$\frac{\partial^2 u}{\partial x^2} = a \frac{\partial u}{\partial t}$$

But given that,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Therefore,  $a=1$

$$k = ah^2$$

$$k = h^2$$

$$k = (0.25)^2 = 0.0625$$

| $\begin{array}{c} t \\ \backslash \\ x \end{array}$ | 0 | 0.25        | 0.5          | 0.75        | 1 |
|-----------------------------------------------------|---|-------------|--------------|-------------|---|
| 0                                                   | 0 | 18.75       | 25           | 18.75       | 0 |
| 0.0625                                              | 0 | a<br>9.8214 | b<br>14.2851 | c<br>9.8214 | 0 |

$$\begin{aligned} u(x, 0) &= 100x(1 - x) \\ &= 100(0.25)(1 - 0.25) \end{aligned}$$

$$u(x, 0) = 18.75$$

$$a = \frac{1}{4}[0 + b + 0 + 25]$$

$$4a - b = 25 \text{ -----(1)}$$

$$b = \frac{1}{4}[a + c + 18.75 + 18.75]$$

$$-a - c + 4b = 37.5 \text{ -----(2)}$$

$$c = \frac{1}{4}[b + 0 + 0 + 25]$$

$$-b + 4c = 25 \text{ -----(3)}$$

Solving the above equations, we get,

$$a = 9.8214$$

$$b = 14.2857$$

$$c = 9.8214$$

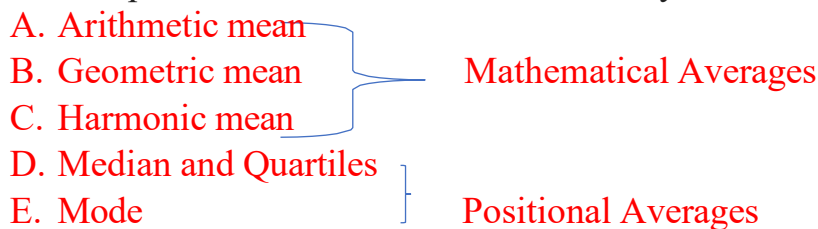


## MEASURES OF CENTRAL TENDENCY

An average or measure of central tendency gives a single representative value for a set of usually unequal values. This value is the point around which all the values cluster. So, the measure of central tendency is also called a measure of central location.

### Definition

An average is a value which is a representative of a set of data  
Various important measures of central tendency are



### Objectives or Functions of an average

- i. Averages provide a quick understanding of complex data.
- ii. Averages enable comparison
- iii. Average facilitate sampling techniques.
- iv. Averages pave the way for further statistical analysis.
- v. Averages establish the relationship between variables.

### A. ARITHMETIC MEAN

#### Definition

Arithmetic mean is the total (sum) of all values divided by the number of observations.

#### Calculation of Arithmetic mean for Raw data

When the observed values are given individually such as  $x_1, x_2, x_3, \dots, x_n$  the arithmetic mean is given by

$$\begin{aligned} \text{Arithmetic mean } \bar{X} &= \frac{\text{Total of all values}}{\text{Number of the observations}} \\ &= \frac{x_1 + x_2 + x_3 + \dots + x_n}{n} = \frac{\sum x_i}{n} \end{aligned}$$

**Example:** Given  $\bar{X}=1600$  and  $n=5$  find the total.

$$\sum x_i = n * \bar{X} = 5 * 1600 = 8000.$$

**Example:** Calculate the arithmetic mean for the following  
1600, 1590, 1560, 1610, 1640, 10.

**Solution:**

$$\text{Arithmetic mean, } \bar{X} = \frac{1600 + 1590 + 1560 + 1610 + 1640 + 10}{6}$$

$$= \frac{8010}{6} = 1335$$

**Example:** Calculate Arithmetic mean

| S.No.              | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|--------------------|----|----|----|----|----|----|----|----|----|----|
| Sales in 1000's(x) | 34 | 55 | 45 | 62 | 48 | 57 | 28 | 57 | 62 | 78 |

$$\text{Arithmetic mean, } \bar{X} = \frac{34 + 55 + 45 + 62 + 48 + 57 + 28 + 57 + 62 + 78}{10}$$

$$= \frac{526}{10} = 52.6 \text{ (average sales)}$$

**Discrete data**

Let  $x_1, x_2, x_3, \dots, x_n$  be the  $n$  values of the variable  $x$  with corresponding frequencies  $f_1, f_2, f_3, \dots, f_n$ . then the arithmetic mean  $\bar{X} = \frac{x_1 \cdot f_1 + x_2 \cdot f_2 + x_3 \cdot f_3 + \dots + x_n \cdot f_n}{f_1 + f_2 + f_3 + \dots + f_n}$

$$= \frac{\sum x_i f_i}{\sum f_i}$$
**Example:** Calculate the arithmetic mean

| x  | f                  | xf                      |
|----|--------------------|-------------------------|
| 2  | 4                  | 2x4= 8                  |
| 4  | 6                  | 24                      |
| 6  | 10                 | 60                      |
| 8  | 12                 | 96                      |
| 10 | 8                  | 80                      |
| 12 | 7                  | 84                      |
| 14 | 3                  | 42                      |
|    | $\sum f_i =$<br>50 | $\sum x_i f_i$<br>= 394 |

$$\text{Arithmetic mean, } \bar{X} = \frac{\sum x_i f_i}{\sum f_i} = \frac{394}{50} = 78.8$$

## Continuous data

Let  $m_1, m_2, m_3, \dots, m_n$  be the mid values of the class interval of the variable  $x$  with corresponding frequency  $f_1, f_2, f_3, \dots, f_n$ . then

$$\text{the arithmetic mean } \bar{X} = \frac{m_1 \cdot f_1 + m_2 \cdot f_2 + m_3 \cdot f_3 + \dots + m_n \cdot f_n}{f_1 + f_2 + f_3 + \dots + f_n}$$

$$= \frac{\sum m_i f_i}{\sum f_i}$$

**Example:** Calculate the arithmetic mean

| Class interval(x) | m                | f               | mf                    |
|-------------------|------------------|-----------------|-----------------------|
| 20-40             | $(20+40)/2 = 30$ | 4               | 120                   |
| 40-60             | 50               | 6               | 300                   |
| 60-80             | 70               | 10              | 700                   |
| 80-100            | 90               | 12              | 1080                  |
| 100-120           | 110              | 8               | 880                   |
|                   |                  | $\sum f_i = 40$ | $\sum m_i f_i = 3080$ |

$$\text{Arithmetic mean, } \bar{X} = \frac{\sum m_i f_i}{\sum f_i} = \frac{3080}{40} = 77.0$$

## Merits and Demerits of Arithmetic Mean:

### Merits:

- It is rigidly defined.
- It is easy to understand and easy to calculate.
- If the number of items is sufficiently large, it is more accurate and more reliable.
- It is a calculated value and is not based on its position in the series.
- It is possible to calculate even if some of the details of the data are lacking.
- Of all averages, it is affected least by fluctuations of sampling.
- It provides a good basis for comparison.

### Demerits:

- It cannot be obtained by inspection nor located through a frequency graph.
- It cannot be in the study of qualitative phenomena not capable of numerical measurement i.e. Intelligence, beauty, honesty etc.,
- It can ignore any single item only at the risk of losing its accuracy.
- It is affected very much by extreme values.
- It cannot be calculated for open-end classes.
- It may lead to fallacious conclusions, if the details of the data from which it is computed are not given.

## B. MEDIAN

- ✓ It is the value which divides the data into two equal parts.
- ✓ 50% of the observations will be less than median value and 50% of the values will be more than the median value.

### Calculation for Raw data

Median = value of  $(n+1)/2$  th observation after the values are arranged in ascending order of magnitude.

For example, the median of 20,30,35,64,23,46,78,34,20

Arranging the data in ascending order; 20,20,23,30,34,35,46,64,78

$$Md = \text{value of } [(9+1)/2 = 5^{\text{th}} \text{ observation}] = 34$$

Suppose the given number of observations is even then median will be the average of two central values

For example, if the data is the median of 20,30,35,64,23,46,78,34,20,56

Arranging the data in ascending order : 20,20,23,30,34,35,46,56,64,78

$$Md = \text{value of } (10+1)/2 = 5.5^{\text{th}} \text{ observation}$$

$$= (\text{value of } 5^{\text{th}} \text{ observation} + \text{value of } 6^{\text{th}} \text{ observation})/2$$

$$= (34+35)/2 = 34.5$$

### Discrete data

Md = value of x corresponding to the cumulative frequency just greater than or equal to  $N/2$

- i) Arrange the data in ascending order
- ii) Find the c.f.; Calculate  $N/2$
- iii) In c.f. column see the value just  $\geq N/2$
- iv) Md = value of x corresponding to this c.f.

**Example:** Find the median

| x  | f                 |  |
|----|-------------------|--|
| 2  | 4                 |  |
| 4  | 6                 |  |
| 6  | 10                |  |
| 8  | 12                |  |
| 10 | 8                 |  |
| 12 | 7                 |  |
| 14 | 3                 |  |
|    | $\Sigma f_i = 50$ |  |

**Solution:**

| x  | f                 | c.f. |
|----|-------------------|------|
| 2  | 4                 | 4    |
| 4  | 6                 | 10   |
| 6  | 10                | 20   |
| 8  | 12                | 32   |
| 10 | 8                 | 40   |
| 12 | 7                 | 47   |
| 14 | 3                 | 50   |
|    | $\Sigma f_i = 50$ |      |

$$N/2 = 50/2 = 25.5, \text{ Md} = 8$$

**Continuous data**

$$\text{Md} = L + \{(N/2 - \text{c.f.}) \times c/f\}$$

where

L= lower limit of the median class

c =class interval of the median class

f=frequency of the median class

c.f.= cumulative frequency of the class preceding the median class

$$N = \Sigma f_i$$

Md class is the class corresponding to the c.f. just  $\geq N/2$ .**Example:** Find the median

| Class interval(x) | f                 |
|-------------------|-------------------|
| 20-40             | 4                 |
| 40-60             | 6                 |
| 60-80             | 10                |
| 80-100            | 12                |
| 100-120           | 8                 |
|                   | $\Sigma f_i = 40$ |

**Solution:**

| Class interval(x) | f                 | c.f |
|-------------------|-------------------|-----|
| 20-40             | 4                 | 4   |
| 40-60             | 6                 | 10  |
| 60-80             | 10                | 20  |
| 80-100            | 12                | 32  |
| 100-120           | 8                 | 40  |
|                   | $\Sigma f_i = 40$ |     |

- $N/2 = 40/2 = 20$
- Median class is 60-80
- $L=60, c=80-60=20, f= 10, c.f= 10$
- $Md = L + \{(N/2 - c.f) \times c/f\} = 60 + \{(20 - 10) \times (20/10)\}$   
 $= 60 + \{10 \times 2\} = 60 + 20 = 80.$

**Example:** Find the median

| Marks  | No. of students    | c.f. |
|--------|--------------------|------|
| 10-25  | 6                  | 6    |
| 25-40  | 20                 | 26   |
| 40-55  | 44                 | 70   |
| 55-70  | 26                 | 96   |
| 70-85  | 3                  | 99   |
| 85-100 | 1                  | 100  |
|        | $\Sigma f_i = 100$ |      |

- $N/2 = 100/2 = 50$
- Median class is 40-55
- $L = 40, f = 44, c = 55 - 40 = 15, c.f. = 26$
- $Md = L + \{(N/2 - c.f) \times c/f\}$   
 $= 40 + \{[50 - 26] \times 15/44\}$   
 $= 40 + \{(24 \times 15)/44\}$   
 $= 40 + [360/44]$   
 $= 40 + 8.18 = 48.18$

## Measures of Dispersion

It helps us to study the extent of the variation of the individual observations from the central value and among themselves

For example, consider the three series

|       | Series A | Series B | Series C |
|-------|----------|----------|----------|
|       | 201      | 200      | 75       |
|       | 198      | 200      | 405      |
|       | 199      | 200      | 150      |
|       | 200      | 200      | 270      |
|       | 202      | 200      | 100      |
| total | 1000     | 1000     | 1000     |
| A.M   | 200      | 200      | 200      |

## Two Broad category of Measures of Dispersion:

- **Absolute measures:** Absolute measures are given in the same units as the

individual data

- **Relative measures:** Relative measures are free from units of measurements and are called as co-efficients and are used to compare two or more series for their variability.

**Absolute measures & their Relative measures (coefficients)**

- [1] Range and co-efficient of range
- [2] Quartile deviation (Q.D) and co-efficient of Q.D.
- [3] Mean deviation (M.D) and co-efficient of M.D.
- [4] Standard deviation (S.D) and co-efficient of variation(C.V).

### [1] RANGE & CO-EFFICIENT OF RANGE

**Definition:**

Range is defined as the difference between the highest and lowest value of x.

Range  $R = \text{largest value} - \text{smallest value} = L - S$

Co-efficient of Range  $= (L - S) / (L + S)$

**Example:**

Find the range and its co-efficient: 234, 22, 425, 325, 78, 236, 120, 422

**Solution:** Largest value  $L = 425$ ; Smallest value  $S = 22$ ; Range  
 $= 425 - 22 = 403$

$$\begin{aligned}\text{Co-efficient of Range} &= (L - S) / (L + S) \\ &= (425 - 22) / (425 + 22) \\ &= 403 / 447 = 0.90\end{aligned}$$

**Example:** Find range and its co-efficient

| X      | f  |
|--------|----|
| 10 -20 | 4  |
| 20-30  | 12 |
| 30-40  | 25 |
| 40-50  | 16 |
| 50-60  | 6  |

**Solution:**

Largest value of X ,  $L = 60$

Smallest value of X ,  $S = 10$

Range =  $L - S = 60 - 10 = 50$

Co-efficient of range =  $(L - S) / (L + S) = (60 - 10) / (60 + 10) = 50 / 70 = 0.71$

## QUARTILES

- ✓ It is the value which divides the data into FOUR equal parts.
- ✓ There are three quartiles.
- ✓  $Q_1$ , the first quartile or the lower quartile divides the data in such a way that 25 percent of the observations will be less than  $Q_1$  value and 75% of the values will be more than the  $Q_1$  value.
- ✓  $Q_3$ , the Third quartile or upper quartile divides the data in such a way that 75 percent of the observations will be less than  $Q_3$  value and 25% of the values will be more than the  $Q_3$  value
- ✓ The second quartile is nothing but the median.
- ✓ 50% of the observations will be less than median value and 50% of the values will be more than the median value.

## Calculation for Raw data

- ✓ Median = value of  $(n+1)/2$  th observation in ascending order data.
- ✓  $Q_1$  = value of  $(n+1)/4$  th observation in ascending order data.
- ✓  $Q_3$  = value of  $3(n+1)/4$  th observation in ascending order data.

For example, the median of 20,30,35,64,23,46,78,34,20

Arranging the data in ascending order : 20,20,23,30,34,35,46,64,78

$$\begin{aligned} \text{Md} &= \text{value of } (9+1)/2 = 5^{\text{th}} \text{ observation} \\ &= 34 \end{aligned}$$

**Example:** Find  $Q_1$  and  $Q_3$ , 20,30,35,64,23,46,78,34,20

Arranging the data in ascending order: 20,20,23,30,34,35,46,64,78

$$\begin{aligned} Q_1 &= \text{value of } (9+1)/4 = 2.5^{\text{th}} \text{ observation} \\ &= \text{value of } 2^{\text{nd}} \text{ observation} + 0.5(3^{\text{rd}} \text{ value} - 2^{\text{nd}} \text{ value}) \\ &= 20 + 0.5(23 - 20) = 20 + 0.5 \times 3 = 20 + 1.5 = 21.5 \end{aligned}$$

$$\begin{aligned} Q_3 &= \text{value of } 3(9+1)/4 = 7.5^{\text{th}} \text{ observation} \\ &= 7^{\text{th}} \text{ observation} + 0.5(8^{\text{th}} \text{ value} - 7^{\text{th}} \text{ value}) \\ &= 46 + 0.5(64 - 46) = 46 + (0.5 \times 18) = 46 + 9 = 55 \end{aligned}$$

Suppose the given number of observations is even then median will be the average of two central values.

**Example:** Find the median of 20,30,35,64,23,46,78,34,20,56.

Arranging the data in ascending order:

20,20,23,30,34,35,46,56,64,78

$$\begin{aligned} \text{Md} &= \text{value of } (10+1)/2 = 5.5^{\text{th}} \text{ observation} \\ &= (\text{value of } 5^{\text{th}} \text{ observation} + \text{value of } 6^{\text{th}} \text{ observation})/2 \\ &= (34+35)/2 = 34.5 \end{aligned}$$

**Example:** Find  $Q_1$  and  $Q_3$  20,30,35,64,23,46,78,34,20,56

**Solution:** Arranging the data in ascending order: 20,20,23,30,34,35,46,56,64,78

$$\begin{aligned} Q_1 &= \text{value of } (10+1)/4 = 2.75^{\text{th}} \text{ observation} \\ &= 2^{\text{nd}} \text{ value} + 0.75(3^{\text{rd}} \text{ value} - 2^{\text{nd}} \text{ value}) \\ &= 20 + 0.75(23 - 20) \\ &= 20 + (0.75 \times 3) = 20 + 2.25 = 22.25 \end{aligned}$$

$$\begin{aligned} Q_3 &= \text{value of } 3(n+1/4)^{\text{th}} \text{ observation} \\ &= (3 \times 2.75 = 8.25^{\text{th}}) \text{ observation} \\ &= 8^{\text{th}} \text{ value} + 0.25(9^{\text{th}} \text{ value} - 8^{\text{th}} \text{ value}) \\ &= 56 + 0.25(64 - 56) \\ &= 56 + 0.25(8) = 56 + 2 = 58 \end{aligned}$$

**Discrete data**

- $Md$  = value of  $x$  corresponding to the cumulative frequency just  $\geq N/2$
- $Q_1$  = value of  $x$  corresponding to the cumulative frequency just  $\geq N/4$
- $Q_3$  = value of  $x$  corresponding to the cumulative frequency just  $\geq 3N/4$
- Arrange the data is in ascending order
- Find the c.f.
- Calculate  $N/2$
- In c.f. column see the value greater than or equal to  $N/2$
- $Md$  = value of  $x$  corresponding to this c.f.

**Example:** Find the median and the quartiles

| x  | f                 |
|----|-------------------|
| 2  | 4                 |
| 4  | 6                 |
| 6  | 10                |
| 8  | 12                |
| 10 | 8                 |
| 12 | 7                 |
| 14 | 3                 |
|    | $\Sigma f_i = 50$ |

Solution:

| x  | f                 | c.f. |
|----|-------------------|------|
| 2  | 4                 | 4    |
| 4  | 6                 | 10   |
| 6  | 10                | 20   |
| 8  | 12                | 32   |
| 10 | 8                 | 40   |
| 12 | 7                 | 47   |
| 14 | 3                 | 50   |
|    | $\Sigma f_i = 50$ |      |

- $N/2 = 50/2 = 25$ ; Therefore  $Md = 8$
- $N/4 = 50/4 = 12.5$

$Q_1$  = value of  $x$  corresponding to the cumulative frequency just greater than or equal to  $N/4 = 20$ .

$$Q_1 = 6$$

- $3N/4 = 37.5$

$Q_3$  = value of  $x$  corresponding to the cumulative frequency just greater than or equal to  $3N/4$ ;  $Q_3$  = value of  $x$  corresponding to the cumulative frequency just greater than 37.5 i.e., 40

$$Q_3 = 10$$

### Continuous data

$$Md = L + \{(N/2 - c.f.) \times c/f\}$$

where  $L$  = lower limit of the median class

$c$  = class interval of the median class

$f$  = frequency of the median class

$c.f.$  = c.f. of the class preceding the median class

$$N = \sum f_i$$

$Md$  class is the class corresponding to the c.f. just  $\geq N/2$ .

$$Q_1 = L_1 + \{(N/4 - c.f_1) \times c_1/f_1\}$$

$L_1$  lower limit of the  $Q_1$  class

$c_1$  class interval of the  $Q_1$  class

$f_1$  frequency of the  $Q_1$  class

$c.f_1$  cumulative frequency of the class preceding the  $Q_1$  class

$$N = \sum f_i$$

$Q_1$  class is the class corresponding to the c.f. just greater than or equal to  $N/4$ .

$$Q_3 = L_3 + \{(3N/4 - c.f_3) \times c_3/f_3\}$$

$L_3$  lower limit of the  $Q_3$  class

$c_3$  class interval of the  $Q_3$  class

$f_3$  frequency of the  $Q_3$  class

$c.f_3$  cumulative frequency of the class preceding the  $Q_3$  class

$N = \sum f_i$   $Q_3$  class is the class corresponding to the c.f. just greater than or equal to  $3N/4$ .

**Example:** Find Median:

| Class interval(x) | f               |
|-------------------|-----------------|
| 20-40             | 4               |
| 40-60             | 6               |
| 60-80             | 10              |
| 80-100            | 12              |
| 100-120           | 8               |
|                   | $\sum f_i = 40$ |

Solution:

| Class interval(x) | f                 | c.f |
|-------------------|-------------------|-----|
| 20-40             | 4                 | 4   |
| 40-60             | 6                 | 10  |
| 60-80             | 10                | 20  |
| 80-100            | 12                | 32  |
| 100-120           | 8                 | 40  |
|                   | $\Sigma f_i = 40$ |     |

- $N/2 = 40/2 = 20$
- Median class is 60-80
- $L=60, c=80-60=20, f=10, c.f=20$
- $Md = L + \{(N/2 - c.f) \times c/f\}$   
 $= 60 + \{(20 - 10) \times (20/10)\}$   
 $= 60 + \{10 \times 2\}$   
 $= 60 + 20 = 80$

| marks  | No. of students    | C.f. |
|--------|--------------------|------|
| 10-25  | 6                  | 6    |
| 25-40  | 20                 | 26   |
| 40-55  | 44                 | 70   |
| 55-70  | 26                 | 96   |
| 70-85  | 3                  | 99   |
| 85-100 | 1                  | 100  |
|        | $\Sigma f_i = 100$ |      |

- $N/2 = 100/2 = 50$   
Median class is 40-55  
 $L = 40, f = 44, c = 55 - 40 = 15, c.f. = 26$   
 $Md = L + \{(N/2 - c.f) \times c/f\}$   
 $= 40 + \{[50 - 26] \times 15/44\}$   
 $= 40 + \{(24 \times 15)/44\}$   
 $= 40 + [360/44]$   
 $= 40 + 8.18 = 48.18$
- $Q_1 = L_1 + \{(N/4 - c.f_1) \times c_1/f_1\}$   
 $Q_1$  class 25-40,  $L_1 = 25, c_1 = 40 - 25 = 15, f_1 = 20, c.f_1 = 26$   
 $Q_1 = 25 + \{(25 - 6) (15/20)\}$   
 $= 25 + \{19 \times 15 / 20\}$   
 $= 25 + 14.25 = 39.25$

$$\begin{aligned}
 \circ \quad Q_3 &= L_3 + \{(3N/4 - c.f_3) \times c_3/f_3\} \\
 3N/4 &= 3 \times 25 = 75 \\
 Q_3 \text{ class is } &55-70 \\
 L_3 &= 55, c_3 = 70 - 55 = 15, f_3 = 26, c.f_3 = 70 \\
 Q_3 &= L_3 + \{(3N/4 - c.f_3) \times c_3/f_3\} \\
 &= 55 + \{(75 - 70) \times 15 / 26\} \\
 &= 55 + \{5 \times 0.57\} \\
 &= 55 + 2.88 = 57.88.
 \end{aligned}$$

### Merits of Median:

- i) Median is not influenced by extreme values because it is a positional average.
- ii) Median can be calculated in case of distribution with open end intervals.
- iii) Median can be located even if the data are incomplete.
- iv) Median can be located even for qualitative factors such as ability, honesty etc.

### Demerits of Median:

- i) A slight change in the series may bring drastic change in median value.
- ii) In case of even number of items or continuous series, median is an estimated value other than any value in the series.
- iii) It is not suitable for further mathematical treatment except its use in mean deviation.
- iv) It does not consider all the observations.

## C. MODE

Mode is the value of x which is repeated more often or more frequently.

### Raw data

Mode is found by observation. The number of times each value occurs is noted and the value which is repeated maximum number of times is the mode.

**Example:** Find mode 20,30,35,64,23,46,78,34,20,56

Mode is 20 as it is repeated twice while other values are repeated only once.

#### Case i) Unimodal – only one mode

In the series 40, 30, 20, 17, 18, 32, 29, 23, 17, 17, 24, 24, 12 mode is 17.

#### Case ii) Bimodal – two modes

In the series 40, 30, 20, 17, 18, 32, 29, 23, 17, 17, 24, 24, 12, 24, 23

Mode-1 = 17, mode-2 is 24,

#### Case iii) No mode:

In the series 40, 34, 45, 45, 34, 40 there is no mode or mode is ill-defined.

## Discrete data

Mode = value of x corresponding to the highest frequency

Case i) Unimodal – only one mode

| x  | f                 |
|----|-------------------|
| 2  | 4                 |
| 4  | 6                 |
| 6  | 10                |
| 8  | 12                |
| 10 | 8                 |
| 12 | 7                 |
| 14 | 3                 |
|    | $\Sigma f_i = 50$ |

Mode = value of x corresponding to the highest frequency 12

Mode = 8.

## Continuous data

Mode =  $l + \left[ \frac{(f_1 - f_0)}{(2f_1 - f_0 - f_2)} \right] \times c$

where  $f_1$  is the frequency of the modal class

$f_0$  is the frequency of the class preceding the modal class

$f_2$  is the frequency of the class succeeding the modal class

$c$  is the class interval of the modal class

$l$  is the lower limit of the modal class

Modal class the class corresponding to the highest frequency.

| marks  | No. of students    |
|--------|--------------------|
| 10-25  | 6                  |
| 25-40  | 20 $f_0$           |
| 40-55  | 44 $f_1$           |
| 55-70  | 26 $f_2$           |
| 70-85  | 3                  |
| 85-100 | 1                  |
|        | $\Sigma f_i = 100$ |

Modal class is 40 -55

$L = 40$ ,  $f_1 = 44$ ,  $f_0 = 20$ ,  $f_2 = 26$ ,  $c = 55 - 40 = 15$ .

$$\begin{aligned}
 \text{Mode} &= l + \left[ \frac{(f_1 - f_0)}{(2f_1 - f_0 - f_2)} \right] \times c \\
 &= 40 + \left[ \frac{(44 - 20)}{(2 \times 44 - 20 - 26)} \right] \times 15 \\
 &= 40 + \left[ \frac{24}{(88 - 46)} \right] \times 15 \\
 &= 40 + \left[ \frac{24}{42} \times 15 \right] \\
 &= 40 + [0.5714 \times 15] \\
 &= 40 + 8.57 \\
 &= 48.57.
 \end{aligned}$$

Relationship between mean, median and mode : **Mode = 3median - 2 mean**

## D. GEOMETRIC MEAN

### Definition:

Geometric mean of  $n$  observations is the  $n^{\text{th}}$  root of product of  $n$  observations.

If  $x_1, x_2, x_3, \dots, x_n$  be the  $n$  observations the G.M is  $(x_1 \cdot x_2 \cdot x_3 \cdot \dots \cdot x_n)^{(1/n)}$

For example, the G.M. of 2,4,8 is  $(2 \times 4 \times 8)^{(1/3)} = (64)^{(1/3)} = 4$ .

But in practice we use log to find G.M.

### Raw Data

If  $x_1, x_2, x_3, \dots, x_n$  be the  $n$  observations

$$\text{G.M.} = (x_1 \cdot x_2 \cdot x_3 \cdot \dots \cdot x_n)^{(1/n)}$$

Taking log on both sides

$$\begin{aligned}
 \log (\text{G.M.}) &= (1/n) [\log x_1 + \log x_2 + \log x_3 + \dots + \log x_n] \\
 &= (1/n) \sum [\log x_i]
 \end{aligned}$$

$$\text{G.M.} = \text{Antilog} \left\{ \frac{1}{n} \sum [\log x_i] \right\}$$

Example: Find the geometric mean for the following  $x$ : 3,6,24,48

| $x$ | $\log x$                     |
|-----|------------------------------|
| 3   | 0.4771                       |
| 6   | 0.7782                       |
| 24  | 1.3802                       |
| 48  | 1.6812                       |
|     | $\sum [\log x_i]$<br>=4.3167 |

$$\begin{aligned}
 \text{G.M.} &= \text{Antilog} \left\{ \frac{1}{n} \sum [\log x_i] \right\} \\
 &= \text{Antilog} \left\{ \frac{1}{4} \times 4.3167 \right\} \\
 &= \text{Antilog} \{1.0792\} = 12.00
 \end{aligned}$$

**Discrete data:**

Let  $x_1, x_2, x_3, \dots, x_n$  be the  $n$  values of the variable  $x$  with corresponding frequency  $f_1, f_2, f_3, \dots, f_n$ , then

$$G.M = \text{Antilog} \left( \frac{\sum [f \log x]}{\sum f} \right)$$

Find the geometric mean for the data given below

|   |    |    |    |    |    |
|---|----|----|----|----|----|
| x | 10 | 15 | 25 | 40 | 50 |
| f | 4  | 6  | 10 | 7  | 3  |

**Solution**

| x  | f         | log x  | f log x                       |
|----|-----------|--------|-------------------------------|
| 10 | 4         | 1.0000 | <b>4.0000</b>                 |
| 15 | 6         | 1.1761 | <b>7.0566</b>                 |
| 25 | 10        | 1.3979 | <b>13.9790</b>                |
| 40 | 7         | 1.6021 | <b>11.2147</b>                |
| 50 | 3         | 1.6990 | <b>5.0970</b>                 |
|    | <b>30</b> |        | $\Sigma [f \log x] = 41.3473$ |

$$G.M. = \text{Antilog} \left( \frac{\Sigma [f \log x]}{\Sigma f} \right) = \text{A.L.} [41.3473/30] = \text{A.L.}(1.3782) = 23.89$$

**Continuous data:**

Let  $m_1, m_2, m_3, \dots, m_n$  be the midpoints of the  $n$  classes of the variable  $x$  with corresponding frequency  $f_1, f_2, f_3, \dots, f_n$ . then

$$G.M = \text{Antilog} \left( \frac{\Sigma [f \log m]}{\Sigma f} \right)$$

**Example:** Compute the geometric mean

|                    |      |       |       |       |       |
|--------------------|------|-------|-------|-------|-------|
| Marks (x)          | 0-10 | 10-20 | 20-30 | 30-40 | 40-50 |
| No. of students(f) | 5    | 7     | 15    | 25    | 8     |

**Solution:**

| x     | f               | m  | log m         | f log m        |
|-------|-----------------|----|---------------|----------------|
| 0-10  | 5               | 5  | <b>0.6990</b> | <b>3.4950</b>  |
| 10-20 | 7               | 15 | <b>1.1761</b> | <b>8.2327</b>  |
| 20-30 | 15              | 25 | <b>1.3979</b> | <b>20.9685</b> |
| 30-40 | 25              | 35 | <b>1.5441</b> | <b>38.6025</b> |
| 40-50 | 8               | 45 | <b>1.6532</b> | <b>13.2256</b> |
|       | $\Sigma f = 60$ |    |               | <b>84.5243</b> |

$$G.M. = \text{Antilog}\left(\frac{\Sigma [f \log m]}{\Sigma f}\right) = A.L. [84.5243/60] = A.L. [1.4087] = 25.63$$

## E. HARMONIC MEAN (HM)

### Definition:

Harmonic mean is **the reciprocal of the arithmetic mean of the reciprocal of observation.**

**Example:** Find HM for the data: **8, 10, 40, 26:**

- Reciprocals: 8 is  $1/8$ , 10 is  $1/10$ , 40 is  $1/40$ , 26 is  $1/26$
- A.M. of  $1/8, 1/10, 1/40$  and  $1/26$  is  $(1/8 + 1/10 + 1/40 + 1/26)/4$
- H.M. =  $4/(1/8 + 1/10 + 1/40 + 1/26)$

### Raw data

If  $x_1, x_2, x_3 \dots x_n$  be the  $n$  observations, then  $H.M. = \frac{n}{\Sigma(1/x)}$

**Example:** Find the harmonic mean for the following  $x$ : 3, 6, 24, 48

| $x$ | $1/x$  |
|-----|--------|
| 3   | 0.3333 |
| 6   | 0.1667 |
| 24  | 0.0417 |
| 48  | 0.0208 |
|     | 0.5625 |

$$H.M. = \frac{n}{\Sigma(1/x)} = \frac{4}{(0.5625)} = 7.11$$

### Discrete data:

Let  $x_1, x_2, x_3 \dots x_n$  be the  $n$  values of the variable  $x$  with corresponding frequencies  $f_1, f_2, f_3 \dots f_n$ , then

$$H.M = \frac{\Sigma f}{\Sigma(f/x)}$$

**Example:** Find the harmonic mean for the data given below

| $x$ | 10 | 15 | 25 | 40 | 50 |
|-----|----|----|----|----|----|
| $f$ | 4  | 6  | 10 | 7  | 3  |

**Solution:**

| $x$ | $f$ | $f/x$  |
|-----|-----|--------|
| 10  | 4   | 0.4000 |

|    |    |               |
|----|----|---------------|
| 15 | 6  | <b>0.4000</b> |
| 25 | 10 | <b>0.4000</b> |
| 40 | 7  | <b>0.1750</b> |
| 50 | 3  | <b>0.0600</b> |

|  |           |               |
|--|-----------|---------------|
|  | <b>30</b> | <b>1.4350</b> |
|--|-----------|---------------|

$$H.M. = \left[ \frac{\Sigma f}{\Sigma(f/x)} \right] = 30/(1.4350) = 20.91$$

### Continuous data:

Let  $m_1, m_2, m_3, \dots, m_n$  be the  $n$  values of the variable  $x$  with corresponding frequency  $f_1, f_2, f_3, \dots, f_n$ . then

$$H.M = \frac{\Sigma f}{\Sigma(f/x)}$$

**Example:** Compute the geometric mean

|                    |      |       |       |       |       |
|--------------------|------|-------|-------|-------|-------|
| Marks (x)          | 0-10 | 10-20 | 20-30 | 30-40 | 40-50 |
| No. of students(f) | 5    | 7     | 15    | 25    | 8     |

**Solution:**

| x     | f               | m  | f / m         |
|-------|-----------------|----|---------------|
| 0-10  | 5               | 5  | <b>1.0000</b> |
| 10-20 | 7               | 15 | <b>0.4667</b> |
| 20-30 | 15              | 25 | <b>0.6000</b> |
| 30-40 | 25              | 35 | <b>0.7143</b> |
| 40-50 | 8               | 45 | <b>0.1778</b> |
|       | $\Sigma f = 60$ |    | <b>2.9588</b> |

s

$$H.M = \frac{\Sigma f}{\Sigma(f/m)} = \frac{60}{2.9588} = 20.28$$

### Weighted averages

The relative importance given to the given to the values is the weights  $W$

### Weighted arithmetic mean

$$\bar{x}_w = \frac{\Sigma [xw]}{\Sigma w}$$

$\Sigma w$

X is the            variable and w is the weights

Find the weighted arithmetic mean for the following data

| <b>x</b>  | <b>w</b> |
|-----------|----------|
| <b>8</b>  | <b>2</b> |
| <b>12</b> | <b>5</b> |
| <b>25</b> | <b>1</b> |
| <b>13</b> | <b>2</b> |
| <b>45</b> | <b>3</b> |

**Solution:**

| x  | w  | xw  |
|----|----|-----|
| 8  | 2  | 16  |
| 12 | 5  | 60  |
| 25 | 1  | 25  |
| 13 | 2  | 26  |
| 45 | 3  | 135 |
|    | 13 | 262 |

$$\bar{x}_w = \frac{\Sigma [xw]}{\Sigma w} = 262 / 13 = 20.15;$$

$$\text{Weighted G.M.} = \text{Antilog}\left(\frac{\Sigma [w \log x]}{\Sigma w}\right)$$

**Example:** Calculate weighted geometric mean

| commodity | A | B  | C  | D  |
|-----------|---|----|----|----|
| weight    | 1 | 6  | 3  | 2  |
| price     | 5 | 17 | 30 | 42 |

**Solution:**

| x  | w  | Log x  | w logx  |
|----|----|--------|---------|
| 5  | 1  | 0.6990 | 0.6990  |
| 17 | 6  | 1.2304 | 7.3824  |
| 30 | 3  | 1.4771 | 4.4313  |
| 42 | 2  | 1.6232 | 3.2464  |
|    | 12 |        | 15.7591 |

$$\begin{aligned} \text{Weighted G.M.} &= \text{Antilog}\left(\frac{\Sigma [w \log x]}{\Sigma w}\right) \\ &= \text{Antilog}\left(\frac{[15.7591]}{12}\right) = \text{A.L. (1.3133)} = 20.57 \end{aligned}$$

$$\text{Weighted H.M.} = \left[ \frac{\Sigma w}{\Sigma (w/x)} \right]$$

**Example:** An aeroplane flies around a square the sides of which measures 100 km each, it covers the first side at an average speed of 100 km. /hr. the second side at 200km/hr and the third with 300 kms/hr and the fourth side at 400 kms./hr. Use the correct mean to find the average speed round the square.

**Solution:**

The average speed round the entire square is the H.M of 100, 200, 300, 400.

$$H.M. = \left[ \frac{n}{\Sigma(1/x)} \right] = \frac{4}{\frac{1}{100} + \frac{1}{200} + \frac{1}{300} + \frac{1}{400}}$$

$$= \frac{4}{0.0100 + 0.0050 + 0.0033 + 0.0025}$$

$$H.M. = 4 / 0.0208 = 192 \text{ kms, /hr}$$

**Example:** You can take a trip which entails travelling 900 kms, by train at an average speed of 60km. /hr., 3000 kms by ship at an average speed of 25km./hr., 400 kms by plane at 350km./hr. and finally 15 kms by taxi at an average speed of 25km./hr, what is the average speed for the entire distance.

| Mode of travel | Distance travelled(w) | Speed (x) | w/x      |
|----------------|-----------------------|-----------|----------|
| Train          | 900                   | 60        | 15.0000  |
| Ship           | 3000                  | 25        | 120.0000 |
| Plane          | 400                   | 350       | 1.1429   |
| taxi           | 15                    | 25        | 0.6000   |
|                | 4315                  |           | 136.7429 |

Weighted H.M. is the best average to find the average speed

$$\text{Weighted H.M.} = \left[ \frac{\Sigma w}{\Sigma(w/x)} \right] = \left[ \frac{4315}{136.7429} \right] = 31.56$$

The average speed of the entire distance is 31.56 kms. /hr.

### Characteristics or desirable properties of a good average.

A measure is said to be a good average if it possesses the following characteristics:

- i) It should be simple to understand and easy to calculate.
- ii) An average should be rigidly defined.
- iii) It should be based on all items.
- iv) It should not be unduly affected by extreme values.
- v) It should lend itself for algebraic manipulation.
- vi) It should have sampling stability.

**Arithmetic Mean** is the best measure among the measures of central tendency, because it possesses almost all the characteristics of a good average.

## MODULE-5

### CURVE FITTING,CORRELATION AND REGRESSION

#### Curve Fitting

This is a method of finding a specific relation connecting the dependent and independent variables for a given data so as to satisfy the data as accurately as possible.

#### Curve fitting by the method of least squares

The method of least squares provides a relationship  $y = f(x)$  such that the sum of the squares of the residuals is least. The difference between observed( $y_i$ ) and expected value ( $Y_i$ ) i.e.,  $R_i = y_i - Y_i$  are called the residuals.

#### Fitting of a Straight line $y = ax + b$

Given  $y = ax + b$

Normal Equations(N.E's):  $a \sum x + nb = \sum y$

$$a \sum x^2 + b \sum x = \sum xy$$

Solving the above equations we get the values of  $a$  and  $b$ , substituting these values in the given equation leads to the curve of best fit.

#### Note:

Normal Equations(N.E's)for the st.line  $y = a + bx$ :

$$\begin{aligned} na + b \sum x &= \sum y \\ a \sum x + b \sum x^2 &= \sum xy \end{aligned}$$

#### Problems

1. Fit a straight line  $y = ax + b$  for the following data

|     |   |   |   |   |   |   |    |    |
|-----|---|---|---|---|---|---|----|----|
| $x$ | 1 | 3 | 4 | 6 | 8 | 9 | 11 | 14 |
| $y$ | 1 | 2 | 4 | 4 | 5 | 7 | 8  | 9  |

Sol: The normal equations for fitting the Straight line  $y = ax + b$  --- (\*) are

$$a \sum x + nb = \sum y$$

$$a \sum x^2 + b \sum x = \sum xy$$

Here  $n=8$

| $x$           | $y$           | $xy$            | $x^2$            |
|---------------|---------------|-----------------|------------------|
| 1             | 1             | 1               | 1                |
| 3             | 2             | 6               | 9                |
| 4             | 4             | 16              | 16               |
| 6             | 4             | 24              | 36               |
| 8             | 5             | 40              | 64               |
| 9             | 7             | 63              | 81               |
| 11            | 8             | 88              | 121              |
| 14            | 9             | 126             | 196              |
| $\sum x = 56$ | $\sum y = 40$ | $\sum xy = 364$ | $\sum x^2 = 524$ |

The N.E's become

$$56a + 8b = 40$$

$$524a + 56b = 364$$

On solving , we get

$$a = 0.6363 \quad \text{and} \quad b = 0.5454$$

Thus, (\*)  $\Rightarrow y = (0.6363)x + (0.5454)$  , is the curve of best fit.

2. Find the equation of the best fitting straight line for the following data and hence estimate the value of dependent variable corresponding to the value 30 of the independent variable.

| $x$ | 5  | 10 | 15 | 20 | 25 |
|-----|----|----|----|----|----|
| $y$ | 16 | 19 | 23 | 26 | 30 |

**Sol:** Let  $y = ax + b$  --- (\*) be the equation of the Straight line.

$$\text{N.E's : } a \sum x + nb = \sum y$$

$$a \sum x^2 + b \sum x = \sum xy$$

Here n=5

| $x$ | $y$ | $xy$ | $x^2$ |
|-----|-----|------|-------|
|-----|-----|------|-------|

|               |                |                  |                   |
|---------------|----------------|------------------|-------------------|
| 5             | 16             | 80               | 25                |
| 10            | 19             | 190              | 100               |
| 15            | 23             | 345              | 225               |
| 20            | 26             | 520              | 400               |
| 25            | 30             | 750              | 625               |
| $\sum x = 75$ | $\sum y = 114$ | $\sum xy = 1885$ | $\sum x^2 = 1375$ |

The N.E's become

$$\begin{aligned} 75a + 5b &= 114 \\ 1375a + 75b &= 1885 \end{aligned}$$

On solving , we get

$$a = 0.7 \quad \text{and} \quad b = 12.3$$

Thus, (\*)  $\Rightarrow y = (0.7)x + (12.3)$  , is the curve of best fit.

When  $x = 30 \Rightarrow y = (0.7)(30) + (12.3)$

$$y = 33.3$$

3. Fit a straight line in the least square sense for the following data

|          |           |           |            |            |
|----------|-----------|-----------|------------|------------|
| <b>x</b> | <b>50</b> | <b>70</b> | <b>100</b> | <b>120</b> |
| <b>y</b> | 12        | 15        | 21         | 25         |

**Sol:** Let  $y = ax + b$  --- (\*) be the equation of the Straight line.

$$\begin{aligned} \text{N.E's : } a \sum x + nb &= \sum y \\ a \sum x^2 + b \sum x &= \sum xy \end{aligned}$$

Here n=4

|          |          |           |                      |
|----------|----------|-----------|----------------------|
| <b>x</b> | <b>y</b> | <b>xy</b> | <b>x<sup>2</sup></b> |
|----------|----------|-----------|----------------------|

|                |               |                  |                    |
|----------------|---------------|------------------|--------------------|
| 50             | 12            | 600              | 2500               |
| 70             | 15            | 1050             | 4900               |
| 100            | 21            | 2100             | 10000              |
| 120            | 25            | 3000             | 14400              |
| $\sum x = 340$ | $\sum y = 73$ | $\sum xy = 6750$ | $\sum x^2 = 31800$ |

The N.E's become

$$\begin{aligned} 340a + 4b &= 73 \\ 13800a + 340b &= 6750 \end{aligned}$$

On solving , we get

$$a = 0.19 \quad \text{and} \quad b = 2.28$$

Thus, (\*)  $\Rightarrow y = (0.19)x + (2.28)$  , is the curve of best fit.

4. A simply supported beam carries a concentrated load  $P$  at its mid point. Corresponding to various values of  $P$  the maximum deflection  $Y$  is measured and is given in the following table.

|     |      |      |      |      |      |      |
|-----|------|------|------|------|------|------|
| $P$ | 100  | 120  | 140  | 160  | 180  | 200  |
| $Y$ | 0.45 | 0.55 | 0.60 | 0.70 | 0.80 | 0.85 |

Find a law of the form  $Y = a + bP$  and estimate  $Y$  when  $P$  is 150.

**Sol:** Let  $Y = a + bP$  --- (\*) be the equation of the Straight line.

$$\begin{aligned} \text{N.E's : } na + b \sum P &= \sum Y \\ a \sum P + b \sum P^2 &= \sum PY \end{aligned}$$

Here  $n = 6$

|     |     |      |       |
|-----|-----|------|-------|
| $P$ | $Y$ | $PY$ | $P^2$ |
|-----|-----|------|-------|

|                |                 |                 |                     |
|----------------|-----------------|-----------------|---------------------|
| 100            | 0.45            | 45              | 10000               |
| 120            | 0.55            | 66              | 14400               |
| 140            | 0.60            | 84              | 19600               |
| 160            | 0.70            | 112             | 25600               |
| 180            | 0.80            | 144             | 32400               |
| 200            | 0.85            | 170             | 40000               |
| $\sum P = 900$ | $\sum Y = 3.95$ | $\sum PY = 621$ | $\sum P^2 = 142000$ |

The N.E's become

$$6a + 900b = 3.95$$

$$900a + 142000b = 621$$

On solving , we get

$$a = 0.0476 \quad \text{and} \quad b = 0.0041$$

Thus, (\*)  $\Rightarrow Y = (0.0476) + (0.0041)P$  , is the curve of best fit.

$$\text{When } P = 150 \Rightarrow Y = (0.0476) + (0.0041)(150)$$

$$Y = 0.6626$$

5. Fit a straight line to the following data.

| Year                 | 1961 | 1971 | 1981 | 1991 | 2001 |
|----------------------|------|------|------|------|------|
| Production (in tons) | 8    | 10   | 12   | 10   | 16   |

Also find the expected production in the year 2006.

**Sol:** Let year and production respectively be represented by the variables **x** and **y**.

Let the equation of Straight be **y = a + bx**

Let **X = x - 1981** and the line of fit will be **y = a + bX** --- (\*)

N.E's are  $na + b \sum X = \sum y$   
 $a \sum X + b \sum X^2 = \sum Xy$  . Here n=5

| $x$  | $X$<br>$= x - 1981$ | $y$           | $Xy$                 | $X^2$                  |
|------|---------------------|---------------|----------------------|------------------------|
| 1961 | -20                 | 8             | -160                 | 400                    |
| 1971 | -10                 | 10            | -100                 | 100                    |
| 1981 | 0                   | 12            | 0                    | 0                      |
| 1991 | 10                  | 10            | 100                  | 100                    |
| 2001 | 20                  | 16            | 320                  | 400                    |
|      | $\sum X = 0$        | $\sum y = 56$ | $\sum Xy$<br>$= 160$ | $\sum X^2$<br>$= 1000$ |

N.E's becomes  $5a + 0.16b = 56$   
 $0.1a + 1000b = 160$

On solving , we get

$$a = 11.2 \quad \text{and} \quad b = 0.16$$

Thus, (\*)  $\Rightarrow y = 11.2 + (0.16)X$

$$y = 11.2 + (0.16)(x - 1981)$$

$y = -305.76 + (0.16)x$  , is the curve of best fit.

When  $x = 2006$

$$y = -305.76 + (0.16)(2006) = 15.2 \text{ tons}$$

Expected production in the year 2006 is 15.2 tons.

### Fitting of a Second degree parabola $y = ax^2 + bx + c$

Given  $y = ax^2 + bx + c$

Normal Equations(N.E's):  $a \sum x^2 + b \sum x + nc = \sum y$

$$a \sum x^3 + b \sum x^2 + c \sum x = \sum xy$$

$$a \sum x^4 + b \sum x^3 + c \sum x^2 = \sum x^2y$$

Solving the above equations we get the values of  $a$  ,  $b$  and  $c$ , substituting these vales in the given equation leads to the curve of best fit.

**Note:** If  $y = a + bx + cx^2$

**Normal Equations(N.E's):**  $na + b \sum x + c \sum x^2 = \sum y$

$$a \sum x + b \sum x^2 + c \sum x^3 = \sum xy$$

$$a \sum x^2 + b \sum x^3 + c \sum x^4 = \sum x^2y$$

### Problems

1. Fit a second degree parabola  $y = ax^2 + bx + c$  in the least square sense for the following

| $x$ | 1  | 2  | 3  | 4  | 5  |
|-----|----|----|----|----|----|
| $y$ | 10 | 12 | 13 | 16 | 19 |

data and hence estimate  $y$  at  $x = 6$ .

**Sol:** Given  $y = ax^2 + bx + c$

**Normal Equations(N.E's):**  $a \sum x^2 + b \sum x + nc = \sum y$

$$a \sum x^3 + b \sum x^2 + c \sum x = \sum xy$$

$$a \sum x^4 + b \sum x^3 + c \sum x^2 = \sum x^2y$$

Here  $n=5$

| $x$           | $y$           | $xy$            | $x^2$           | $x^2y$            | $x^3$            | $x^4$            |
|---------------|---------------|-----------------|-----------------|-------------------|------------------|------------------|
| 1             | 10            | 10              | 1               | 10                | 1                | 1                |
| 2             | 12            | 24              | 4               | 48                | 8                | 16               |
| 3             | 13            | 39              | 9               | 117               | 27               | 81               |
| 4             | 16            | 64              | 16              | 256               | 64               | 256              |
| 5             | 19            | 95              | 25              | 475               | 125              | 625              |
| $\sum x = 15$ | $\sum y = 70$ | $\sum xy = 232$ | $\sum x^2 = 55$ | $\sum x^2y = 906$ | $\sum x^3 = 225$ | $\sum x^4 = 979$ |

The N.E's become

$$\begin{aligned} 55a + 15b + 5c &= 70 \\ 225a + 55b + 15c &= 232 \\ 979a + 225b + 55c &= 906 \end{aligned}$$

On Solving, we have

$$a = 0.2857 \quad b = 0.4857 \quad \text{and} \quad c = 9.4$$

Thus, (\*)  $\Rightarrow y = (0.2857)x^2 + (0.4857)x + (9.4)$ , is the curve of best fit .

At  $x = 6$ ,

$$y = (0.2857)(6)^2 + (0.4857)(6) + (9.4)$$

$$y = 22.78$$

2. Fit a second degree parabola  $y = a + bx + cx^2$  for the data:

| $x$ | 0 | 1   | 2   | 3   | 4   |
|-----|---|-----|-----|-----|-----|
| $y$ | 1 | 1.8 | 1.3 | 2.5 | 6.3 |

**Sol:** Given  $y = a + bx + cx^2$

Normal Equations(N.E's):  $na + b \sum x + c \sum x^2 = \sum y$

$$a \sum x + b \sum x^2 + c \sum x^3 = \sum xy$$

$$a \sum x^2 + b \sum x^3 + c \sum x^4 = \sum x^2y$$

Here  $n=5$

| $x$           | $y$             | $xy$             | $x^2$           | $x^2y$              | $x^3$            | $x^4$            |
|---------------|-----------------|------------------|-----------------|---------------------|------------------|------------------|
| 0             | 1               | 0                | 0               | 0                   | 0                | 0                |
| 1             | 1.8             | 1.8              | 1               | 1.8                 | 1                | 1                |
| 2             | 1.3             | 2.6              | 4               | 5.2                 | 8                | 16               |
| 3             | 2.5             | 7.5              | 9               | 22.5                | 27               | 81               |
| 4             | 6.3             | 25.2             | 16              | 100.8               | 64               | 256              |
| $\sum x = 10$ | $\sum y = 12.9$ | $\sum xy = 37.1$ | $\sum x^2 = 30$ | $\sum x^2y = 130.3$ | $\sum x^3 = 100$ | $\sum x^4 = 354$ |

The N.E's become

$$\begin{aligned}5a + 10b + 30c &= 12.9 \\10a + 30b + 100c &= 37.1 \\30a + 100b + 354c &= 130.3\end{aligned}$$

On Solving, we have

$$a = 1.42 \quad b = -1.07 \quad \text{and} \quad c = 0.55$$

Thus, (\*)  $\Rightarrow y = (1.42) + (-1.07)x + (0.55)x^2$ , is the curve of best fit .

3. Fit a parabola  $y = ax^2 + bx + c$  in the least square sense for the following data

| $x$ | 2    | 4     | 6     | 8     | 10    |
|-----|------|-------|-------|-------|-------|
| $y$ | 3.07 | 12.85 | 31.47 | 57.38 | 91.29 |

**Sol:** Given  $y = ax^2 + bx + c$

Normal Equations(N.E's):

$$\begin{aligned}a \sum x^2 + b \sum x + nc &= \sum y \\a \sum x^3 + b \sum x^2 + c \sum x &= \sum xy \\a \sum x^4 + b \sum x^3 + c \sum x^2 &= \sum x^2y\end{aligned}$$

Here  $n=5$

| $x$              | $y$                  | $xy$                  | $x^2$               | $x^2y$                    | $x^3$                | $x^4$                 |
|------------------|----------------------|-----------------------|---------------------|---------------------------|----------------------|-----------------------|
| 2                | 3.07                 | 6.14                  | 4                   | 12.28                     | 8                    | 16                    |
| 4                | 12.85                | 51.40                 | 16                  | 205.60                    | 64                   | 256                   |
| 6                | 31.47                | 188.82                | 36                  | 1132.92                   | 216                  | 1296                  |
| 8                | 57.38                | 459.04                | 64                  | 3672.32                   | 512                  | 4096                  |
| 10               | 91.29                | 912.90                | 100                 | 9129.00                   | 1000                 | 10000                 |
| $\sum x$<br>= 30 | $\sum y$<br>= 196.06 | $\sum xy$<br>= 1618.3 | $\sum x^2$<br>= 220 | $\sum x^2y$<br>= 14152.12 | $\sum x^3$<br>= 1800 | $\sum x^4$<br>= 15664 |

The N.E's become

$$\begin{aligned}220a + 30b + 5c &= 196.06 \\1800a + 220b + 30c &= 1618.3 \\15664a + 1800b + 220c &= 14152.12\end{aligned}$$

On Solving, we have

$$a = 0.992 \quad b = -0.8551 \quad \text{and} \quad c = 0.696$$

Thus, (\*)  $\Rightarrow y = (0.992)x^2 + (-0.8551)x + (0.696)$ , is the curve of best fit.

4. Find the best values of a, b, c if the equation  $y = a + bx + cx^2$  is to fit most closely to the following observations

| $x$ | -2     | -1     | 0     | 1     | 2     |
|-----|--------|--------|-------|-------|-------|
| $y$ | -3.150 | -1.390 | 0.620 | 2.880 | 5.378 |

**Sol:** Equation of the parabola  $y = a + bx + cx^2$

Normal Equations(N.E's):

$$na + b \sum x + c \sum x^2 = \sum y$$

$$a \sum x + b \sum x^2 + c \sum x^3 = \sum xy$$

$$a \sum x^2 + b \sum x^3 + c \sum x^4 = \sum x^2 y$$

Here  $n=5$

| $x$          | $y$              | $xy$               | $x^2$           | $x^2 y$               | $x^3$          | $x^4$           |
|--------------|------------------|--------------------|-----------------|-----------------------|----------------|-----------------|
| -2           | -3.150           | 6.30               | 4               | -12.60                | -8             | 16              |
| -1           | -1.390           | 1.39               | 1               | -1.39                 | -1             | 1               |
| 0            | 0.620            | 0                  | 0               | 0                     | 0              | 0               |
| 1            | 2.880            | 2.88               | 1               | 2.88                  | 1              | 1               |
| 2            | 5.378            | 10.756             | 4               | 21.512                | 8              | 16              |
| $\sum x = 0$ | $\sum y = 4.338$ | $\sum xy = 21.326$ | $\sum x^2 = 10$ | $\sum x^2 y = 10.402$ | $\sum x^3 = 0$ | $\sum x^4 = 34$ |

The N.E's become

$$5a + 0.b + 10c = 4.338$$

$$0.a + 10b + 0.c = 21.326$$

$$10a + 0.b + 34c = 10.402$$

On Solving, we have

$$a = 0.6210 \quad b = 2.1326 \quad \text{and} \quad c = 0.1233$$

5. Fit a parabola for the following data.

| $x$ | 1 | 2 | 3 | 4 | 5  | 6  | 7  | 8  | 9 |
|-----|---|---|---|---|----|----|----|----|---|
| $y$ | 2 | 6 | 7 | 8 | 10 | 11 | 11 | 10 | 9 |

**Sol:** Equation of the parabola  $y = a + bx + cx^2$

Normal Equations(N.E's):

$$\begin{aligned} na + b \sum x + c \sum x^2 &= \sum y \\ a \sum x + b \sum x^2 + c \sum x^3 &= \sum xy \\ a \sum x^2 + b \sum x^3 + c \sum x^4 &= \sum x^2 y \end{aligned}$$

Here  $n=9$

| $x$              | $y$              | $xy$               | $x^2$               | $x^2y$             | $x^3$                | $x^4$                 |
|------------------|------------------|--------------------|---------------------|--------------------|----------------------|-----------------------|
| 1                | 2                | 2                  | 1                   | 2                  | 1                    | 1                     |
| 2                | 6                | 12                 | 4                   | 24                 | 8                    | 16                    |
| 3                | 7                | 21                 | 9                   | 63                 | 27                   | 81                    |
| 4                | 8                | 32                 | 16                  | 128                | 64                   | 256                   |
| 5                | 10               | 50                 | 25                  | 250                | 125                  | 625                   |
| 6                | 11               | 66                 | 36                  | 396                | 216                  | 1296                  |
| 7                | 11               | 77                 | 49                  | 539                | 343                  | 2401                  |
| 8                | 10               | 80                 | 64                  | 640                | 512                  | 4096                  |
| 9                | 9                | 81                 | 81                  | 729                | 729                  | 6561                  |
| $\sum x$<br>= 45 | $\sum y$<br>= 74 | $\sum xy$<br>= 421 | $\sum x^2$<br>= 285 | $\sum x^2y = 2771$ | $\sum x^3$<br>= 2025 | $\sum x^4$<br>= 15333 |

The N.E's becomes

$$\begin{aligned}9a + 45b + 285c &= 74 \\45a + 285b + 2025c &= 421 \\285a + 2025b + 15333c &= 2771\end{aligned}$$

On Solving, we have

$$a = -0.9286 \quad b = 3.5232 \quad \text{and} \quad c = -0.2673$$

Thus, (\*)  $\Rightarrow y = (-0.9286) + (3.5232)x + (-0.2673)x^2$ , is the curve of best fit .

6. Fit a second degree parabola to the following data

| x | 1   | 1.5 | 2   | 2.5 | 3   | 3.5 | 4   |
|---|-----|-----|-----|-----|-----|-----|-----|
| y | 1.1 | 1.3 | 1.6 | 2   | 2.7 | 3.4 | 4.1 |

**Sol:** Equation of the parabola  $y = a + bx + cx^2$

Normal Equations(N.E's):

$$\begin{aligned}na + b \sum x + c \sum x^2 &= \sum y \\a \sum x + b \sum x^2 + c \sum x^3 &= \sum xy \\a \sum x^2 + b \sum x^3 + c \sum x^4 &= \sum x^2 y\end{aligned}$$

Here n=7

| x                  | y                  | xy                   | $x^2$                 | $x^2y$                | $x^3$                   | $x^4$                    |
|--------------------|--------------------|----------------------|-----------------------|-----------------------|-------------------------|--------------------------|
| 1                  | 1.1                | 1.1                  | 1                     | 1.1                   | 1                       | 1                        |
| 1.5                | 1.3                | 1.95                 | 2.25                  | 2.95                  | 3.375                   | 5.0625                   |
| 2                  | 1.6                | 3.2                  | 4                     | 6.4                   | 8                       | 16                       |
| 2.5                | 2                  | 5                    | 6.25                  | 12.5                  | 15.625                  | 39.0625                  |
| 3                  | 2.7                | 8.1                  | 9                     | 24.3                  | 27                      | 81                       |
| 3.5                | 3.4                | 11.9                 | 12.25                 | 41.65                 | 42.875                  | 150.0625                 |
| 4                  | 4.1                | 16.4                 | 16                    | 65.6                  | 64                      | 256                      |
| $\sum x$<br>= 17.5 | $\sum y$<br>= 16.2 | $\sum xy$<br>= 47.65 | $\sum x^2$<br>= 50.75 | $\sum x^2y$ = 154.475 | $\sum x^3$<br>= 161.875 | $\sum x^4$<br>= 548.1875 |

The N.E's becomes

$$\begin{aligned}7a + 17.5b + 50.75c &= 16.2 \\17.5a + 50.75b + 161.875c &= 47.65 \\50.75a + 161.875b + 548.1875c &= 154.475\end{aligned}$$

On Solving, we have

$$a = 1.0357 \quad b = -0.1928 \quad \text{and} \quad c = 0.2428$$

Thus, (\*)  $\Rightarrow y = (1.0357) + (-0.1928)x + (0.2428)x^2$ , is the curve of best fit .

### **Fitting of a curve of the form $y = ax^b$**

Consider  $y = ax^b$

Taking  $\log_e$  on both sides we get,

$$\log_e y = \log_e(ax^b)$$

$$\log_e y = \log_e a + \log_e x^b$$

$$\log_e y = \log_e a + b \log_e x$$

$$Y = A + bX \quad , \text{ where } Y = \log_e y \quad , A = \log_e a \quad X = \log_e x$$

Normal Equations(N.E's)for the st.line  $Y = A + bX$ :

$$\begin{aligned}nA + b \sum X &= \sum Y \\A \sum X + b \sum X^2 &= \sum XY\end{aligned}$$

Solving above equations we get  $A, b$ . But  $\log_e a = A \Rightarrow a = e^A$

### **Problems**

1. Fit a least square geometric curve  $y = ax^b$  for the following data

| $x$ | 1   | 2 | 3   | 4 | 5    |
|-----|-----|---|-----|---|------|
| $y$ | 0.5 | 2 | 4.5 | 8 | 12.5 |

**Sol:** Consider  $y = ax^b$  --- (\*)

Taking  $\log_e$  on both sides we get,

$$\log_e y = \log_e(ax^b)$$

$$\log_e y = \log_e a + \log_e x^b$$

$$\log_e y = \log_e a + b \log_e x$$

$$Y = A + bX \quad , \text{ where } Y = \log_e y \quad , A = \log_e a \quad X = \log_e x$$

Normal Equations(N.E's)for the st.line  $Y = A + bX$

$$\begin{aligned} nA + b \sum X &= \sum Y \\ A \sum X + b \sum X^2 &= \sum XY \end{aligned}$$

Here n=5

| $x$ | $y$  | $X = \log_e x$       | $Y = \log_e y$       | $XY$                  | $X^2$                  |
|-----|------|----------------------|----------------------|-----------------------|------------------------|
| 1   | 0.5  | 0                    | -0.6931              | 0                     | 0                      |
| 2   | 2    | 0.6931               | 0.6931               | 0.4804                | 0.4804                 |
| 3   | 4.5  | 1.0986               | 1.5041               | 1.6524                | 1.2069                 |
| 4   | 8    | 1.3863               | 2.0794               | 2.8827                | 1.9218                 |
| 5   | 12.5 | 1.6094               | 2.5257               | 4.0649                | 2.5902                 |
|     |      | $\sum X$<br>= 4.7874 | $\sum Y$<br>= 6.1092 | $\sum XY$<br>= 9.0804 | $\sum X^2$<br>= 6.1933 |

N.E's becomes  $5A + 4.7874b = 6.1092$   
 $4.7874A + 6.1993b = 9.0804$

On Solving, we have

$$A = -0.6932 \quad b = 2$$

But  $\log_e a = -0.6932 \Rightarrow a = e^{-0.6932} = 0.5$

Thus, (\*)  $\Rightarrow y = ax^b$

$y = (0.5)x^2$ , is the curve of best fit.

2. An experiment of lifetime  $t$  of cutting tool at different cutting speeds  $v$  (units) are given below

| Speed ( $v$ ) | 350 | 400 | 500 | 600 |
|---------------|-----|-----|-----|-----|
| Life ( $t$ )  | 61  | 26  | 7   | 2.6 |

Fit a relation of the form  $v = at^b$ .

**Sol: Consider  $v = at^b$  --- (\*)**

Taking  $\log_e$  on both sides we get,

$$\log_e v = \log_e(at^b)$$

$$\log_e v = \log_e a + \log_e t^b$$

$$\log_e v = \log_e a + b \log_e t$$

$$V = A + bT \quad , \text{ where } V = \log_e v \quad , A = \log_e a \quad T = \log_e t$$

Normal Equations(N.E's)for the st.line  $V = A + bT$

$$\begin{aligned} nA + b \sum T &= \sum V \\ A \sum T + b \sum T^2 &= \sum VT \end{aligned}$$

Here  $n=4$

| v   | t   | $V = \log_e v$        | $T = \log_e t$        | VT                     | $T^2$                   |
|-----|-----|-----------------------|-----------------------|------------------------|-------------------------|
| 350 | 61  | 5.8579                | 4.1109                | 24.0812                | 16.8995                 |
| 400 | 26  | 5.9915                | 3.2581                | 19.5209                | 10.6152                 |
| 500 | 7   | 6.2146                | 1.9459                | 12.093                 | 3.7865                  |
| 600 | 2.6 | 6.3969                | 0.9555                | 6.1122                 | 0.913                   |
|     |     | $\sum V$<br>= 24.4609 | $\sum T$<br>= 10.2704 | $\sum VT$<br>= 61.8073 | $\sum T^2$<br>= 32.2142 |

$$\begin{aligned} \text{N.E's becomes} \quad 4A + 10.2704b &= 24.4609 \\ 10.2704A + 32.2142b &= 61.8073 \end{aligned}$$

Solving above equations we get,

$$\mathbf{A = 6.5539} \quad \mathbf{b = -0.1709}$$

$$\text{But } \log_e a = 6.5539 \Rightarrow \mathbf{a = e^{6.5539} = 701.9766}$$

$$\text{Thus , (*) } \Rightarrow \mathbf{v = at^b}$$

$$\mathbf{v = (701.9766)t^{-0.1709} , \text{ is the curve of best fit.}}$$

3. Fit a curve of the form  $y = ax^b$  for the following data

|     |      |      |      |     |     |     |
|-----|------|------|------|-----|-----|-----|
| $x$ | 1    | 2    | 3    | 4   | 5   | 6   |
| $y$ | 2.98 | 4.26 | 5.21 | 6.1 | 6.8 | 7.5 |

## Statistical Methods

Statistics is a science of facts and figures which may be readily available or obtained through the process of direct enquiry or enumeration. It deals with the methods of collecting, classifying and analysing the data so as to draw some valid conclusion.

‘Correlation’ can be understood as co-relation. Co-variance of two independent magnitudes is known as correlation. ‘Regression’ is an estimation of one independent variable in terms of the other.

### Correlation and Regression

This topic deals with data concerning independent observations.

Examples

1. Marks of individuals in two subjects
2. Height and weight of individuals
3. Amount involved in advertising a product and sale of product etc.,

We discuss the aspect of inter-relation between the independent variables

### Correlation and Correlation Coefficient

Co-variation of two independent magnitudes is known as **correlation**. If two variables  $x$  and  $y$  are related in such a way that increase or decrease in one of them corresponds to increase or decrease in the other, we say that the variables are positively correlated. Also if increase or decrease in one of them corresponds to decrease or increase in the other, the variables are negatively correlated.

The numerical measure of correlation between two variables  $x$  and  $y$  is known as Karl Pearson’s coefficient of correlation usually by  $r$  and is defined as follows

$$r = \frac{\sum XY}{\sqrt{\sum X^2} \cdot \sqrt{\sum Y^2}} \quad \text{where, } X = x - \bar{x}, Y = y - \bar{y}, \bar{x} = \frac{\sum x}{n}, \bar{y} = \frac{\sum y}{n}$$

**Note:**

$$1. \sigma_x^2 = \frac{\sum X^2}{n} \quad ; \quad \sigma_y^2 = \frac{\sum Y^2}{n} \quad ; \quad \sigma_x \sigma_y = \frac{\sqrt{\sum X^2} \cdot \sqrt{\sum Y^2}}{n} \quad \text{or} \quad n \sigma_x \sigma_y = \sqrt{\sum X^2} \cdot \sqrt{\sum Y^2}$$

2. The coefficient of correlation numerically does not exceed unity i.e.,  $-1 \leq r \leq 1$

3. If  $r = \pm 1$ , we say that  $x$  and  $y$  are perfectly correlated and if  $r = 0$  we say that  $x$  and  $y$

are non correlated.

$$4. r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y}$$

## Regression

Regression is an estimation of one independent variable in terms of the other. If x and y are correlated, the best fitting straight line in the least square sense gives reasonably a good relation between x and y .

The best fitting straight line of the form  $y = ax + b$  is called the regression line of y on x and  $x = ay + b$  is called the regression line of x on y .

The regression line of **y on x** is

$$(y - \bar{y}) = r \frac{\sigma_y}{\sigma_x} (x - \bar{x})$$

The regression line of **x on y** is

$$(x - \bar{x}) = r \frac{\sigma_x}{\sigma_y} (y - \bar{y})$$

$r \frac{\sigma_x}{\sigma_y}$  ;  $r \frac{\sigma_y}{\sigma_x}$  are known as regression coefficients.

**Note:**

1.  $r = \pm \sqrt{(\text{coeff. of } x) \cdot (\text{coeff. of } y)}$  , the sign of r will be positive or negative according as the regression coefficients are positive or negative.
2. The lines of regression are also of the form  $Y = \frac{\sum XY}{\sum X^2} (X)$  and  $X = \frac{\sum XY}{\sum Y^2} (Y)$

## Problems

1. Compute the coefficient of correlation and the equation of the lines of regression for the data

| x | 1 | 2 | 3  | 4  | 5  | 6  | 7  |
|---|---|---|----|----|----|----|----|
| y | 9 | 8 | 10 | 12 | 11 | 13 | 14 |

**Sol:** Now  $\bar{x} = \frac{\sum x}{n}$  ,  $\bar{y} = \frac{\sum y}{n}$

Here  $\sum x = 28$   $\sum y = 77$  ,  $n = 7$

$$\bar{x} = \frac{28}{7} \quad \bar{y} = \frac{77}{7}$$

$$\bar{x} = 4 \quad \bar{y} = 11$$

| <b>x</b> | <b>y</b> | <b>X = x - <math>\bar{x}</math><br/>= x - 4</b> | <b>Y = y - <math>\bar{y}</math><br/>= y - 11</b> | <b>XY</b>         | <b>X<sup>2</sup></b> | <b>Y<sup>2</sup></b> |
|----------|----------|-------------------------------------------------|--------------------------------------------------|-------------------|----------------------|----------------------|
| 1        | 9        | -3                                              | -2                                               | 6                 | 9                    | 4                    |
| 2        | 8        | -2                                              | -3                                               | 6                 | 4                    | 9                    |
| 3        | 10       | -1                                              | -1                                               | 1                 | 1                    | 1                    |
| 4        | 12       | 0                                               | 1                                                | 0                 | 0                    | 1                    |
| 5        | 11       | 1                                               | 0                                                | 0                 | 1                    | 0                    |
| 6        | 13       | 2                                               | 2                                                | 4                 | 4                    | 4                    |
| 7        | 14       | 3                                               | 3                                                | 9                 | 9                    | 9                    |
|          |          |                                                 |                                                  | $\sum XY$<br>= 26 | $\sum X^2$<br>= 28   | $\sum Y^2$<br>= 28   |

Wkt,  $r = \frac{\sum XY}{\sqrt{\sum X^2} \cdot \sqrt{\sum Y^2}}$

$$r = \frac{26}{\sqrt{28} \cdot \sqrt{28}}$$

$$r = \frac{26}{28}$$

$$r = 0.9286$$

The lines regression are given by

$$(y - \bar{y}) = r \frac{\sigma_y}{\sigma_x} (x - \bar{x}) \qquad r \frac{\sigma_y}{\sigma_x} = \frac{\sum XY}{\sum X^2} = \frac{26}{28} = 0.9286$$

$$(y - 11) = 0.9286(x - 4)$$

$$y - 11 = 0.9286x - 3.7144$$

$$y = 0.9286x + 7.2856$$

$$\text{Also, } (x - \bar{x}) = r \frac{\sigma_x}{\sigma_y} (y - \bar{y}) \qquad r \frac{\sigma_x}{\sigma_y} = \frac{\sum XY}{\sum Y^2} = \frac{26}{28} = 0.9286$$

$$(x - 4) = 0.9286(y - 11)$$

$$x - 4 = 0.9286y - 10.2146$$

$$x = 0.9286y - 6.2146$$

2. Obtain the lines of regression and hence find the coefficient of correlation for the following data

| $x$ | 1 | 2 | 3  | 4  | 5  | 6  | 7  |
|-----|---|---|----|----|----|----|----|
| $y$ | 9 | 8 | 10 | 12 | 11 | 13 | 14 |

**Sol:** Now  $\bar{x} = \frac{\sum x}{n}$  ,  $\bar{y} = \frac{\sum y}{n}$

Here  $\sum x = 28$   $\sum y = 77$  ,  $n = 7$

$$\bar{x} = \frac{28}{7} , \bar{x} = 4 ; \bar{y} = \frac{77}{7} , \bar{y} = 11$$

| $x$ | $y$ | $X = x - \bar{x}$<br>$= x - 4$ | $Y = y - \bar{y}$<br>$= y - 11$ | $XY$                | $X^2$                | $Y^2$                |
|-----|-----|--------------------------------|---------------------------------|---------------------|----------------------|----------------------|
| 1   | 9   | -3                             | -2                              | 6                   | 9                    | 4                    |
| 2   | 8   | -2                             | -3                              | 6                   | 4                    | 9                    |
| 3   | 10  | -1                             | -1                              | 1                   | 1                    | 1                    |
| 4   | 12  | 0                              | 1                               | 0                   | 0                    | 1                    |
| 5   | 11  | 1                              | 0                               | 0                   | 1                    | 0                    |
| 6   | 13  | 2                              | 2                               | 4                   | 4                    | 4                    |
| 7   | 14  | 3                              | 3                               | 9                   | 9                    | 9                    |
|     |     |                                |                                 | $\sum XY$<br>$= 26$ | $\sum X^2$<br>$= 28$ | $\sum Y^2$<br>$= 28$ |

The lines regression are given by

$$(y - \bar{y}) = \frac{\sum XY}{\sum X^2} (x - \bar{x})$$

$$\frac{\sum XY}{\sum X^2} = \frac{26}{28} = 0.9286$$

$$(y - 11) = 0.9286(x - 4)$$

$$y - 11 = 0.9286x - 3.7144$$

$$y = \mathbf{0.9286x + 7.2856}$$

$$\text{Also, } (x - \bar{x}) = \frac{\sum XY}{\sum Y^2} (y - \bar{y})$$

$$\frac{\sum XY}{\sum Y^2} = \frac{26}{28} = \mathbf{0.9286}$$

$$(x - 4) = 0.9286(y - 11)$$

$$x - 4 = 0.9286y - 10.2146$$

$$x = \mathbf{0.9286y - 6.2146}$$

$$\text{Wkt } r = \sqrt{(\text{coeff. of } x) \cdot (\text{coeff. of } y)} \quad r \text{ must be positive since regression coefficients are +ve}$$

$$r = \sqrt{(0.9286) \cdot (0.9286)}$$

$$r = \mathbf{0.9286}$$

3. Find the correlation coefficient for the following two groups

|          |           |           |           |           |           |           |           |           |           |           |
|----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| <b>A</b> | <b>92</b> | <b>89</b> | <b>87</b> | <b>86</b> | <b>83</b> | <b>77</b> | <b>71</b> | <b>63</b> | <b>53</b> | <b>50</b> |
| <b>B</b> | <b>86</b> | <b>83</b> | <b>91</b> | <b>77</b> | <b>68</b> | <b>85</b> | <b>52</b> | <b>82</b> | <b>37</b> | <b>57</b> |

$$\text{Sol: Now } \bar{x} = \frac{\sum x}{n}, \quad \bar{y} = \frac{\sum y}{n}$$

$$\text{Here } \sum x = 751 \quad \sum y = 718, \quad n = 10$$

$$\bar{x} = \frac{751}{10}, \quad \bar{x} = \mathbf{75.1} \quad ; \quad \bar{y} = \frac{718}{10}, \quad \bar{y} = \mathbf{71.8}$$

| <b>x</b> | <b>y</b> | <b>X</b><br><b>= x - <math>\bar{x}</math></b><br><b>= x</b><br><b>- 75.1</b> | <b>Y</b><br><b>= y - <math>\bar{y}</math></b><br><b>= y</b><br><b>- 71.8</b> | <b>XY</b> | <b>X<sup>2</sup></b> | <b>Y<sup>2</sup></b> |
|----------|----------|------------------------------------------------------------------------------|------------------------------------------------------------------------------|-----------|----------------------|----------------------|
|----------|----------|------------------------------------------------------------------------------|------------------------------------------------------------------------------|-----------|----------------------|----------------------|

|    |    |       |       |                       |                         |                        |
|----|----|-------|-------|-----------------------|-------------------------|------------------------|
| 92 | 86 | 16.9  | 14.2  | 239.98                | 285.61                  | 201.64                 |
| 89 | 83 | 13.9  | 11.2  | 155.68                | 193.21                  | 125.44                 |
| 87 | 91 | 11.9  | 19.2  | 228.48                | 141.61                  | 368.64                 |
| 86 | 77 | 10.9  | 5.2   | 56.68                 | 118.81                  | 27.04                  |
| 83 | 68 | 7.9   | -3.8  | -30.02                | 62.41                   | 14.44                  |
| 77 | 85 | 1.9   | 13.2  | 25.08                 | 3.61                    | 174.24                 |
| 71 | 52 | -4.1  | -19.8 | 81.18                 | 16.81                   | 392.04                 |
| 63 | 82 | -12.1 | 10.2  | -123.42               | 146.41                  | 104.04                 |
| 53 | 37 | -22.1 | -34.8 | 769.08                | 488.41                  | 1211.04                |
| 50 | 57 | -25.1 | -14.8 | 371.48                | 630.01                  | 219.04                 |
|    |    |       |       | $\sum XY$<br>= 1774.2 | $\sum X^2$<br>= 2086.59 | $\sum Y^2$<br>= 2837.6 |

Wkt,  $r = \frac{\sum XY}{\sqrt{\sum X^2} \cdot \sqrt{\sum Y^2}}$

$$r = \frac{1774.2}{\sqrt{2086.59} \cdot \sqrt{2837.6}}$$

$$r = \frac{1774.2}{2433.29}$$

$$r = \mathbf{0.7291}$$

4. Find the correlation coefficient and the equation of the lines of regression for the following values of  $x$  and  $y$

|                       |          |          |          |          |          |
|-----------------------|----------|----------|----------|----------|----------|
| <b><math>x</math></b> | <b>1</b> | <b>2</b> | <b>3</b> | <b>4</b> | <b>5</b> |
| <b><math>y</math></b> | <b>2</b> | <b>5</b> | <b>3</b> | <b>8</b> | <b>7</b> |

**Sol:** Now  $\bar{x} = \frac{\sum x}{n}$  ,  $\bar{y} = \frac{\sum y}{n}$

Here  $\sum x = 15$   $\sum y = 25$  ,  $n = 5$

$$\bar{x} = \frac{15}{5} , \bar{x} = 3 \quad ; \quad \bar{y} = \frac{25}{5} , \bar{y} = 5$$

| $x$ | $y$ | $X = x - \bar{x}$<br>$= x - 3$ | $Y = y - \bar{y}$<br>$= y - 5$ | $XY$                | $X^2$                | $Y^2$                |
|-----|-----|--------------------------------|--------------------------------|---------------------|----------------------|----------------------|
| 1   | 2   | -2                             | -3                             | 6                   | 4                    | 9                    |
| 2   | 5   | -1                             | 0                              | 0                   | 1                    | 0                    |
| 3   | 3   | 0                              | -2                             | 0                   | 0                    | 4                    |
| 4   | 8   | 1                              | 3                              | 3                   | 1                    | 9                    |
| 5   | 7   | 2                              | 2                              | 4                   | 4                    | 4                    |
|     |     |                                |                                | $\sum XY$<br>$= 13$ | $\sum X^2$<br>$= 10$ | $\sum Y^2$<br>$= 26$ |

**Wkt,**  $r = \frac{\sum XY}{\sqrt{\sum X^2} \cdot \sqrt{\sum Y^2}}$

$$r = \frac{13}{\sqrt{10} \cdot \sqrt{26}}$$

$$r = 0.8062$$

Now,  $\sigma_x^2 = \frac{\sum X^2}{n} = \frac{10}{5} = 2$  ;  $\sigma_x^2 = 2$  ;  $\sigma_x = \sqrt{2}$

$$\sigma_y^2 = \frac{\sum Y^2}{n} = \frac{26}{5} = 5.2 \quad ; \quad \sigma_y^2 = 5.2 \quad ; \quad \sigma_y = \sqrt{5.2}$$

The lines regression are given by

$$(y - \bar{y}) = r \frac{\sigma_y}{\sigma_x} (x - \bar{x})$$

$$(y - 5) = (0.8062) \frac{\sqrt{2}}{\sqrt{5.2}} (x - 3)$$

$$(y - 5) = (1.3) (x - 3)$$

$$y - 5 = 1.3x - 3.9$$

$$y = 1.3x - 3.9 + 5$$

$$y = 1.3x + 1.1$$

$$\text{Also, } (x - \bar{x}) = r \frac{\sigma_x}{\sigma_y} (y - \bar{y})$$

$$(x - 3) = (0.8062) \frac{\sqrt{2}}{\sqrt{5.2}} (y - 5)$$

$$x - 3 = 0.5(y - 5)$$

$$x = 0.5y - 2.5 + 3$$

$$x = 0.5y + 0.5$$

5. The following data gives the age of husband ( $x$ ) and the age of wife ( $y$ ) in years. Form the two regression lines and calculate the age of husband corresponding to 16 years age of wife

| $x$ | 36 | 23 | 27 | 28 | 28 | 29 | 30 | 31 | 33 | 35 |
|-----|----|----|----|----|----|----|----|----|----|----|
| $y$ | 29 | 18 | 20 | 22 | 27 | 21 | 29 | 27 | 29 | 28 |

$$\text{Sol: Now } \bar{x} = \frac{\sum x}{n}, \quad \bar{y} = \frac{\sum y}{n}$$

$$\text{Here } \sum x = 300 \quad \sum y = 250 \quad , \quad n = 10$$

$$\bar{x} = \frac{300}{10} \quad \bar{y} = \frac{250}{10}$$

$$\bar{x} = 30 \quad \bar{y} = 25$$

| $x$ | $y$ | $X$<br>$= x - \bar{x}$<br>$= x - 30$ | $Y$<br>$= y - \bar{y}$<br>$= y - 25$ | $XY$ | $X^2$ | $Y^2$ |
|-----|-----|--------------------------------------|--------------------------------------|------|-------|-------|
| 36  | 29  | 6                                    | 4                                    | 24   | 36    | 16    |

|    |    |    |    |                    |                     |                     |
|----|----|----|----|--------------------|---------------------|---------------------|
| 23 | 18 | -7 | -7 | 49                 | 49                  | 49                  |
| 27 | 20 | -3 | -5 | 15                 | 9                   | 25                  |
| 28 | 22 | -2 | -3 | 6                  | 4                   | 9                   |
| 28 | 27 | -2 | 2  | -4                 | 4                   | 4                   |
| 29 | 21 | -1 | -4 | 4                  | 1                   | 16                  |
| 30 | 29 | 0  | 4  | 0                  | 0                   | 16                  |
| 31 | 27 | 1  | 2  | 2                  | 1                   | 4                   |
| 33 | 29 | 3  | 4  | 12                 | 9                   | 16                  |
| 35 | 28 | 5  | 3  | 15                 | 25                  | 9                   |
|    |    |    |    | $\sum XY$<br>= 123 | $\sum X^2$<br>= 138 | $\sum Y^2$<br>= 164 |

The lines regression are given by

$$(y - \bar{y}) = \frac{\sum XY}{\sum X^2} (x - \bar{x}) \qquad \frac{\sum XY}{\sum X^2} = \frac{123}{138} = 0.8913$$

$$(y - 25) = 0.8913(x - 30)$$

$$y - 25 = 0.8913x - 26.739$$

$$\mathbf{y = 0.8913x - 1.739}$$

$$\text{Also, } (x - \bar{x}) = \frac{\sum XY}{\sum Y^2} (y - \bar{y}) \qquad \frac{\sum XY}{\sum Y^2} = \frac{123}{164} = 0.75$$

$$(x - 30) = 0.75(y - 25)$$

$$x - 30 = 0.75y - 18.75$$

$$\mathbf{x = 0.75y + 11.25}$$

$$\text{When } y = 16, x = 0.75(16) + 11.25 \quad \mathbf{x = 23.25 \cong 23}$$

Thus husband's age is 23 years corresponding to wife age of 16 years.

6. In a partially destroyed lab record, only the lines of regression of  $y$  on  $x$  and  $x$  on  $y$  are available as  $4x - 5y + 33 = 0$  and  $20x - 9y = 107$  respectively. Calculate the coefficient of correlation between  $x$  and  $y$ .

**Sol:** Lines of regression of  $y$  on  $x$  and  $x$  on  $y$  are

$$4x - 5y + 33 = 0$$

$$20x - 9y = 107$$

$$5y = 4x + 33$$

$$20x = 9y + 107$$

$$y = \frac{4}{5}x + \frac{33}{5}$$

$$x = \frac{9}{20}y + \frac{107}{20}$$

$$\text{Coefficient of } x = \frac{4}{5}$$

$$\text{Coefficient of } y = \frac{9}{20}$$

The coefficient of correlation is

$$r = \sqrt{(\text{coeff. of } x) \cdot (\text{coeff. of } y)} \quad r \text{ must be positive since regression coefficients are +ve}$$

$$r = \sqrt{\left(\frac{4}{5}\right) \cdot \left(\frac{9}{20}\right)}$$

$$r = \sqrt{\frac{36}{100}}$$

$$r = \frac{6}{10}$$

$$r = 0.6$$

7. Given the equation of the regression lines  $x = 19.13 - 0.87y$ ,  $y = 11.64 - 0.5x$ .

Compute the mean of  $x$ 's, mean of  $y$ 's and the coefficient of correlation.

**Sol: To find  $\bar{x}$ ,  $\bar{y}$  and  $r$ .**

We know that the regression lines passes through  $(\bar{x}, \bar{y})$ .

$$\therefore \bar{x} = 19.13 - 0.87\bar{y} \quad \text{and} \quad \bar{y} = 11.64 - 0.5\bar{x}$$

$$\begin{aligned} \text{That is, } \bar{x} + 0.87\bar{y} &= 19.13 \\ 0.5\bar{x} + \bar{y} &= 11.64 \end{aligned}$$

Solving above equations, we get

$$\bar{x} = 15.94 \quad \text{and} \quad \bar{y} = 3.67$$

Rewriting the equations

$$y = -0.5x + 11.64 \quad ; \quad x = -0.87y + 19.13$$

$$\text{Coefficient of } x = -0.5$$

$$\text{Coefficient of } y = -0.87$$

The coefficient of correlation is

$r = -\sqrt{(\text{coeff. of } x) \cdot (\text{coeff. of } y)}$   $r$  must be negative since regression coefficients are -ve

$$r = -\sqrt{(-0.5) \cdot (-0.87)}$$

$$r = -0.6595$$

8. Compute  $\bar{x}$ ,  $\bar{y}$  and  $r$  from the following equation of the regression lines  $2x + 3y + 1 = 0$  and  $x + 6y - 4 = 0$

**Sol:** We know that the regression lines passes through  $(\bar{x}, \bar{y})$ .

$$\therefore 2\bar{x} + 3\bar{y} = -1 \quad \text{and} \quad \bar{x} + 6\bar{y} = 4$$

Solving above equations , we get

$$\bar{x} = -2 \quad \text{and} \quad \bar{y} = 1$$

Consider the regression lines

$$2x + 3y + 1 = 0 \quad \text{and} \quad x + 6y - 4 = 0$$

$$3y = -2x - 1 \quad \text{and} \quad x = -6y + 4$$

$$y = -\frac{2}{3}x - \frac{1}{3}$$

$$\text{Coefficient of } x = -\frac{2}{3}$$

$$\text{Coefficient of } y = -6$$

The coefficient of correlation is

$r = -\sqrt{(\text{coeff. of } x) \cdot (\text{coeff. of } y)}$   $r$  must be negative since regression coefficients are -ve

$$r = -\sqrt{\left(-\frac{2}{3}\right) \cdot (-6)}$$

$$r = -\sqrt{4}$$

$r = -2$  , this is absurd. Since  $r$  lies between -1 and +1.

Rewriting the regression lines

$$x + 6y - 4 = 0 \quad \text{and} \quad 2x + 3y + 1 = 0$$

$$6y = -x + 4 \quad \text{and} \quad 2x = -3y - 1$$

$$y = -\frac{1}{6}x + \frac{4}{6} \quad \text{and} \quad x = -\frac{3}{2}y - \frac{1}{2}$$

$$\text{Coefficient of } x = -\frac{1}{6}$$

$$\text{Coefficient of } y = -\frac{3}{2}$$

The coefficient of correlation is

$r = -\sqrt{(\text{coeff. of } x) \cdot (\text{coeff. of } y)}$   $r$  must be negative since regression coefficients are – ve.

$$r = -\sqrt{\left(-\frac{1}{6}\right) \cdot \left(-\frac{3}{2}\right)}$$

$$r = -\sqrt{\frac{1}{4}}$$

$$r = -0.5$$

Thus,  $\bar{x} = -2$  ,  $\bar{y} = 1$  and  $r = -0.5$

9. Given the following table and  $r = 0.8$  , write down the equation of the lines of

|      | x-series | y-series |
|------|----------|----------|
| Mean | 18       | 100      |
| SD   | 14       | 20       |

regression and hence find the most probable value of  $y$  when  $x = 70$ .

**Sol:** By data  $\bar{x} = 18$  and  $\bar{y} = 100$  ;  $\sigma_x = 14$  and  $\sigma_y = 20$  ;  $r = 0.8$

The equations of regression lines are

$$(y - \bar{y}) = r \frac{\sigma_y}{\sigma_x} (x - \bar{x}) ; \quad (x - \bar{x}) = r \frac{\sigma_x}{\sigma_y} (y - \bar{y})$$

$$(y - 100) = (0.8) \frac{20}{14} (x - 18) ; \quad (x - 18) = (0.8) \frac{14}{20} (y - 100)$$

$$y - 100 = 1.14(x - 18) ; \quad x - 18 = 0.56(y - 100)$$

$$y = 1.14x + 79.48 ; \quad x = 0.56y - 38$$

When  $x = 70$ ,  $y = 1.14(70) + 79.48$  ;  $y = 159.28$

10. Compute the coefficient of correlation using the formula  $r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y}$  and the lines of

regression to the following data

|   |   |   |    |    |    |    |    |
|---|---|---|----|----|----|----|----|
| x | 1 | 2 | 3  | 4  | 5  | 6  | 7  |
| y | 9 | 8 | 10 | 12 | 11 | 13 | 14 |

**Sol:** We have  $r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y}$

We shall prepare the table showing the columns of  $x, y, z, x^2, y^2, z^2$  where  $z = x - y$  so as to compute SD's of  $x, y, z$ .

| $x$           | $y$           | $z = x - y$    | $x^2$            | $y^2$            | $z^2$            |
|---------------|---------------|----------------|------------------|------------------|------------------|
| 1             | 9             | -8             | 1                | 81               | 64               |
| 2             | 8             | -6             | 4                | 64               | 36               |
| 3             | 10            | -7             | 9                | 100              | 49               |
| 4             | 12            | -8             | 16               | 144              | 64               |
| 5             | 11            | -6             | 25               | 121              | 36               |
| 6             | 13            | -7             | 36               | 169              | 49               |
| 7             | 14            | -7             | 49               | 196              | 49               |
| $\sum x = 28$ | $\sum y = 77$ | $\sum z = -49$ | $\sum x^2 = 140$ | $\sum y^2 = 875$ | $\sum z^2 = 347$ |

**Wkt,**  $\bar{x} = \frac{\sum x}{n}$        $\bar{y} = \frac{\sum y}{n}$       ,  $\bar{z} = \frac{\sum z}{n}$

**Here**  $\sum x = 28$        $\sum y = 77$        $\sum z = -49$  ,       $n = 7$

$\bar{x} = \frac{28}{7}$        $\bar{y} = \frac{77}{7}$        $\bar{z} = \frac{-49}{7}$

$\bar{x} = 4$        $\bar{y} = 11$        $\bar{z} = -7$

$\sigma_x^2 = \frac{\sum x^2}{n} - (\bar{x})^2$        $\sigma_y^2 = \frac{\sum y^2}{n} - (\bar{y})^2$        $\sigma_z^2 = \frac{\sum z^2}{n} - (\bar{z})^2$

$$\sigma_x^2 = \frac{140}{7} - 16$$

$$\sigma_x^2 = 4$$

$$\sigma_y^2 = \frac{875}{7} - 121$$

$$\sigma_y^2 = 4$$

$$\sigma_z^2 = \frac{347}{7} - 49$$

$$\sigma_z^2 = 0.57$$

$$\text{Now, } r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y}$$

$$r = \frac{4+4-0.57}{2.2.2}$$

$$r = .9287$$

The lines regression are given by

$$(y - \bar{y}) = r \frac{\sigma_y}{\sigma_x} (x - \bar{x})$$

$$(y - 11) = 0.9286(x - 4)$$

$$y - 11 = 0.9286x - 3.7144$$

$$y = 0.9286x + 7.2856$$

$$\text{Also, } (x - \bar{x}) = r \frac{\sigma_x}{\sigma_y} (y - \bar{y})$$

$$(x - 4) = 0.9286(y - 11)$$

$$x - 4 = 0.9286y - 10.2146$$

$$x = 0.9286y - 6.2146$$

### **Rank Correlation**

The coefficient of correlation in respect of the ranks of some two characteristics of an individual or an observation is called **the rank correlation coefficient** usually denoted by ' $\rho$ '.

$$\rho = 1 - \frac{6 \sum (x-y)^2}{n(n^2-1)} \quad \text{or} \quad \rho = 1 - \frac{6 \sum d^2}{n(n^2-1)}$$

$d = x - y$ , being the difference in ranks.

#### **NOTE:**

1. If the rankings of  $x$ ,  $y$  are entirely in the same order like for example,

$x: 1,2,3,4,5$ ;  $y: 1,2,3,4,5$  then  $\sum d^2 = \sum (x-y)^2 = 0$ . This will give us  $\rho = +1$  and is called **perfect direct correlation**.

If the rankings of  $x$ ,  $y$  are entirely in the opposite order like for example,

$x: 1,2,3,4,5$ ;  $y: 5,4,3,2,1$  then  $\sum d^2 = \sum (x-y)^2 = (-4)^2 + (-2)^2 + (0)^2 + (2)^2 + (4)^2 = 40$ .

This will give us  $\rho = 1 - \frac{6(40)}{5(5^2-1)} = -1$  and is called **perfect inverse correlation**.

2.  $\rho$  in case of repeated ranks

If two or more magnitudes are repeated, then there will be a 'tie' for a particular rank. In such cases we assign the average rank to all those magnitudes and use the correlation factor  $\frac{m(m^2-1)}{12}$  along with  $\sum d^2$  in the formula for  $\rho$  where  $m$  denotes the number of times a magnitude is repeated. The correction factor must be added every time the 'tie' occurs for a particular rank.

$$\rho = 1 - \frac{6 \left[ \sum d^2 + \frac{m(m^2-1)}{12} + \dots \right]}{n(n^2-1)}$$

For example, if there is a 'tie' for the second rank in two magnitudes, the rank  $\frac{2+3}{2} = 2.5$  is assigned to both the magnitudes thereby weeding both 2<sup>nd</sup> and 3<sup>rd</sup> ranks. (Here  $m = 2$ )

Again, suppose there is a tie for 5<sup>th</sup> rank among three magnitudes we take the average of the three ranks in a row starting from 5. That is  $\frac{5+6+7}{3} = 6$  which will be assigned to all the three magnitudes and the next ranking will only be from 8. (Here  $m = 3$ )

## Problems

1. Ten competitors in a beauty contest are ranked by two judges in the following order. Compute the coefficient of rank correlation.

|    |   |   |   |   |    |   |   |    |   |   |
|----|---|---|---|---|----|---|---|----|---|---|
| I  | 1 | 6 | 5 | 3 | 10 | 2 | 4 | 9  | 7 | 8 |
| II | 6 | 4 | 9 | 8 | 1  | 2 | 3 | 10 | 5 | 7 |

**Sol:** From the given data,  $n = 10$  and we have

$$\sum d^2 = (1-6)^2 + (6-4)^2 + (5-9)^2 + (3-8)^2 + (10-1)^2 + (2-2)^2 + (4-3)^2 + (9-10)^2 + (7-5)^2 + (8-7)^2$$

$$\sum d^2 = 25 + 4 + 16 + 25 + 81 + 0 + 1 + 1 + 4 + 1 = 158$$

$$\text{Wkt, } \rho = 1 - \frac{6 \sum d^2}{n(n^2-1)}$$

$$\rho = 1 - \frac{6(158)}{10(10^2-1)}$$

$$\rho = 0.042$$

2. Ten competitors in music contest are ranked by 3 judges A, B, C in the following order. Use the rank correlation coefficient to decide which pair of judges have the nearest approach to common taste of music.

|   |   |   |   |    |   |   |   |   |   |   |
|---|---|---|---|----|---|---|---|---|---|---|
| A | 1 | 6 | 5 | 10 | 3 | 2 | 4 | 9 | 7 | 8 |
|---|---|---|---|----|---|---|---|---|---|---|

|          |          |          |          |          |          |           |          |           |          |          |
|----------|----------|----------|----------|----------|----------|-----------|----------|-----------|----------|----------|
| <b>B</b> | <b>3</b> | <b>5</b> | <b>8</b> | <b>4</b> | <b>7</b> | <b>10</b> | <b>2</b> | <b>1</b>  | <b>6</b> | <b>9</b> |
| <b>C</b> | <b>6</b> | <b>4</b> | <b>9</b> | <b>8</b> | <b>1</b> | <b>2</b>  | <b>3</b> | <b>10</b> | <b>5</b> | <b>7</b> |

**Sol:** We shall compute  $\rho_{AB}, \rho_{BC}, \rho_{CA}$  with the help of the following table where  $d$  is the difference in ranks.

| <b>A</b> | <b>B</b> | <b>C</b> | <b><math>d_{AB}^2</math></b>            | <b><math>d_{BC}^2</math></b>            | <b><math>d_{CA}^2</math></b>           |
|----------|----------|----------|-----------------------------------------|-----------------------------------------|----------------------------------------|
| 1        | 3        | 6        | 4                                       | 9                                       | 25                                     |
| 6        | 5        | 4        | 1                                       | 1                                       | 4                                      |
| 5        | 8        | 9        | 9                                       | 1                                       | 16                                     |
| 10       | 4        | 8        | 36                                      | 16                                      | 4                                      |
| 3        | 7        | 1        | 16                                      | 36                                      | 4                                      |
| 2        | 10       | 2        | 64                                      | 64                                      | 0                                      |
| 4        | 2        | 3        | 4                                       | 1                                       | 1                                      |
| 9        | 1        | 10       | 64                                      | 81                                      | 1                                      |
| 7        | 6        | 5        | 1                                       | 1                                       | 4                                      |
| 8        | 9        | 7        | 1                                       | 4                                       | 1                                      |
|          |          |          | <b><math>\sum d_{AB}^2 = 200</math></b> | <b><math>\sum d_{BC}^2 = 214</math></b> | <b><math>\sum d_{CA}^2 = 60</math></b> |

Wkt,  $\rho = 1 - \frac{6\sum d^2}{n(n^2-1)}$  and from the given data  $n = 10$

$$\text{Now, } \rho_{AB} = 1 - \frac{6(200)}{10(10^2-1)} = -0.21$$

$$\rho_{BC} = 1 - \frac{6(214)}{10(10^2-1)} = -0.297$$

$$\rho_{CA} = 1 - \frac{6(60)}{10(10^2-1)} = +0.636$$

It may be observed that  $\rho_{AB}$  and  $\rho_{BC}$  are negative which means that their tastes (A&B ; B&C) are opposite. But  $\rho_{CA}$  is positive and is near to 1.(Perfect correlation)

**Thus, we conclude that the judges C and A have the nearest approach to common taste of music.**

**3. Ten students got the following percentage of marks in two subjects  $x$  and  $y$ . Compute their rank correlation coefficient.**

|                                |           |           |           |           |           |           |           |           |           |           |
|--------------------------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| <b>Marks in <math>x</math></b> | <b>78</b> | <b>36</b> | <b>98</b> | <b>25</b> | <b>75</b> | <b>82</b> | <b>90</b> | <b>62</b> | <b>65</b> | <b>39</b> |
| <b>Marks in <math>y</math></b> | <b>84</b> | <b>51</b> | <b>91</b> | <b>60</b> | <b>68</b> | <b>62</b> | <b>86</b> | <b>58</b> | <b>53</b> | <b>47</b> |

**Sol:** We prepare the table consisting of the given data along with the ranks assigned according to their order of the magnitude. In the subject  $x$ , 98 will be awarded rank 1, 90 as rank 2 and so on.

| <b>Marks in <math>x</math></b> | <b>Rank (<math>x</math>)</b> | <b>Marks in <math>y</math></b> | <b>Rank (<math>y</math>)</b> | <b><math>d = (x - y)</math></b> | <b><math>d^2 = (x - y)^2</math></b> |
|--------------------------------|------------------------------|--------------------------------|------------------------------|---------------------------------|-------------------------------------|
| 78                             | 4                            | 84                             | 3                            | 1                               | 1                                   |
| 36                             | 9                            | 51                             | 9                            | 0                               | 0                                   |
| 98                             | 1                            | 91                             | 1                            | 0                               | 0                                   |
| 25                             | 10                           | 60                             | 6                            | 4                               | 16                                  |
| 75                             | 5                            | 68                             | 4                            | 1                               | 1                                   |

|    |   |    |    |    |                 |
|----|---|----|----|----|-----------------|
| 82 | 3 | 62 | 5  | -2 | 4               |
| 90 | 2 | 86 | 2  | 0  | 0               |
| 62 | 7 | 58 | 7  | 0  | 0               |
| 65 | 6 | 53 | 8  | -2 | 4               |
| 39 | 8 | 47 | 10 | -2 | 4               |
|    |   |    |    |    | $\sum d^2 = 30$ |

Wkt,  $\rho = 1 - \frac{6\sum d^2}{n(n^2-1)}$  and from the given data  $n = 10$

$$\rho = 1 - \frac{6(30)}{10(10^2-1)}$$

$$\rho = 0.8182$$

4. Compute the rank correlation coefficient for the following data

|          |    |    |    |    |    |    |    |    |    |    |
|----------|----|----|----|----|----|----|----|----|----|----|
| <i>x</i> | 68 | 64 | 75 | 50 | 64 | 80 | 75 | 40 | 55 | 64 |
| <i>y</i> | 62 | 58 | 68 | 45 | 81 | 60 | 68 | 48 | 50 | 70 |

Sol:

| <i>x</i> | Rank ( <i>x</i> ) | <i>y</i> | Rank ( <i>y</i> ) | <i>d</i> = ( <i>x</i> - <i>y</i> ) | <i>d</i> <sup>2</sup> = ( <i>x</i> - <i>y</i> ) <sup>2</sup> |
|----------|-------------------|----------|-------------------|------------------------------------|--------------------------------------------------------------|
| 68       | 4                 | 62       | 5                 | -1                                 | 1                                                            |
| 64       | 6                 | 58       | 7                 | -1                                 | 1                                                            |
| 75       | 2.5               | 68       | 3.5               | -1                                 | 1                                                            |
| 50       | 9                 | 45       | 10                | -1                                 | 1                                                            |
| 64       | 6                 | 81       | 1                 | 5                                  | 25                                                           |
| 80       | 1                 | 60       | 6                 | -5                                 | 25                                                           |
| 75       | 2.5               | 68       | 3.5               | -1                                 | 1                                                            |
| 40       | 10                | 48       | 9                 | 1                                  | 1                                                            |
| 55       | 8                 | 50       | 8                 | 0                                  | 0                                                            |
| 64       | 6                 | 70       | 2                 | 4                                  | 16                                                           |
|          |                   |          |                   |                                    | $\sum d^2 = 72$                                              |

Here in *x* the magnitude 75 is repeated twice for rank 2 and hence the average rank of  $2,3 = \frac{2+3}{2} = 2.5$  is assigned to both ( $m = 2$ ); Also in *x* the magnitude 64 is repeated thrice for rank 5 and hence the average rank of  $5,6,7 = \frac{5+6+7}{3} = 6$  is assigned to the magnitudes 64 ( $m = 3$ ). Further in *y* the magnitude 68 is repeated twice for rank 3 and hence the average rank of  $3,4 = \frac{3+4}{2} = 3.5$  is assigned to both of them ( $m = 2$ ).

Wkt,  $\rho = 1 - \frac{6\left[\sum d^2 + \frac{m(m^2-1)}{12} + \dots\right]}{n(n^2-1)}$  and from the given data  $n = 10$

$$\rho = 1 - \frac{6\left[72 + \frac{2(2^2-1)}{12} + \frac{3(3^2-1)}{12} + \frac{2(2^2-1)}{12}\right]}{10(10^2-1)}$$

$$\rho = 1 - \frac{6[72+0.5+2+0.5]}{990}$$

$$\rho = 0.545$$

5. The coefficient of rank correlation obtained by ten students in Statistics and Accountancy was 0.2. It was later discovered that the difference in ranks in the two subjects of one of the students was wrongly taken as 9 instead of 7. Find the correct rank correlation coefficient.

**Sol:** Wkt,  $\rho = 1 - \frac{6\sum d^2}{n(n^2-1)}$

Here  $n = 10$  and  $\rho = 0.2$

Hence,  $0.2 = 1 - \frac{6\sum d^2}{10(10^2-1)}$

$$\frac{6\sum d^2}{990} = 1 - 0.2$$

$$\sum d^2 \text{ (incorrect)} = \frac{990 \times 0.8}{6} = 132$$

Now, correct  $\sum d^2 = 132 - 9^2 + 7^2 = 100$

$\therefore$  Correct  $\rho = 1 - \frac{6(100)}{10(10^2-1)}$

$$\rho = 0.394$$