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College of Engineering



Transform and Statistics

1BMATME301



Course outcome

CO1: Apply Fourier series techniques to represent periodic functions and perform harmonic analysis relevant to engineering applications.

CO2: Utilize Fourier Transforms and Z-Transforms to analyze engineering systems and solve difference equations arising in mechanical and allied engineering problems.

CO3: Formulate and solve partial differential equations and apply numerical methods for the solution of heat, wave, and Laplace equations encountered in engineering practice.

CO4: Employ Laplace Transform techniques to solve ordinary differential equations and engineering problems involving dynamic systems.

CO5: Analyze and interpret engineering data using statistical tools, measures of central tendency, dispersion, and data visualization techniques for informed decision-making.



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Module-1: Fourier Series

Module-2: Fourier transforms and Z transforms

Module-3: Partial differential equations

Module-4: Statistics-I

Module-5: Statistics-II



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Module-1: Fourier Series



TOPICS



Fourier Series: Periodic functions, Dirichlet's condition. Fourier series of periodic functions

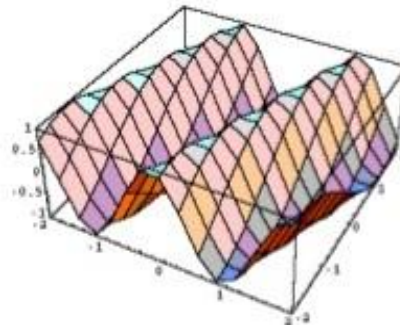
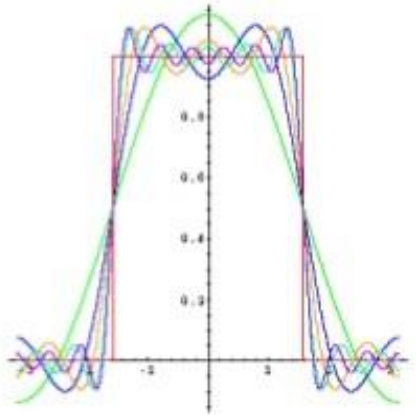


Half range Fourier series.



Practical Harmonic Analysis

Fourier waves



FOURIER SERIES can be generally written as,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx),$$

Where,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \quad \text{..... (1.1)}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \quad \text{..... (1.2)}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \quad \text{..... (1.3)}$$

Fourier series make use of the orthogonality relationships of the sine and cosine functions.

BASIS FORMULAE OF FOURIER SERIES

The Fourier series of a periodic function $f(x)$ with period 2π is defined as the trigonometric series with the coefficient a_0 , a_n and b_n , known as **FOURIER COEFFICIENTS**, determined by formulae (1.1), (1.2) and (1.3).

The individual terms in Fourier Series are known as **HARMONICS**.

Every function $f(x)$ of period 2π satisfying following conditions known as **DIRICHLET'S CONDITIONS**, can be expressed in the form of Fourier series.

FOURIER SERIES FOR EVEN AND ODD FUNCTIONS

EVEN FUNCTIONS



If function $f(x)$ is an even periodic function with the period $2L$ ($-L \leq x \leq L$), then $f(x)\cos(n\pi x/L)$ is even while $f(x)\sin(n\pi x/L)$ is odd.

Thus the Fourier series expansion of an even periodic function $f(x)$ with period $2L$ ($-L \leq x \leq L$) is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

Where, $a_0 = \frac{2}{L} \int_0^L f(x) dx$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx \quad n = 1, 2, \dots$$

$$b_n = 0$$

ODD FUNCTIONS



If function $f(x)$ is an even periodic function with the period $2L$ ($-L \leq x \leq L$), then $f(x)\cos(n\pi x/L)$ is even while $f(x)\sin(n\pi x/L)$ is odd.

Thus the Fourier series expansion of an odd periodic function $f(x)$ with period $2L$ ($-L \leq x \leq L$) is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \left(\frac{n\pi x}{L} \right)$$

Where, $b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \quad n = 1, 2, \dots$



EXAMPLES..

Question.: Find the fourier series of $f(x) = x^2 + x$, $-\pi \leq x \leq \pi$.

Solution.: The fourier series of $f(x)$ is given by,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx),$$

Using above,

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} (x^2 + x) dx \\ &= \frac{1}{\pi} \left(\frac{x^3}{3} + \frac{x^2}{2} \right)_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left(\frac{\pi^3}{3} + \frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} \right) = \frac{2\pi^3}{3} = a_0 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} (x^2 + x) \cos nx dx \\ &= \frac{1}{\pi} \left[(x^2 + x) \left(\frac{\sin nx}{n} \right) - (2x + 1) \left(\frac{-\cos nx}{n^2} \right) + (2) \left(\frac{-\sin nx}{n^3} \right) \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left[(2\pi + 1) \frac{\cos n\pi}{n^2} - (-2\pi + 1) \frac{\cos n\pi}{n^2} \right] \\ &= \frac{1}{\pi} \left[(2\pi + 1) \frac{(-1)^n}{n^2} - (-2\pi + 1) \frac{(-1)^n}{n^2} \right] \\ &= \frac{4(-1)^n}{n^2} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} (x^2 + x) \sin nx dx \\ &= \frac{1}{\pi} \left[(x^2 + x) \left(-\frac{\cos nx}{n} \right) - (2x + 1) \left(-\frac{\sin nx}{n^2} \right) + (2) \left(\frac{\cos nx}{n^3} \right) \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left[-\frac{(\pi^2 + \pi)}{n} (-1)^n + \frac{(\pi^2 + \pi)}{n} (-1)^n \right] \\ &= \frac{(-1)^n}{\pi n} [-\pi^2 - \pi + \pi^2 - \pi] \\ &= -\frac{2(-1)^n}{n} \end{aligned}$$

Hence fourier series of, $f(x) = x^2 + x$,

$$x^2 + x = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left[\frac{4(-1)^n}{n^2} \cos nx - \frac{2(-1)^n}{n} \sin nx \right]$$

Half Range Fourier Series

If a function is defined over half the range, say 0 to L , instead of the full range from $-L$ to L , it may be expanded in a series of sine terms only or of cosine terms only. The series produced is then called a **half range Fourier series**.

Conversely, the Fourier Series of an even or odd function can be analyzed using the half range definition

HALF RANGE SINE SERIES

- Suppose that $f(x)$ satisfies Dirichlet's condition in the interval $0 < x < \tau$
- $f(x)$ is an even function of period 2τ , then even periodic $G(x)$ will have a convergent Fourier series representation consisting of cosine terms only and given by

$$G(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\tau}\right)$$

$$b_n = \frac{2}{\tau} \int_0^{\tau} f(x) \sin \frac{n\pi x}{\tau} dx, (n = 0, 1, 2, \dots)$$

HALF RANGE COSINE SERIES

- Suppose that $f(x)$ satisfies Dirichlet's condition in the interval $0 < x < \tau$
- $f(x)$ is an even function of period 2τ , then even periodic $F(x)$ will have a convergent Fourier series representation consisting of cosine terms only and given by

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\tau}\right)$$

$$a_0 = \frac{2}{\tau} \int_0^{\tau} f(x) dx \text{ and } a_n = \frac{2}{\tau} \int_0^{\tau} f(t) \cos \frac{n\pi x}{\tau} dx, \\ \text{for } (n = 0, 1, 2, \dots)$$

Example 1

Find the half-range sine series of the function

$$f(x) = \begin{cases} 4, & \text{if } 0 < x < \pi/2 \\ 0, & \text{if } \pi/2 < x < \pi. \end{cases}$$

Solution: $L = \pi$, so that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \quad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx.$$

where

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx = \\ &= \frac{8}{\pi} \int_0^{\pi/2} \sin(nx) dx = \\ &= \frac{8}{n\pi} \int_0^{n\pi/2} \sin u du = \\ &= -\frac{8}{n\pi} [\cos u]_0^{n\pi/2} \\ &= \frac{8}{n\pi} [1 - \cos(n\pi/2)]. \end{aligned}$$

Example 2

Find the half-range cosine series of the function

$$f(x) = \begin{cases} 4, & \text{if } 0 < x < \pi/2 \\ 0, & \text{if } \pi/2 < x < \pi. \end{cases}$$

NB same function as in the previous example!

Solution: again, $L = \pi$, so that

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx), \quad a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) dx,$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx.$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi/2} 4 dx = 2.$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx = \frac{8}{\pi} \int_0^{\pi/2} \cos(nx) dx \\ &= \frac{8}{n\pi} \int_0^{n\pi/2} \cos u du = \frac{8}{n\pi} [\sin u]_0^{n\pi/2} \\ &= \frac{8}{n\pi} \sin(n\pi/2). \end{aligned}$$

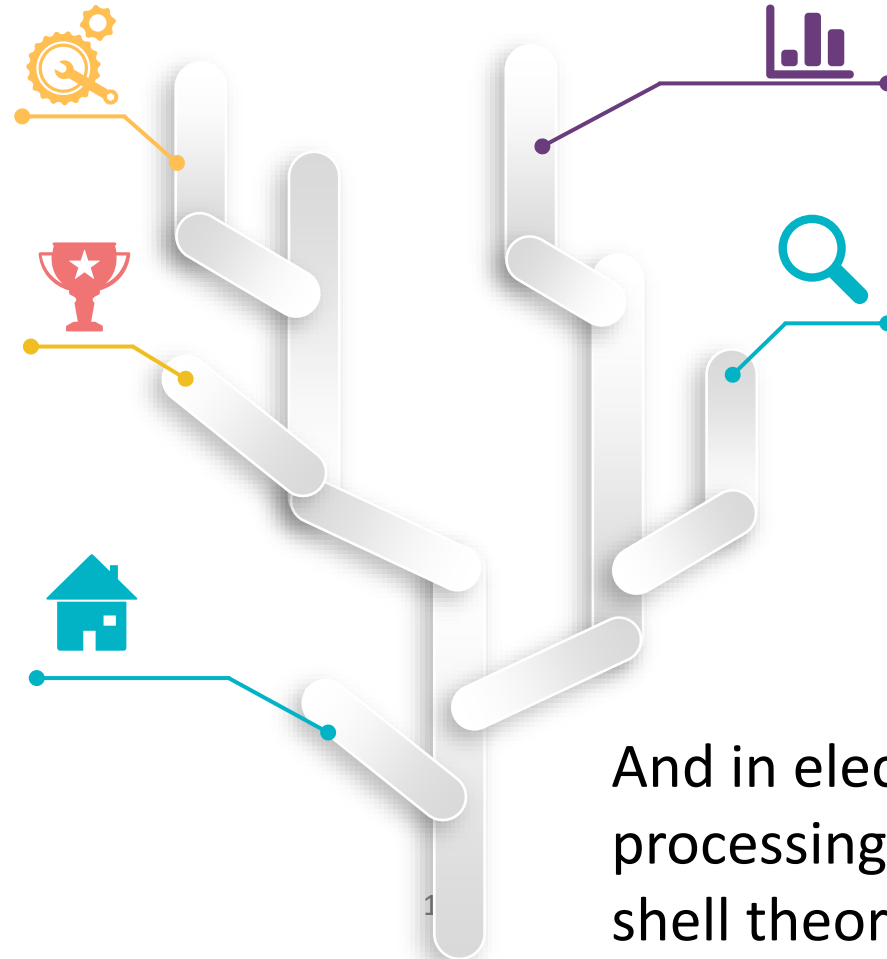


APPLICATIONS OF FOURIER SERIES

Quantum Mechanics

Acoustics

Image Processing



Vibration Analysis

Optics

And in electrical engineering, signal processing, econometrics, thin-walled shell theory, etc.



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Module-2: Fourier transforms and Z transforms

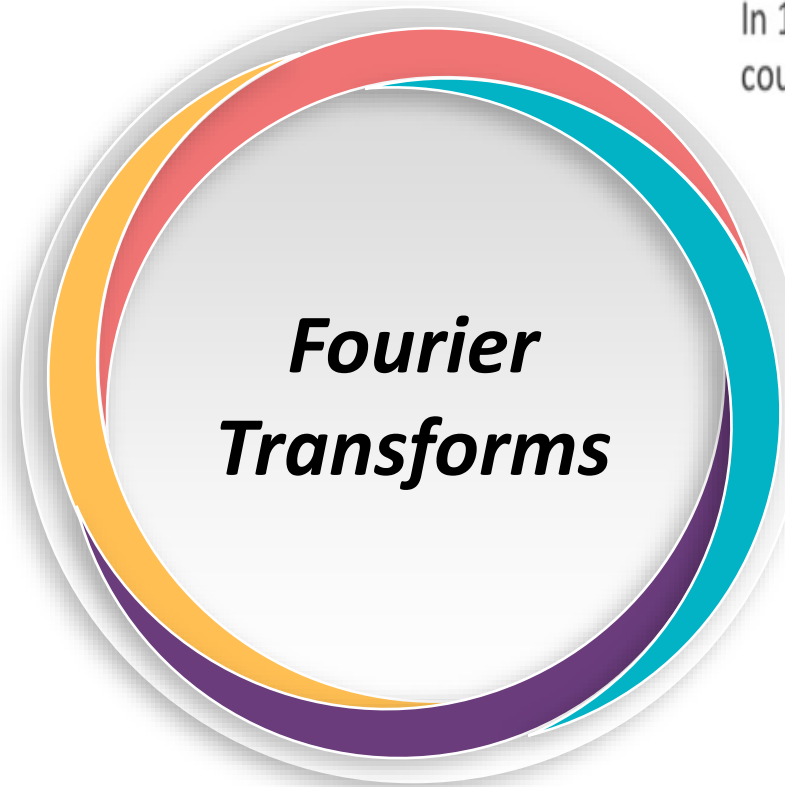
- ***Fourier Transforms:*** Infinite Fourier transforms, Fourier sine and
- cosine transforms.
- Inverse Fourier transforms.
- Problems

Fourier Transform (FT) involves a whole new paradigm of viewing signals in a context different from the frequency domain of their occurrence.
Transform:

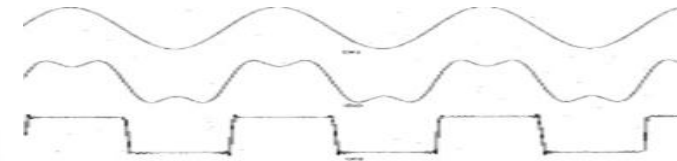
- Four Types of transformation in mathematics
- I. Translation
- II. Rotation
- III. Reflection
- IV. Dilation

What is Fourier Transform?

- The Fourier Transform is a type of mathematical transform.
- The Fourier Transform transforms a non-periodic function $f(t)$ in time domain into a function $F(s)$ in its corresponding frequency domain.
- It is a tool to break a function into a sinusoidal forms characterized by sine and cosine terms.
- To understand the conduction of heat, wave propagation, digital signal processing, image processing, filtering, etc.



In 1807, Joseph Fourier showed that any **periodic** signal could be represented by a series of **sinusoidal** functions



What is Fourier Transform?

- defined on $(-\infty, \infty)$
- piecewise continuous in each finite interval
- absolutely integrable in $(-\infty, \infty)$

$$F(w) = \int_{-\infty}^{\infty} f(t)e^{-iwt} dt$$

The function $F(w)$ is called the Fourier Transformation of $f(t)$

Fourier sine and cosine transform (Infinite)

If $f(t)$ is a function in $0 < t < \infty$ then

➤ $f_s\{w\} = \int_0^{\infty} f(t) \sin wt \, dt$ > Infinite Fourier sine transform

➤ $F_s^{-1}[f_s\{w\}] = \frac{2}{\pi} \int_0^{\infty} f_s\{w\} \sin wt \, dw$ > Inverse form of Infinite Fourier sine transform

➤ $f_c\{w\} = \int_0^{\infty} f(t) \cos wt \, dt$ > Infinite Fourier cosine transform

➤ $F_c^{-1}[f_c\{w\}] = \frac{2}{\pi} \int_0^{\infty} f_c\{w\} \cos wt \, dw$ > Inverse form of Infinite Fourier cosine transform

Fourier sine and cosine transform (finite)

➤ Finite Fourier sine transform of $F(x)$ over $0 < x < l$

$$f_s(n) = \int_0^l F(x) \sin \frac{n\pi x}{l} dx \text{ where } n \text{ is integer.}$$

The function $F(x)$ is called inverse finite Fourier sine transform of $f_s(n)$

$$F(x) = \frac{2}{l} \sum_{n=1}^{\infty} f_s(n) \sin \frac{n\pi x}{l}$$

➤ Finite Fourier cosine transform of $F(x)$ over $0 < x < l$

$$f_c(n) = \int_0^l F(x) \cos \frac{n\pi x}{l} dx \text{ where } n \text{ is integer.}$$

The function $F(x)$ is called inverse finite Fourier cosine transform of $f_c(n)$

$$F(x) = \frac{1}{l} f_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} f_c(n) \cos \frac{n\pi x}{l}$$



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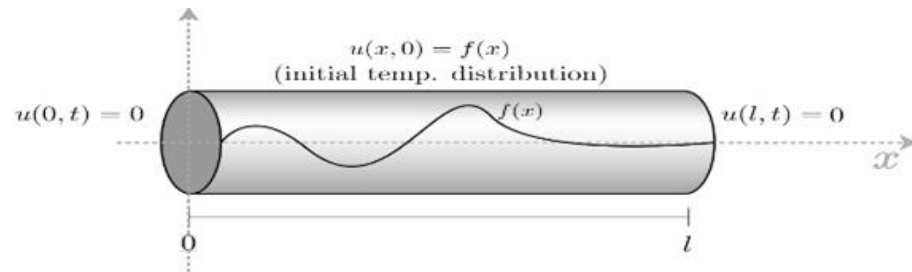
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Using FT to Solve Heat

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}; U(0, t) = 0; U(4, t) = 0;$$
$$U(x, 0) = 2x \text{ where } 0 < x < 4, t > 0.$$

$u(x, t)$ represents temperature at any point x at time t , $l = 0 \rightarrow 4$



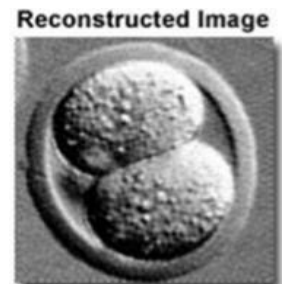
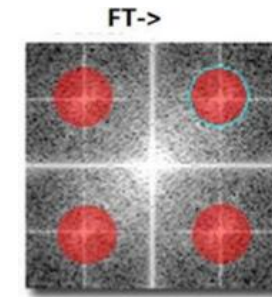
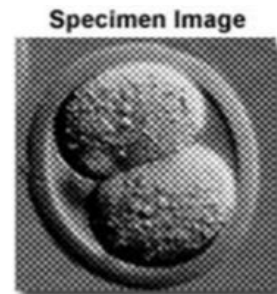
Real life Examples:

1. A doctor examines a patient's heart beat, is done by ECG (Electro-Cardio-Gram)
2. For a musician, all the ragas played are actually in time domain but frequency is more important for him than time
3. For a circuit, the input and output signals are functions of time.

Other Applications of the FT

- Signal analysis
- Sound filtering
- Partial differential equations
- Image processing
- Conduction of heat
- Wave propagation.

Applications: Image Filtering





Difference equations and Z-transforms

Difference Equations

Difference equations arise in problems where the time is assumed to have a discrete set of possible values.

$$y(k+n) + a_{n-1}y(k+n-1) + \dots + a_1y(k+1) + a_0y(k) \\ = b_nu(k+n) + b_{n-1}u(k+n-1) + \dots + b_1u(k+1) + b_0u(k)$$

Where coefficients $a_{n-1}, a_{n-2}, \dots$ and $b_n, b_{n-1}, \dots$ are constant. $u(k)$ is forcing function

Example 1: For the given difference equation, determine the (a) order of the equation. Is the equation (b) linear, (c) time invariant, or (d) homogeneous? $y(k+2) + 0.8y(k+1) + 0.07y(k) = u(k)$

Solution:

- a) The equation is second order.
- b) All terms enter the equation linearly
- c) All the terms if the equation have constant coefficients. Therefore the equation is therefore LTI.
- d) A forcing function appears in the equation, so it is nonhomogeneous.



Z-transform Definition

Given the causal sequence $\{x(kT)\}$, its z-transform is defined as
$$X(z) = \sum_{k=0}^{\infty} x(kT) z^{-k}$$

$$X(z) = x(0) + x(T) z^{-1} + x(2T) z^{-2} + \dots + x(kT) z^{-k}$$

The variable z^{-1} in the above equation can be regarded as a time delay operator.

Example

- Obtain the Z-transform of the sequence

$$x(kT) = \{1, 1, 3, 2, 0, 4, 0, 0, 0, \dots\}$$

- Solution:

$$X(z) = 1 + z^{-1} + 3z^{-2} + 2z^{-3} + 4z^{-5}$$

z-transform of a Unit-Step Function

Let $x(kT) = u(kT)$

$$u(kT) = \begin{cases} 1 & \text{for } k \geq 0 \\ 0 & \text{for } k < 0 \end{cases}$$

Then

$$X(z) = \sum_{k=0}^{\infty} x(kT) z^{-k} = \sum_{k=0}^{\infty} u(kT) z^{-k} = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

Z-Transform Properties: Linearity

Notation $x_1(kT) \xleftrightarrow{Z} X_1(z)$ & $x_2(kT) \xleftrightarrow{Z} X_2(z)$

Linearity $a x_1(kT) + b x_2(kT) \xleftrightarrow{Z} a X_1(z) + b X_2(z)$

Example:

$$x(kT) = 4u(kT) - 3a^k u(kT)$$

$$X(z) = \frac{4}{1-z^{-1}} - \frac{3}{1-az^{-1}} = \frac{1-(3-4a)z^{-1}}{(1-z^{-1})(1-az^{-1})}$$

Complex Differentiation $k x(kT) \xleftrightarrow{Z} -z \frac{dX(z)}{dz}$
 $k^m x(kT) \xleftrightarrow{Z} \left(-z \frac{d}{dz}\right)^m X(z)$

Multiplication by Exponential

$$a^k x(kT) \xleftrightarrow{Z} X(z/a)$$



Time Right Shifting (Time Delay)

Use the time delay property of the Laplace Transform

$$x(t - T_d) \xrightarrow{Z} e^{-sT} X(s)$$

Then

$$x(kT - nT) \xrightarrow{Z} z^{-n} X(z)$$

Example:

$$x(kT) = (0.2)^{(k-3)T} u((k-3)T)$$

$$X(Z) = z^{-3} \frac{1}{1 - 0.2z^{-1}}$$

Time Left Shifting (Time Advance)

$$x(kT + nT) \xrightarrow{Z} z^n [X(z) - \sum_{k=0}^{n-1} x(kT) z^{-k}]$$

Example:

$$x(kT) = (0.2)^{(k+2)T} u((k+2)T)$$

$$X(Z) = z^2 \left[\frac{1}{1 - 0.2z^{-1}} - \sum_{k=0}^1 x(kT) z^{-k} \right]$$

$$X(Z) = z^2 \left[\frac{1}{1 - 0.2z^{-1}} - (1 + 0.2z^{-1}) \right] = \frac{0.04}{1 - 0.2z^{-1}}$$



Convolution

$$x_1(kT) \xleftrightarrow{Z} X_1(z) \quad \& \quad x_2(kT) \xleftrightarrow{Z} X_2(z)$$

Convolution in time domain is multiplication in z-domain

$$x_1(kT) * x_2(kT) \xleftrightarrow{Z} X_1(z) X_2(z)$$

Example: Let's calculate the convolution of

$$x_1(kT) = a^{kT} u(kT) \quad \text{and} \quad x_2(kT) = u(kT)$$

Multiplications of z-transforms is

$$Y(z) = X_1(z) X_2(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})}$$

The Inverse Z-Transform

Formal inverse z-transform is based on a Cauchy integral

$$x(kT) = Z^{-1}[X(z)] = \frac{1}{2\pi j} \oint_C X(z) z^{k-1} dz$$

Initial and Final Value Theorems

Initial Value Theorem:

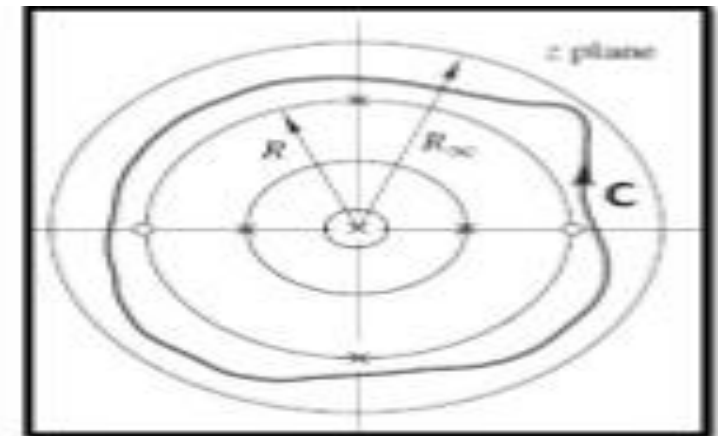
- The value of $x(n)$ as $k \rightarrow 0$ is given by:

$$x(0) = \lim_{k \rightarrow 0} x(kT) = \lim_{z \rightarrow \infty} X(z)$$

Final Value Theorem:

- The value of $x(n)$ as $k \rightarrow \infty$ is given by:

$$x(\infty) = \lim_{k \rightarrow \infty} x(kT) = \lim_{z \rightarrow 1} [(z - 1) X(z)]$$



APPLICATIONS OF Z- TRANSFORM

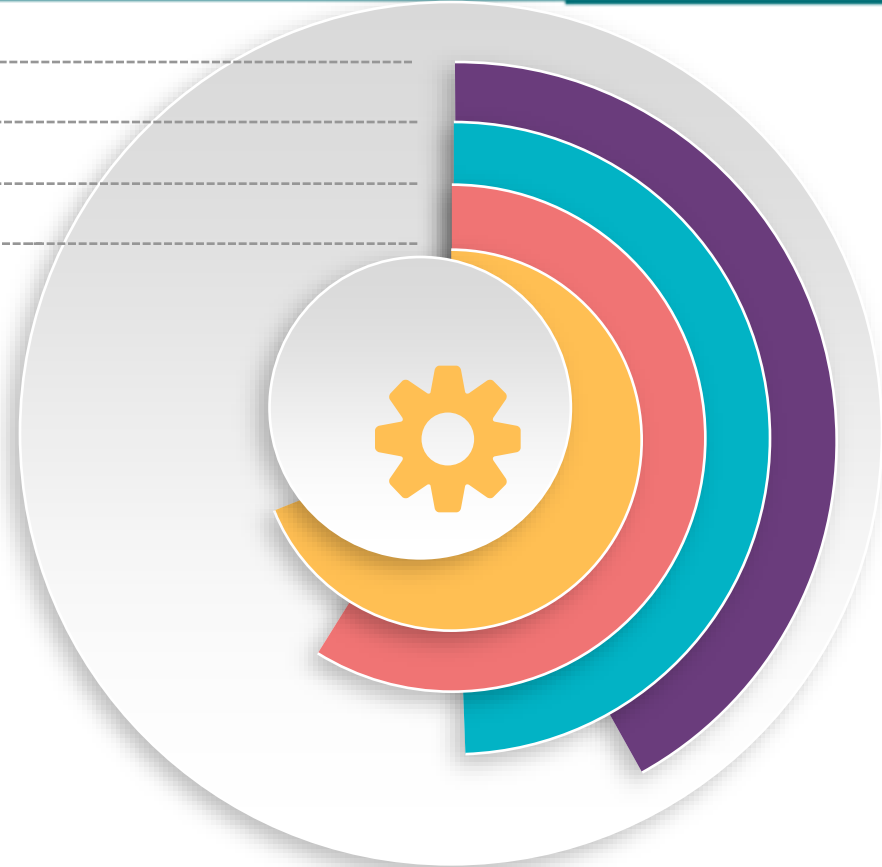
Z transform is used to convert discrete time domain signal into discrete frequency domain signal.

It has wide range of applications in mathematics and digital signal processing.

It is mainly used to analyze and process digital data. For example to analyze JPEG images, MP3 and MP4 songs, ZIP files etc,

Applications of Z transform in digital signal processing are,

1. Used in system designing
- 2.Used to find out stability of a system
- 3.Used to find frequency response of a signal
- 4.Analysis of linear discrete system
- 5.For designing digital filters





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Module-3: Partial Differential Equations

Definition

An equation involving one or more partial derivatives of a function of two or more variables is called a *Partial differential equation[PDE]*.

The *order* of a PDE is the order of the highest derivative and the *degree* of highest order derivative after clearing the equation of fractional powers.

A PDE is said to be linear if it is of first first degree in the dependent variable and its partial derivatives.

If each term of the PDE contains either the dependent variable or one of its partial derivatives, the PDE is said to be *homogeneous*. Otherwise it is said to be a *nonhomogeneous* PDE.

Examples

$$1. \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$$

[Order – 1, Degree – 1, Homogeneous PDE]

2. $\frac{\partial^2 z}{\partial x \partial y} = xy$

[Order – 2, Degree – 1, NonHomogeneous PDE]

3. $\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y} + 2z$

[Order – 2, Degree – 1, Homogeneous PDE]

4. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

[Order – 2, Degree – 1, Homogeneous PDE]

5. $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$

[Order – 3, Degree – 1, NonHomogeneous PDE]

6. $\frac{\partial^2 u}{\partial x^2} = x + y$

[Order – 2, Degree – 1, NonHomogeneous PDE]

7. $\frac{\partial^2 z}{\partial x^2} - 16z = 0$

[Order – 2, Degree – 1, Homogeneous PDE]

8. $\frac{\partial^2 u}{\partial x \partial y} + 4 \frac{\partial u}{\partial y} = 0$

[Order – 2, Degree – 1, NonHomogeneous PDE]



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Formation of PDE by eliminating arbitrary constants and arbitrary functions

Note:

If $z = f(x, y)$, then

$$z_x = \frac{\partial z}{\partial x} = p$$

$$z_y = \frac{\partial z}{\partial y} = q$$

$$z_{xx} = \frac{\partial^2 z}{\partial x^2} = r$$

$$z_{xy} = \frac{\partial^2 z}{\partial x \partial y} = s$$

$$z_{yy} = \frac{\partial^2 z}{\partial y^2} = t$$

Problems

Form the PDE by eliminating the arbitrary constant in the following

1. $z = (x + a)(y + b)$

Sol: Given $z = (x + a)(y + b)$ --- (1)

Differentiate (1) w.r.t 'x' and 'y' partially

$$\frac{\partial z}{\partial x} = (y + b).1 \quad \Rightarrow \quad p = (y + b)$$

$$\text{Also, } \frac{\partial z}{\partial y} = (x + a).1 \quad \Rightarrow \quad q = (x + a)$$

$$(1) \Rightarrow z = q.p$$

$z = pq$, is the required PDE



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$$2. 2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Sol: Given $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ --- (1)

Differentiate (1) w.r.t 'x' and 'y' partially

$$2 \frac{\partial z}{\partial x} = \frac{2x}{a^2}$$

$$\Rightarrow p = \frac{x}{a^2}$$

Multiply by 'x'

$$px = \frac{x^2}{a^2}$$

Also, $2 \frac{\partial z}{\partial y} = \frac{2y}{b^2}$

$$\Rightarrow q = \frac{y}{b^2}$$

Multiply by 'y'

$$qy = \frac{y^2}{b^2}$$

(1) $\Rightarrow 2z = px + qy$, is the required PDE.

$$3. ax^2 + by^2 + z^2 = 1$$

Sol: Given $ax^2 + by^2 + z^2 = 1$

$$z^2 = 1 - ax^2 - by^2 \quad \text{--- (1)}$$

Differentiate (1) w.r.t 'x' and 'y' partially

$$2z \frac{\partial z}{\partial x} = 0 - 2ax - 0$$

$$\Rightarrow 2zp = -2ax$$



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$$ax = -zpx$$

Multiply by 'x'

$$ax^2 = -zpx$$

$$\text{Also, } 2z \frac{\partial z}{\partial y} = 0 - 0 - 2by$$

$$\Rightarrow 2zq = -2by$$

$$by = -zq$$

Multiply by 'y'

$$by^2 = -zqy$$

$$(1) \Rightarrow z^2 = 1 - (-zpx) - (-zqy)$$

$$z^2 - 1 = zpx + zqy$$

$$z^2 - 1 = z(px + qy), \text{ is the required PDE.}$$

4. $z = a \log(x^2 + y^2) + b$

Sol: Given $z = a \log(x^2 + y^2) + b$ --- (1)

Differentiate (1) w.r.t 'x' and 'y' partially

$$\frac{\partial z}{\partial x} = a \cdot \frac{1}{(x^2 + y^2)} \cdot 2x$$

$$\Rightarrow p = \frac{2xa}{(x^2 + y^2)}$$

$$\Rightarrow \frac{p}{x} = \frac{2a}{(x^2 + y^2)} \quad \text{--- (2)}$$

Also, $\frac{\partial z}{\partial y} = a \cdot \frac{1}{(x^2 + y^2)} \cdot 2y$

$$\Rightarrow \frac{q}{y} = \frac{2a}{(x^2 + y^2)} \quad \text{--- (3)}$$

From (2) and (3), we get

$$\frac{p}{x} = \frac{q}{y}$$

$$py = qx$$

$py - qx = 0$, is the required PDE.

$$5. z = xy + y\sqrt{x^2 - a^2} + b$$

Sol: Given $z = xy + y\sqrt{x^2 - a^2} + b$ --- (1)

Differentiate (1) w.r.t 'x' and 'y' partially

$$\frac{\partial z}{\partial x} = y + y \frac{1}{2\sqrt{x^2 - a^2}} \cdot 2x$$

$$\Rightarrow p = y + \frac{xy}{\sqrt{x^2 - a^2}}$$

$$\Rightarrow p - y = \frac{xy}{\sqrt{x^2 - a^2}} \quad \text{--- (2)}$$

Also, $\frac{\partial z}{\partial y} = x.1 + \sqrt{x^2 - a^2}.1$

$$q - x = \sqrt{x^2 - a^2} \quad \text{--- (3)}$$

Using (3) in (2), we get

$$(2) \Rightarrow p - y = \frac{xy}{(q-x)}$$

$$(p - y)(q - x) = xy$$

$$pq - px - qy + xy = xy$$

$pq = px + qy$, is the required PDE.

6. Find the PDE of the family of spheres whose centre lie on the plane $z = 0$ and have a constant radius 'r'.

Sol: The co-ordinates of the centre of the sphere can be taken as $(a, b, 0)$ where a and b are arbitrary. r is the constant radius.

The equation of the sphere is given by

$$(x - a)^2 + (y - b)^2 + (z - 0)^2 = r^2$$

$$(x - a)^2 + (y - b)^2 + z^2 = r^2$$

$$z^2 = r^2 - (x - a)^2 - (y - b)^2 \quad \text{--- (1)}$$

Differentiate (1) w.r.t 'x' and 'y' partially

$$2z \frac{\partial z}{\partial x} = 0 - 2(x - a) - 0$$

$$\Rightarrow 2zp = -2(x - a)$$

$$\Rightarrow (x - a) = -zp$$

$$\text{Also } 2z \frac{\partial z}{\partial y} = 0 - 0 - 2(y - b)$$

$$\Rightarrow 2zq = -2(y - b)$$

$$\Rightarrow (y - b) = -zq$$

$$(1) \Rightarrow z^2 = r^2 - (-zp)^2 - (-zq)^2$$

$$z^2 = r^2 - z^2 p^2 - z^2 q^2$$

$$z^2 + z^2 p^2 + z^2 q^2 = r^2$$

$$z^2(1 + p^2 + q^2) = r^2, \text{ is the required PDE.}$$

$$7. \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Sol: Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \quad \text{--- (1)}$$

Differentiate (1) w.r.t 'x' and 'y' partially

$$\frac{2z}{c^2} \frac{\partial z}{\partial x} = 0 - \frac{2x}{a^2} - 0$$

$$\frac{2z}{c^2} p = -\frac{2x}{a^2}$$

$$\frac{zp}{c^2} = -\frac{x}{a^2} \quad \text{--- (2)}$$

$$\begin{aligned} \text{Also, } \frac{2z}{c^2} \frac{\partial z}{\partial y} &= 0 - 0 - \frac{2y}{b^2} \\ \frac{2z}{c^2} q &= -\frac{2y}{b^2} \\ \frac{zq}{c^2} &= -\frac{y}{b^2} \quad \text{--- (3)} \end{aligned}$$

Differentiate (2) w.r.t 'x' partially

$$\begin{aligned} \frac{1}{c^2} \frac{\partial}{\partial x} \left[z \cdot \frac{\partial z}{\partial x} \right] &= -\frac{1}{a^2} \frac{\partial}{\partial x} [x] \\ \frac{1}{c^2} \left[\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial x} + z \cdot \frac{\partial^2 z}{\partial x^2} \right] &= -\frac{1}{a^2} \cdot 1 \\ \frac{1}{c^2} [p^2 + z \cdot r] &= -\frac{1}{a^2} \end{aligned}$$

Multiply by 'x'

$$\text{Since } p = \frac{\partial z}{\partial x}$$

$$\frac{x}{c^2} [p^2 + zr] = -\frac{x}{a^2}$$

$$\frac{x}{c^2} [p^2 + zr] = \frac{zp}{c^2} \quad \text{From (2)}$$

$$x[p^2 + zr] = zp$$

$$xp^2 + xzr = zp$$

$$xp^2 + xzr - zp = 0$$

$xp^2 + z(xr - p) = 0$, is the required PDE.

8. Form the PDE of the family of spheres whose centre lies on $x - axis$ with constant radius 3.

Sol: Equation of sphere with centre on $x - axis$ i.e., $C = (a, 0, 0)$ and radius $r = 3$.

The equation of the sphere is given by

$$(x - a)^2 + (y - 0)^2 + (z - 0)^2 = 3^2$$

$$(x - a)^2 + y^2 + z^2 = 9$$

$$z^2 = 9 - (x - a)^2 - y^2 \quad \text{--- (1)}$$

Differentiate (1) w.r.t 'x' and 'y' partially

$$2z \frac{\partial z}{\partial x} = 0 - 2(x - a) - 0$$

$$\Rightarrow 2zp = -2(x - a)$$

$$\Rightarrow (x - a) = -zp$$

$$\text{Also } 2z \frac{\partial z}{\partial y} = 0 - 0 - 2y \Rightarrow 2zq = -2y \Rightarrow y = -zq$$

$$(1) \Rightarrow z^2 = 9 - (-zp)^2 - (y)^2$$

$$z^2 = 9 - z^2p^2 - y^2$$

$$z^2 + z^2p^2 + y^2 = 9$$

$$z^2(1 + p^2) + y^2 = 9, \text{ is the required PDE.}$$

Form the PDE by eliminating arbitrary functions in the following

1. $z = f(x^2 + y^2)$

Sol: Given $z = f(x^2 + y^2)$ --- (1)

Differentiating (1) partially w.r.t 'x' and 'y', we have

$$\frac{\partial z}{\partial x} = p = f'(x^2 + y^2) \cdot 2x$$

$$\frac{\partial z}{\partial y} = q = f'(x^2 + y^2) \cdot 2y$$

$$\frac{p}{x} = 2f'(x^2 + y^2) \quad \text{--- (2)}$$

$$\frac{q}{y} = 2f'(x^2 + y^2) \quad \text{--- (3)}$$

From (2) and (3), we have

$$\frac{p}{x} = \frac{q}{y}$$

$$py = qx$$

$py - qx = 0$, is the required PDE.

2. $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

Sol: Given $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ --- (1)

Differentiating (1) partially w.r.t 'x' and 'y', we have

$$\frac{\partial z}{\partial x} = p = 0 + 2f'\left(\frac{1}{x} + \log y\right) \cdot \left(\frac{-1}{x^2}\right)$$

$$p = \frac{-1}{x^2} \cdot 2f'\left(\frac{1}{x} + \log y\right)$$

$$x^2 p = -2f'\left(\frac{1}{x} + \log y\right) \quad \text{--- (2)}$$

From (2) and (3), we have

$$x^2 p = -(q - 2y)y$$

$$x^2 p = -qy + 2y^2$$

$$x^2 p + qy = 2y^2, \text{ is the required PDE.}$$

$$\frac{\partial z}{\partial y} = q = 2y + 2f'\left(\frac{1}{x} + \log y\right) \cdot \left(\frac{1}{y}\right)$$

$$q - 2y = \frac{1}{y} \cdot 2f'\left(\frac{1}{x} + \log y\right)$$

$$(q - 2y)y = 2f'\left(\frac{1}{x} + \log y\right) \quad \text{--- (3)}$$

3. $z = e^y f(x + y)$

Sol: Given $z = e^y f(x + y)$ --- (1)

Differentiating (1) partially w.r.t 'x' and 'y', we have

$$\frac{\partial z}{\partial x} = p = e^y f'(x + y)$$

$$\frac{\partial z}{\partial y} = q = e^y \cdot f'(x + y) + f(x + y) \cdot e^y$$

$$p = e^y f'(x + y) \quad \text{--- (2)}$$

$$q = e^y \cdot f'(x + y) + z$$

$$q - z = e^y \cdot f'(x + y) \quad \text{--- (3)}$$

From (2) and (3), we have

$$p = q - z$$

$p + z = q$, is the required PDE.

4. $z = e^{ax+by} f(ax - by)$

Sol: Given $z = e^{ax+by} f(ax - by)$ --- (1)

Differentiating (1) partially w.r.t 'x', we have

$$\frac{\partial z}{\partial x} = e^{ax+by} f'(ax - by) \cdot a + f(ax - by) \cdot e^{ax+by} \cdot a$$

$$p = e^{ax+by} f'(ax - by) \cdot a + az$$

$$p - az = ae^{ax+by} f'(ax - by)$$

Multiply by 'b'

$$pb - abz = abe^{ax+by} f'(ax - by) \text{ --- (2)}$$

Differentiating (1) partially w.r.t 'y', we have

$$\frac{\partial z}{\partial y} = e^{ax+by} f'(ax - by) \cdot (-b) + f(ax - by) \cdot e^{ax+by} \cdot b$$

$$q = -be^{ax+by} f'(ax - by) + bz$$

$$q - bz = -be^{ax+by} f'(ax - by)$$

Multiply by 'a'

$$qa - abz = -abe^{ax+by} f'(ax - by) \text{ --- (3)}$$

From (2) and (3) , we get

$$pb - abz = -(qa - abz)$$

$$pb - abz = -qa + abz$$

$$pb + qa = abz + abz$$

$pb + qa = 2abz$, is the required PDE.

5. $lx + my + nz = \varphi(x^2 + y^2 + z^2)$

Sol: Given $lx + my + nz = \varphi(x^2 + y^2 + z^2)$ --- (1)

Differentiating (1) partially w.r.t 'x', we have

$$l \cdot 1 + 0 + n \frac{\partial z}{\partial x} = \varphi'(x^2 + y^2 + z^2) \cdot \left(2x + 0 + 2z \frac{\partial z}{\partial x} \right)$$

$$l + np = 2\varphi'(x^2 + y^2 + z^2) \cdot (x + zp)$$

$$\frac{(l+np)}{(x+zp)} = 2\varphi'(x^2 + y^2 + z^2) \quad \text{--- (2)}$$

Differentiating (1) partially w.r.t 'y', we have

$$\begin{aligned}0 + m \cdot 1 + n \frac{\partial z}{\partial y} &= \varphi'(x^2 + y^2 + z^2) \cdot \left(0 + 2y + 2z \frac{\partial z}{\partial y}\right) \\m + nq &= 2\varphi'(x^2 + y^2 + z^2) \cdot (y + zq) \\ \frac{(m+nq)}{(y+zq)} &= 2\varphi'(x^2 + y^2 + z^2) \quad \text{--- (3)}\end{aligned}$$

From (2) and (3), we get

$$\frac{(l + np)}{(x + zp)} = \frac{(m + nq)}{(y + zq)}$$

$$(l + np)(y + zq) = (x + zp)(m + nq)$$

$$ly + lzq + npy + npzq = mx + nqx + mzp + zpnq$$

$$ly - mx = nqx + mzp - lzq - npy$$

$$ly - mx = p(mz - ny) + q(nx - lz), \text{ is the required PDE.}$$

6. $\varphi(x + y + z, x^2 + y^2 - z^2) = 0$

Sol: Given $\varphi(x + y + z, x^2 + y^2 - z^2) = 0$

Let $u = x + y + z$

$$\frac{\partial u}{\partial x} = 1 + 0 + \frac{\partial z}{\partial x}$$

$$\frac{\partial u}{\partial x} = 1 + p$$

$$\frac{\partial u}{\partial y} = 0 + 1 + \frac{\partial z}{\partial y}$$

$$\frac{\partial u}{\partial y} = 1 + q$$

$$\text{Now } \varphi(u, v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$v = x^2 + y^2 - z^2$$

$$\frac{\partial v}{\partial x} = 2x + 0 - 2z \frac{\partial z}{\partial x}$$

$$\frac{\partial v}{\partial x} = 2x - 2zp$$

$$\frac{\partial v}{\partial y} = 0 + 2y - 2z \frac{\partial z}{\partial y}$$

$$\frac{\partial v}{\partial y} = 2y - 2zq$$

$$\varphi(u, v) = 0$$
$$\left| \begin{matrix} (1+p) & (1+q) \\ (2x-2zp) & (2y-2zq) \end{matrix} \right| = 0$$

$$(1+p)(2y-2zq) - (1+q)(2x-2zp) = 0$$

$$2y - 2zq + 2py - 2zpq - 2x + 2zp - 2qx + 2zpq = 0 \quad \div \text{ by } 2$$

$$y - zq + py - x + zp - qx = 0$$

$$-zq + py + zp - qx = x - y$$

$p(y+z) - q(x+z) = x - y$, is the required PDE.

$$7. \varphi(xy + z^2, x + y + z) = 0$$

Sol: Given $\varphi(xy + z^2, x + y + z) = 0$

$$\text{Let } u = xy + z^2$$

$$v = x + y + z$$

$$\frac{\partial u}{\partial x} = y \cdot 1 + 2z \frac{\partial z}{\partial x}$$

$$\frac{\partial v}{\partial x} = 1 + 0 + \frac{\partial z}{\partial x}$$



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$$\frac{\partial u}{\partial x} = y + 2zp$$

$$\frac{\partial v}{\partial x} = 1 + p$$

$$\frac{\partial u}{\partial y} = x \cdot 1 + 0 + 2z \frac{\partial z}{\partial y}$$

$$\frac{\partial v}{\partial y} = 1 + 0 + \frac{\partial z}{\partial y}$$

$$\frac{\partial u}{\partial y} = x + 2zq$$

$$\frac{\partial v}{\partial y} = 1 + q$$

$$\text{Now } \varphi(u, v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\varphi(u, v) = 0$$

$$\begin{vmatrix} (y + 2zp) & (x + 2zq) \\ (1 + p) & (1 + q) \end{vmatrix} = 0$$

$$(y + 2zp)(1 + q) - (x + 2zq)(1 + p) = 0$$



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$$y + yq + 2zp + 2zpq - x - 2zq - px - 2pzq = 0$$



$$y + yq + 2zp - x - 2zq - px = 0$$

$$yq + 2zp - 2zq - px = x - y$$

$p(2z - x) + q(y - 2z) = x - y$, is the required PDE.

$$8. f(x^2 + y^2, z - xy) = 0$$

Sol: Given $f(x^2 + y^2, z - xy) = 0$

$$\text{Let } u = x^2 + y^2$$

$$\frac{\partial u}{\partial x} = 2x + 0$$

$$\frac{\partial u}{\partial x} = 2x$$

$$\frac{\partial u}{\partial y} = 0 + 2y$$

$$v = z - xy$$

$$\frac{\partial v}{\partial x} = \frac{\partial z}{\partial x} - y. 1$$

$$\frac{\partial v}{\partial x} = p - y$$

$$\frac{\partial v}{\partial y} = \frac{\partial z}{\partial y} - x. 1$$



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$$\frac{\partial u}{\partial y} = 2y$$

$$\frac{\partial v}{\partial y} = q - x$$

$$\text{Now } f(u, v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$f(u, v) = 0$$

$$\begin{vmatrix} (2x) & (2y) \\ (p - y) & (q - x) \end{vmatrix} = 0$$

$$(2x)(q - x) - (2y)(p - y) = 0$$

$$2xq - 2x^2 - 2yp + 2y^2 = 0 \quad \div \text{ by } 2$$

$$xq - yp = x^2 - y^2, \text{ is the required PDE.}$$

9. $f(x^2 + 2yz, y^2 + 2zx) = 0$

Sol: Given $f(x^2 + 2yz, y^2 + 2zx) = 0$

Let $u = x^2 + 2yz$

$$\frac{\partial u}{\partial x} = 2x + 2y \frac{\partial z}{\partial x}$$

$$\frac{\partial u}{\partial x} = 2x + 2yp$$

$$\frac{\partial u}{\partial y} = 0 + 2y \cdot \frac{\partial z}{\partial y} + 2z \cdot 1$$

$$\frac{\partial u}{\partial y} = 2yq + 2z$$

$$\text{Now } f(u, v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$v = y^2 + 2zx$$

$$\frac{\partial v}{\partial x} = 0 + 2x \cdot \frac{\partial z}{\partial x} + 2z \cdot 1$$

$$\frac{\partial v}{\partial x} = 2xp + 2z$$

$$\frac{\partial v}{\partial y} = 2y + 2x \frac{\partial z}{\partial y}$$

$$\frac{\partial v}{\partial y} = 2y + 2xq$$



ATME

$$f(u, v) = 0$$

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$$\begin{vmatrix} (2x + 2yp) & (2yq + 2z) \\ (2z + 2xp) & (2y + 2xq) \end{vmatrix} = 0$$

$$(2x + 2yp)(2y + 2xq) - (2yq + 2z)(2z + 2xp) = 0$$

$$4xy + 4x^2q + 4y^2p + 4xypq - 4zyq - 4xypq - 4z^2 - 4xzp = 0$$

$\div \text{by } 4$

$$xy + x^2q + y^2p - zyq - z^2 - xzp = 0$$

$$x^2q + y^2p - zyq - xzp = z^2 - xy$$

$$p(y^2 - xz) + q(x^2 - yz) = z^2 - xy, \text{ is the required PDE.}$$

$$10. f\left(\frac{xy}{z}, z\right) = 0$$

$$\text{Sol: Given } f\left(\frac{xy}{z}, z\right) = 0$$



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Let $u = \frac{xy}{z}$

$$v = z$$

$$\frac{\partial u}{\partial x} = y \cdot \frac{\left(z \cdot 1 - x \cdot \frac{\partial z}{\partial x}\right)}{z^2}$$

$$\frac{\partial u}{\partial x} = \frac{y(z - xp)}{z^2}$$

$$\frac{\partial u}{\partial y} = x \cdot \frac{\left(z \cdot 1 - y \cdot \frac{\partial z}{\partial y}\right)}{z^2}$$

$$\frac{\partial u}{\partial y} = \frac{x(z - yq)}{z^2}$$

$$\frac{\partial v}{\partial x} = \frac{\partial z}{\partial x}$$

$$\frac{\partial v}{\partial x} = p$$

$$\frac{\partial v}{\partial y} = \frac{\partial z}{\partial y}$$

$$\frac{\partial v}{\partial y} = q$$

$$\text{Now } f(u, v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$



ATME

$$f(u, v) = 0$$

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$$\left| \frac{y(z - xp)}{z^2 p} - \frac{x(z - yq)}{z^2 q} \right| = 0$$

$$\frac{y(z - xp)}{z^2 p} \cdot q - \frac{x(z - yq)}{z^2 q} \cdot p = 0$$
$$\frac{yq(z - xp)}{z^2} - \frac{xp(z - yq)}{z^2} = 0$$

$$\frac{yq(z - xp) - xp(z - yq)}{z^2} = 0$$
$$\frac{yqz - xppq - xzq + xypq}{z^2} = 0$$

$$yqz - xzq = 0 \quad ; \quad z(yq - xp) = 0$$

$yq - xp = 0$, is the required PDE.

11. $z = yf(x) + x\varphi(y)$

Sol: Given $z = yf(x) + x\varphi(y)$ --- (1)

Differentiate (1) partially

$$\frac{\partial z}{\partial x} = y \cdot f'(x) + \varphi(y) \cdot 1$$

$$p = yf'(x) + \varphi(y) \quad \text{--- (2)}$$

$$\frac{\partial z}{\partial y} = f(x) \cdot 1 + x \cdot \varphi'(y)$$

$$q = f(x) + x\varphi'(y) \quad \text{--- (3)}$$

$$\frac{\partial^2 z}{\partial x^2} = y \cdot f''(x) + 0$$

$$r = yf''(x) \quad \text{--- (4)}$$

$$\frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = f'(x) \cdot 1 + \varphi'(y)$$



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$$\frac{\partial^2 z}{\partial y \cdot \partial x} = f'(x) + \varphi'(y)$$

$$s = f'(x) + \varphi'(y) \quad \text{--- (5)}$$

$$\frac{\partial^2 z}{\partial y^2} = 0 + x\varphi''(y)$$

$$t = x\varphi''(y) \quad \text{--- (6)}$$

$$(1)*x + (2)*y$$

$$px + qy = xyf'(x) + x\varphi(y) + yf(x) + xy\varphi'(y)$$

$$px + qy = [x\varphi(y) + yf(x)] + xy[f'(x) + \varphi'(y)]$$

$$px + qy = z + xys \quad \text{From (1) and (5)}$$

$px + qy - xys = z$, is the required PDE.

$$12. z = f(y + x) + g(y + 2x)$$

$$\text{Sol: Given } z = f(y + x) + g(y + 2x) \quad \text{--- (1)}$$



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Differentiate (1) partially

$$\frac{\partial z}{\partial x} = f'(y + x) + 2g'(y + 2x)$$

$$p = f' + 2g' \quad \text{--- (2)}$$

$$\frac{\partial z}{\partial y} = f'(y + x) + g'(y + 2x)$$

$$q = f' + g' \quad \text{--- (3)}$$

$$\frac{\partial^2 z}{\partial x^2} = f''(y + x) + 4g''(y + 2x)$$

$$r = f'' + 4g'' \quad \text{--- (4)}$$

$$\frac{\partial^2 z}{\partial x \partial y} = f''(y + x) + 4g''(y + 2x)$$

$$s = f'' + 2g'' \quad \text{--- (5)}$$



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$$\frac{\partial^2 z}{\partial y^2} = f''(y+x) + g''(y+2x)$$

$$t = f'' + g'' \quad \text{--- (6)}$$

$$(4) - (5)$$

$$r - s = f'' + 4g'' - f'' - 2g''$$

$$r - s = 2g'' \quad \text{--- (7)}$$

$$(5) - (6)$$

$$s - t = f'' + 2g'' - f'' - g''$$

$$s - t = g'' \quad \text{--- (8)}$$

$$(7) \Rightarrow r - s = 2(s - t) \quad \text{From (8)}$$

$$r - s = 2s - 2t$$

$$r - s - 2s + 2t = 0$$

$r - 3s + 2t = 0$, is the required PDE.

13. $z = f_1(y - 2x) + f_2(2y - x)$

Sol: Given $z = f_1(y - 2x) + f_2(2y - x)$ --- (1)

Differentiate (1) partially

$$\frac{\partial z}{\partial x} = -2f_1'(y - 2x) - f_2'(2y - x)$$

$$p = -2f_1' - f_2' \quad \text{--- (2)}$$

$$\frac{\partial z}{\partial y} = f_1'(y - 2x) + 2f_2'(2y - x)$$

$$q = f_1' + 2f_2' \quad \text{--- (3)}$$

$$\frac{\partial^2 z}{\partial x^2} = 4f_1''(y - 2x) + f_2''(2y - x)$$

$$r = 4f_1'' + f_2'' \quad \text{--- (4)}$$

$$\frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = -2f_1''(y - 2x) - 2f_2''(2y - x)$$

$$s = -2f_1'' - 2f_2'' \quad \text{--- (5)}$$

$$\frac{\partial^2 z}{\partial y^2} = f_1''(y - 2x) + 4f_2''(2y - x)$$

$$t = f_1'' + 4f_2'' \quad \text{--- (6)}$$

Now

$$r + 2s = 4f_1'' + f_2'' - 4f_1'' - 4f_2''$$

$$r + 2s = -3f_2'' \quad \text{--- (7)}$$

Also

$$s + 2t = -2f_1'' - 2f_2'' + 2f_1'' + 8f_2''$$

$$s + 2t = 6f_2'' \quad \text{--- (8)}$$

$$(8) \div (7)$$

$$\frac{s + 2t}{r + 2s} = \frac{6f_2''}{-3f_2''}$$



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$$\frac{s + 2t}{r + 2s} = -2$$

$$s + 2t = -2(r + 2s)$$

$$s + 2t = -2r - 4s$$

$$s + 2t + 2r + 4s = 0$$

$5s + 2t + 2r = 0$, is the required PDE.

14. $z = f(x) + e^y g(x)$

Sol: Given $z = f(x) + e^y g(x)$ --- (1)

Differentiate (1) partially

$$\frac{\partial z}{\partial x} = f'(x) + e^y g'(x)$$

$p = f'(x) + e^y g'(x)$ --- (2)

$$\frac{\partial z}{\partial y} = e^y g(x)$$

$q = e^y g(x)$ --- (3)



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$$\frac{\partial^2 z}{\partial x^2} = f''(x) + e^y g''(x)$$

$$r = f''(x) + e^y g''(x) \quad \text{--- (4)}$$

$$\frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = 0 + e^y g'(x)$$

$$s = e^y g'(x) \quad \text{--- (5)}$$

$$\frac{\partial^2 z}{\partial y^2} = e^y g(x)$$

$$t = e^y g(x) \quad \text{--- (6)}$$

From (3) and (6)

$$q = t$$

$t - q = 0$, is the required the PDE.

15. $z = f(x + ct) + g(x - ct)$

Sol: Given $z = f(x + ct) + g(x - ct)$ --- (1)

$$\frac{\partial z}{\partial x} = f'(x + ct) + g'(x - ct)$$

$p = f' + g'$ --- (2)

$$\frac{\partial z}{\partial t} = cf'(x + ct) - cg'(x - ct)$$

$q = cf' - cg'$ --- (3)

$$\frac{\partial^2 z}{\partial x^2} = f''(x + ct) + g''(x - ct)$$

$r = f'' + g''$ --- (4)

$$\frac{\partial}{\partial t} \left(\frac{\partial z}{\partial x} \right) = cf''(x + ct) - cg''(x - ct)$$

$s = cf'' - cg''$ --- (5)



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$$\frac{\partial^2 z}{\partial t^2} = c^2 f''(x + ct) + c^2 g''(x - ct)$$

$$t = c^2 f'' + c^2 g'' \quad \text{--- (6)}$$

Consider

$$t = c^2 f'' + c^2 g''$$

$$t = c^2 [f'' + g'']$$

$$t = c^2 r$$

$t - c^2 r = 0$, is the required PDE.

1. Solve $\frac{\partial^2 u}{\partial x^2} = x + y$

Sol: Given $\frac{\partial^2 u}{\partial x^2} = x + y$

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = x + y$$

Integrate w.r.t 'x'

$$\frac{\partial u}{\partial x} = \int x \, dx + y \int 1 \, dx$$

$$\frac{\partial u}{\partial x} = \frac{x^2}{2} + yx + f(y)$$

Again integrating w.r.t 'x'

$$u = \frac{1}{2} \int x^2 \, dx + y \int x \, dx + f(y) \int 1 \, dx$$



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$$u = \frac{1}{2} \left[\frac{x^3}{3} \right] + y \left[\frac{x^2}{2} \right] + f(y) \cdot x + g(y)$$

$u = \frac{x^3}{6} + \frac{x^2 y}{2} + x f(y) + g(y)$, is the required solution.

2. Solve $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + a$

Sol: Given $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + a$

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{x}{y} + a$$

Integrate w.r.t 'x'

$$\frac{\partial z}{\partial y} = \frac{1}{y} \int x \, dx + a \int 1 \, dx$$

$$\frac{\partial z}{\partial y} = \frac{x^2}{2y} + ax + f(y)$$

$$z = \frac{x^2}{2} \int \frac{1}{y} dy + ax \int 1 dy + \int f(y) dy$$

$$z = \frac{x^2}{2} \log y + axy + F(y) + g(x), \text{ is the required solution.}$$

3. Solve $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$

Sol: Given $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$

$$\frac{\partial}{\partial x} \left(\frac{\partial^2 z}{\partial x \partial y} \right) = \cos(2x + 3y)$$

Integrate w.r.t 'x'

$$\frac{\partial^2 z}{\partial x \partial y} = \int \cos(2x + 3y) dx$$



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$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\sin(2x + 3y)}{2} + f(y)$$

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\sin(2x + 3y)}{2} + f(y)$$

Integrating w.r.t 'x'

$$\frac{\partial z}{\partial y} = \frac{1}{2} \int \sin(2x + 3y) dx + f(y) \int 1 dx$$

$$\frac{\partial z}{\partial y} = \frac{-\cos(2x + 3y)}{2.2} + f(y).x + g(y)$$

Integrating w.r.t 'y'

$$z = -\frac{1}{4} \int \cos(2x + 3y) dy + x \int f(y) dy + \int g(y) dy$$

$$z = -\frac{1}{4} \cdot \frac{\sin(2x + 3y)}{3} + xF(y) + G(y) + h(x)$$

$z = -\frac{\sin(x+3y)}{12} + xF(y) + G(y) + h(x)$, is the required solution.

4. Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ for which $\frac{\partial z}{\partial y} = -2 \sin y$ when $x = 0$ and $z = 0$ if y is an odd multiple of $\frac{\pi}{2}$. {or $z = 0$ if $y = (2n + 1) \frac{\pi}{2}$ }

Sol: Given $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \sin x \sin y$$

Integrate w.r.t 'x'

$$\frac{\partial z}{\partial y} = \sin y \int \sin x \, dx$$

$$\frac{\partial z}{\partial y} = \sin y (-\cos x) + f(y) \quad \text{--- (1)}$$

Integrate w.r.t 'y'



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$$z = -\cos x \int \sin y \, dy + \int f(y) \, dy$$

$$z = -\cos x (-\cos y) + F(y) + g(x)$$

$$z = \cos x \cdot \cos y + F(y) + g(x) \quad \text{--- (2)}$$

By data $\frac{\partial z}{\partial y} = -2\sin y$ when $x = 0$

$$(1) \Rightarrow -2\sin y = \sin y (-\cos 0) + f(y)$$

$$-2\sin y = -\sin y + f(y)$$

$$f(y) = -2\sin y + \sin y$$

$$f(y) = -\sin y$$

Now

$$F(y) = \int f(y) \, dy$$

Since $\cos 0 = 1$



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$$F(y) = - \int \sin y \, dy$$

$$F(y) = -(-\cos y)$$

$$F(y) = \cos y$$

$$(2) \Rightarrow z = \cos x \cdot \cos y + \cos y + g(x)$$

Again by data $z = 0$ if $y = (2n + 1)\frac{\pi}{2}$

$$0 = \cos x \cdot \cos \left[(2n + 1)\frac{\pi}{2} \right] + \cos \left[(2n + 1)\frac{\pi}{2} \right] + g(x)$$

$$\cos \left[(2n + 1)\frac{\pi}{2} \right] = 0$$

$$g(x) = 0$$

Thus,

$$(2) \Rightarrow z = \cos x \cdot \cos y + \cos y + 0$$

$$z = \cos x \cdot \cos y + \cos y$$

$$z = \cos y (\cos x + 1), \text{ is the solution.}$$



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5. Solve $\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$ given that $u = 0$ when $t = 0$ and $\frac{\partial u}{\partial t} = 0$ at $x = 0$. Also show that $u \rightarrow \sin x$ as $t \rightarrow \infty$.

Sol: Given $\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} \right) = e^{-t} \cos x$$

Integrate w.r.t ' x '

$$\frac{\partial u}{\partial t} = e^{-t} \int \cos x \, dx$$

$$\frac{\partial u}{\partial t} = e^{-t} \sin x + f(t) \quad \text{--- (1)}$$

Integrate w.r.t ' t '

$$u = \sin x \int e^{-t} dt + \int f(t) dt + g(x)$$

$$u = \sin x \left[\frac{e^{-t}}{-1} \right] + F(t) + g(x)$$



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$$u = -e^{-t} \sin x + F(t) + g(x) \quad \text{--- (2)}$$

By data $\frac{\partial u}{\partial t} = 0$ at $x = 0$

$$(1) \Rightarrow 0 = e^{-t}(\sin 0) + f(t)$$

$$f(t) = 0$$

Now

$$F(t) = \int f(t) dt$$

$$F(t) = 0$$

$$(2) \Rightarrow u = -e^{-t} \sin x + 0 + g(x)$$

$$u = -e^{-t} \sin x + g(x)$$

Also given $u = 0$ when $t = 0$

$$0 = -e^{-0} \sin x + g(x)$$

Since $\sin 0 = 0$

Since $e^{-0} = 1$



$$g(x) = \sin x$$

Thus,

$$(2) \Rightarrow u = -e^{-t} \sin x + 0 + \sin x$$

$u = \sin x(1 - e^{-t})$, is the solution.

As $t \rightarrow \infty$

$$u = \sin x(1 - e^{-\infty})$$

$$u = \sin x(1 - 0)$$

$$u = \sin x$$

$$\therefore u \rightarrow \sin x \quad \text{as } t \rightarrow \infty$$



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6. Solve $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y}$ subject to $z = 0$ when $x = 1$ and $\frac{\partial z}{\partial x} = \log x$ when $y = 1$.

Sol: Given $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y}$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{x}{y}$$

$$\frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{x}{y}$$

Integrate w.r.t 'y'

$$\frac{\partial z}{\partial x} = x \int \frac{1}{y} dy$$

$$\frac{\partial z}{\partial x} = x \log y + f(x) \quad \text{--- (1)}$$

Integrate w.r.t 'x'

$$z = \log y \int x dx + \int f(x) dx$$



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$$z = \log y \left[\frac{x^2}{2} \right] + F(x) + g(y) \quad \text{--- (2)}$$

By data $\frac{\partial z}{\partial x} = \log x$ when $y = 1$

$$(1) \Rightarrow \log x = x \log 1 + f(x)$$

$$\mathbf{f(x) = \log x}$$

Now

$$F(x) = \int f(x) dx$$

$$F(x) = \int \log x \cdot 1 dx$$

$$F(x) = x \log x - \int \frac{1}{x} \cdot x dx$$

$$\mathbf{F(x) = x \log x - x}$$

Since $\log 1 = 0$



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$$(2) \Rightarrow z = \frac{x^2 \log y}{2} + x \log x - x + g(y)$$

Also given $z = 0$ when $x = 1$

$$0 = \frac{1 \cdot \log y}{2} + 1 \cdot \log 1 - 1 + g(y)$$

Since $\log 1 = 0$

$$0 = \frac{\log y}{2} - 1 + g(y)$$

$$g(y) = 1 - \frac{\log y}{2}$$

Thus,

$$z = \frac{x^2 \log y}{2} + x \log x - x + 1 - \frac{\log y}{2}$$

$$z = \frac{\log y}{2} (x^2 - 1) + x(\log x - 1) + 1, \text{ is the required solution.}$$



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7. Solve $\frac{\partial^2 z}{\partial x^2} = xy$ subject to the conditions that $\frac{\partial z}{\partial x} = \log(1 + y)$ when $x = 1$ and $z = 0$ when $x = 0$.

Sol: Given $\frac{\partial^2 z}{\partial x^2} = xy$

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = xy$$

Integrate w.r.t 'x'

$$\frac{\partial z}{\partial x} = y \int x \, dx$$

$$\frac{\partial z}{\partial x} = y \cdot \frac{x^2}{2} + f(y) \quad \text{--- (1)}$$

Integrate w.r.t 'x'

$$z = \frac{y}{2} \int x^2 \, dx + f(y) \int 1 \, dx$$



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$$z = \frac{y}{2} \left[\frac{x^3}{3} \right] + xf(y) + g(y)$$

$$z = \frac{x^3 y}{6} + xf(y) + g(y) \quad \text{--- (2)}$$

By data $\frac{\partial z}{\partial x} = \log(1 + y)$ when $x = 1$

$$(1) \Rightarrow \log(1 + y) = \frac{y}{2} + f(y)$$

$$f(y) = \log(1 + y) - \frac{y}{2}$$

$$(2) \Rightarrow z = \frac{x^3 y}{6} + x \left[\log(1 + y) - \frac{y}{2} \right] + g(y)$$

Also by data $z = 0$ when $x = 0$

$$0 = 0 + 0 + g(y)$$

$$g(y) = 0$$



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Thus,

$$z = \frac{x^3 y}{6} + x \left[\log(1 + y) - \frac{y}{2} \right] + 0$$

$$z = \frac{x^3 y}{6} + x \left[\log(1 + y) - \frac{y}{2} \right] , \text{ is the required solution.}$$



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Module-4: Statistics-I

MEASURES OF CENTRAL TENDENCY

An average or measure of central tendency gives a single representative value for a set of usually unequal values. This value is the point around which all the values cluster. So, the measure of central tendency is also called a measure of central location.

Definition

An average is a value which is a representative of a set of data

Various important measures of central tendency are

A. Arithmetic mean

B. Geometric mean

C. Harmonic mean

D. Median and Quartiles

E. Mode

Mathematical Averages

Positional Averages

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Mathematical Averages

Positional Averages

Calculation of Arithmetic mean for Raw data

When the observed values are given individually such as $x_1, x_2, x_3, \dots, x_n$ the arithmetic mean is given by

$$\begin{aligned}\text{Arithmetic mean } \bar{X} &= \frac{\text{Total of all values}}{\text{Number of the observations}} \\ &= \frac{x_1 + x_2 + x_3 + \dots + x_n}{n} = \frac{\sum x_i}{n}\end{aligned}$$

Discrete Data

Let $x_1, x_2, x_3, \dots, x_n$ be the n values of the variable x with corresponding frequencies $f_1, f_2, f_3, \dots, f_n$. then the arithmetic mean $\bar{X} = \frac{x_1 \cdot f_1 + x_2 \cdot f_2 + x_3 \cdot f_3 + \dots + x_n \cdot f_n}{f_1 + f_2 + f_3 + \dots + f_n}$
$$= \frac{\sum x_i f_i}{\sum f_i}$$

Continuous data

Let $m_1, m_2, m_3, \dots, m_n$ be the mid values of the class interval of the variable x with corresponding frequency $f_1, f_2, f_3, \dots, f_n$. then

$$\text{the arithmetic mean } \bar{X} = \frac{m_1 \cdot f_1 + m_2 \cdot f_2 + m_3 \cdot f_3 + \dots + m_n \cdot f_n}{f_1 + f_2 + f_3 + \dots + f_n}$$

$$= \frac{\sum m_i f_i}{\sum f_i}$$

Calculate the arithmetic mean

x	f	xf
2	4	2x4= 8
4	6	24
6	10	60
8	12	96
10	8	80
12	7	84
14	3	42
	$\Sigma f_i =$ 50	$\Sigma x_i f_i$ = 394

$$\text{Arithmetic mean, } \bar{X} = \frac{\Sigma x_i f_i}{\Sigma f_i} = \frac{394}{50} = 78.8$$

Calculate the arithmetic mean

Class interval(x)	m	f	mf
20-40	$(20+40)/2 = 30$	4	120
40-60	50	6	300
60-80	70	10	700
80-100	90	12	1080
100-120	110	8	880
		$\Sigma f_i = 40$	$\Sigma m_i f_i = 3080$

$$\text{Arithmetic mean, } \bar{X} = \frac{\Sigma m_i f_i}{\Sigma f_i} = \frac{3080}{40} = 77.0$$



MEDIAN

- It is the value which divides the data into two equal parts.
- 50% of the observations will be less than median value and 50% of the values will be more than the median value.

Calculation for Raw data

Median = value of $(n+1)/2$ th observation after the values are arranged in ascending order of magnitude.



Discrete data

M_d = value of x corresponding to the cumulative frequency just greater than or equal to $N/2$

- Arrange the data in ascending order
- Find the c.f.; Calculate $N/2$
- In c.f. column see the value just $\geq N/2$
- M_d = value of x corresponding to this c.f.

Continuous data

$$M_d = L + \{(N/2 - c.f) \times c/f\}$$

where

L = lower limit of the median class

c = class interval of the median class

f = frequency of the median class

$c.f$ = cumulative frequency of the class preceding the median class $N = \sum f_i$

M_d class is the class corresponding to the c.f. just $\geq N/2$.

Find the median

Marks	No. of students	c.f.
10-25	6	6
25-40	20	26
40-55	44	70
55-70	26	96
70-85	3	99
85-100	1	100
	$\Sigma f_i = 100$	

- $N/2 = 100/2 = 50$
- Median class is 40-55
- $L = 40, f = 44, c = 55 - 40 = 15, c.f. = 26$
- $Md = L + \left\{ \frac{(N/2 - c.f.) \times c}{f} \right\}$
 $= 40 + \left\{ \frac{[50 - 26] \times 15}{44} \right\}$
 $= 40 + \left\{ \frac{(24 \times 15)}{44} \right\}$
 $= 40 + [360/44]$
 $= 40 + 8.18 = 48.18$



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Module-5: Statistics-II

Curve Fitting

This is a method of finding a specific relation connecting the dependent and independent variables for a given data so as to satisfy the data as accurately as possible.

Curve fitting by the method of least squares

The method of least squares provides a relationship $y = f(x)$ such that the sum of the squares of the residuals is least. The difference between observed(y_i) and expected value (Y_i) i.e., $R_i = y_i - Y_i$ are called the residuals.

Fitting of a Straight line $y = ax + b$

Given $y = ax + b$

Normal Equations(N.E's): $a \sum x + nb = \sum y$

$$a \sum x^2 + b \sum x = \sum xy$$

Solving the above equations we get the values of a and b , substituting these vales in the given equation leads to the curve of best fit.

Note:

Normal Equations(N.E's)for the st.line $y = a + bx$:

$$\begin{aligned}na + b \sum x &= \sum y \\ a \sum x + b \sum x^2 &= \sum xy\end{aligned}$$

Problems

1. Fit a straight line $y = ax + b$ for the following data

x	1	3	4	6	8	9	11	14
y	1	2	4	4	5	7	8	9

Sol: The normal equations for fitting the Straight line $y = ax + b$ --- (*) are

$$a \sum x + nb = \sum y$$

$$a \sum x^2 + b \sum x = \sum xy$$

Here n=8

x	y	xy	x^2
1	1	1	1
3	2	6	9
4	4	16	16
6	4	24	36
8	5	40	64
9	7	63	81
11	8	88	121
14	9	126	196
$\sum x = 56$	$\sum y = 40$	$\sum xy = 364$	$\sum x^2 = 524$

The N.E's become

$$56a + 8b = 40$$

$$524a + 56b = 364$$

On solving , we get

$$a = 0.6363 \quad \text{and} \quad b = 0.5454$$

Thus, (*) $\Rightarrow y = (0.6363)x + (0.5454)$, is the curve of best fit.

2. Find the equation of the best fitting straight line for the following data and hence estimate the value of dependent variable corresponding to the value 30 of the independent variable.

x	5	10	15	20	25
y	16	19	23	26	30

Sol: Let $y = ax + b$ --- (*) be the equation of the Straight line.

N.E's : $a \sum x + nb = \sum y$

$$a \sum x^2 + b \sum x = \sum xy$$

Here n=5

x	y	xy	x^2
5	16	80	25
10	19	190	100
15	23	345	225
20	26	520	400
25	30	750	625
$\sum x = 75$	$\sum y = 114$	$\sum xy = 1885$	$\sum x^2 = 1375$

The N.E's become

$$75a + 5b = 114$$

$$1375a + 75b = 1885$$

On solving , we get

$$a = 0.7 \quad \text{and} \quad b = 12.3$$

Thus, (*) $\Rightarrow y = (0.7)x + (12.3)$, is the curve of best fit.

$$\text{When } x = 30 \Rightarrow y = (0.7)(30) + (12.3)$$

$$y = 33.3$$

3. Fit a straight line in the least square sense for the following data

x	50	70	100	120
y	12	15	21	25



Sol: Let $y = ax + b$ --- (*) be the equation of the Straight line.

N.E's : $a \sum x + nb = \sum y$

$$a \sum x^2 + b \sum x = \sum xy$$

Here $n=4$

x	y	xy	x^2
50	12	600	2500
70	15	1050	4900
100	21	2100	10000
120	25	3000	14400
$\sum x = 340$	$\sum y = 73$	$\sum xy = 6750$	$\sum x^2 = 31800$

The N.E's become

$$340a + 4b = 73$$

$$13800a + 340b = 6750$$

On solving , we get

$$a = 0.19 \quad \text{and} \quad b = 2.28$$

Thus, (*) $\Rightarrow y = (0.19)x + (2.28)$, is the curve of best fit.

4. A simply supported beam carries a concentrated load P at its mid point. Corresponding to various values of P the maximum deflection Y is measured and is given in the following table.

P	100	120	140	160	180	200
Y	0.45	0.55	0.60	0.70	0.80	0.85

Find a law of the form $Y = a + bP$ and estimate Y when P is 150.

Sol: Let $Y = a + bP$ --- (*) be the equation of the Straight line.

N.E's : $na + b \sum P = \sum Y$

$$a \sum P + b \sum P^2 = \sum PY$$

Here $n = 6$

P	Y	PY	P^2
100	0.45	45	10000
120	0.55	66	14400
140	0.60	84	19600
160	0.70	112	25600
180	0.80	144	32400
200	0.85	170	40000
$\sum P = 900$	$\sum Y = 3.95$	$\sum PY = 621$	$\sum P^2 = 142000$

The N.E's become

$$6a + 900b = 3.95$$

$$900a + 142000b = 621$$

On solving , we get

$$a = 0.0476 \quad \text{and} \quad b = 0.0041$$

Thus, (*) $\Rightarrow Y = (0.0476)P + (0.0041)$, is the curve of best fit.

When $P = 150 \Rightarrow Y = (0.0476)(150) + (0.0041)$

$$Y= 0.6626$$

5. Fit a straight line to the following data.

Year	1961	1971	1981	1991	2001
Production (in tons)	8	10	12	10	16

Also find the expected production in the year 2006.

Sol: Let year and production respectively be represented by the variables ***x*** and ***y***.

Let the equation of Straight be ***y = a + bx***

Let ***X = x – 1981*** and the line of fit will be ***y = a + bX*** --- (*)

N.E's are ***na + b ∑ X = ∑ y***
a ∑ X + b ∑ X² = ∑ Xy . Here n=5

<i>x</i>	<i>X = x – 1981</i>	<i>y</i>	<i>Xy</i>	<i>X²</i>
1961	-20	8	-160	400
1971	-10	10	-100	100
1981	0	12	0	0
1991	10	10	100	100
2001	20	16	320	400
	<i>∑ X = 0</i>	<i>∑ y = 56</i>	<i>∑ Xy = 160</i>	<i>∑ X² = 1000</i>



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N.E's becomes

$$5a + 0.b = 56$$
$$0.a + 1000b = 160$$

On solving , we get

$$a = 11.2 \quad \text{and} \quad b = 0.16$$

Thus, (*) $\Rightarrow y = 11.2 + (0.16)X$

$$y = 11.2 + (0.16)(x - 1981)$$

$$y = -305.76 + (0.16)x, \text{ is the curve of best fit.}$$

When $x = 2006$

$$y = -305.76 + (0.16)(2006) = 15.2 \text{ tons}$$

Expected production in the year 2006 is 15.2 tons.

Fitting of a Second degree parabola $y = ax^2 + bx + c$

Given $y = ax^2 + bx + c$

Normal Equations(N.E's):

$$a \sum x^2 + b \sum x + nc = \sum y$$
$$a \sum x^3 + b \sum x^2 + c \sum x = \sum xy$$
$$a \sum x^4 + b \sum x^3 + c \sum x^2 = \sum x^2y$$

Solving the above equations we get the values of a , b and c , substituting these values in the given equation leads to the curve of best fit.

Note: If $y = a + bx + cx^2$

Normal Equations(N.E's):

$$na + b \sum x + c \sum x^2 = \sum y$$
$$a \sum x + b \sum x^2 + c \sum x^3 = \sum xy$$
$$a \sum x^2 + b \sum x^3 + c \sum x^4 = \sum x^2y$$

Problems

1. Fit a second degree parabola $y = ax^2 + bx + c$ in the least square sense for the following data and hence estimate y at $x = 6$.

x	1	2	3	4	5
y	10	12	13	16	19

Sol: Given $y = ax^2 + bx + c$

Normal Equations(N.E's):

$$\begin{aligned}a \sum x^2 + b \sum x + nc &= \sum y \\a \sum x^3 + b \sum x^2 + c \sum x &= \sum xy \\a \sum x^4 + b \sum x^3 + c \sum x^2 &= \sum x^2 y\end{aligned}$$

Here $n=5$

x	y	xy	x^2	x^2y	x^3	x^4
1	10	10	1	10	1	1
2	12	24	4	48	8	16
3	13	39	9	117	27	81
4	16	64	16	256	64	256
5	19	95	25	475	125	625
$\sum x = 15$	$\sum y = 70$	$\sum xy = 232$	$\sum x^2 = 55$	$\sum x^2y = 906$	$\sum x^3 = 225$	$\sum x^4 = 979$



The N.E's become

$$\begin{aligned}55a + 15b + 5c &= 70 \\225a + 55b + 15c &= 232 \\979a + 225b + 55c &= 906\end{aligned}$$

On Solving, we have

$$a = 0.2857 \quad b = 0.4857 \quad \text{and} \quad c = 9.4$$

Thus, (*) $\Rightarrow y = (0.2857)x^2 + (0.4857)x + (9.4)$, is the curve of best fit .

At $x = 6$,

$$y = (0.2857)(6)^2 + (0.4857)(6) + (9.4)$$

$$y = 22.78$$

2. Fit a second degree parabola $y = a + bx + cx^2$ for the data:

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

Sol: Given $y = a + bx + cx^2$

Normal Equations(N.E's):

$$\begin{aligned} na + b \sum x + c \sum x^2 &= \sum y \\ a \sum x + b \sum x^2 + c \sum x^3 &= \sum xy \\ a \sum x^2 + b \sum x^3 + c \sum x^4 &= \sum x^2 y \end{aligned}$$

Here $n=5$



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x	y	xy	x^2	x^2y	x^3	x^4
0	1	0	0	0	0	0
1	1.8	1.8	1	1.8	1	1
2	1.3	2.6	4	5.2	8	16
3	2.5	7.5	9	22.5	27	81
4	6.3	25.2	16	100.8	64	256
$\sum x = 10$	$\sum y = 12.9$	$\sum xy = 37.1$	$\sum x^2 = 30$	$\sum x^2y = 130.3$	$\sum x^3 = 100$	$\sum x^4 = 354$

The N.E's become

$$\begin{aligned}5a + 10b + 30c &= 12.9 \\10a + 30b + 100c &= 37.1 \\30a + 100b + 354c &= 130.3\end{aligned}$$

On Solving, we have

$a = 1.42$ **$b = -1.07$** and **$c = 0.55$**

Thus, (*) $\Rightarrow y = (1.42) + (-1.07)x + (0.55)x^2$, is the curve of best fit .

3. Fit a parabola $y = ax^2 + bx + c$ in the least square sense for the following data

x	2	4	6	8	10
y	3.07	12.85	31.47	57.38	91.29

Sol: Given $y = ax^2 + bx + c$
Normal Equations(N.E's):

$$\begin{aligned}
 a \sum x^2 + b \sum x + nc &= \sum y \\
 a \sum x^3 + b \sum x^2 + c \sum x &= \sum xy \\
 a \sum x^4 + b \sum x^3 + c \sum x^2 &= \sum x^2y
 \end{aligned}$$

Here n=5

x	y	xy	x^2	x^2y	x^3	x^4
2	3.07	6.14	4	12.28	8	16
4	12.85	51.40	16	205.60	64	256
6	31.47	188.82	36	1132.92	216	1296
8	57.38	459.04	64	3672.32	512	4096
10	91.29	912.90	100	9129.00	1000	10000
$\sum x = 30$	$\sum y = 196.06$	$\sum xy = 1618.3$	$\sum x^2 = 220$	$\sum x^2y = 14152.12$	$\sum x^3 = 1800$	$\sum x^4 = 15664$

The N.E's become

$$\begin{aligned}220a + 30b + 5c &= 196.06 \\1800a + 220b + 30c &= 1618.3 \\15664a + 1800b + 220c &= 14152.12\end{aligned}$$

On Solving, we have

$$a = 0.992 \qquad b = -0.8551 \quad \text{and} \qquad c = 0.696$$

Thus, (*) $\Rightarrow y = (0.992)x^2 + (-0.8551)x + (0.696)$, is the curve of best fit.

4. Find the best values of a , b , c if the equation $y = a + bx + cx^2$ is to fit most closely to the following observations

x	-2	-1	0	1	2
y	-3.150	-1.390	0.620	2.880	5.378

Sol: Equation of the parabola $y = a + bx + cx^2$

Normal Equations(N.E's):

$$na + b \sum x + c \sum x^2 = \sum y$$

$$a \sum x + b \sum x^2 + c \sum x^3 = \sum xy$$

$$a \sum x^2 + b \sum x^3 + c \sum x^4 = \sum x^2y$$

Here n=5

x	y	xy	x^2	x^2y	x^3	x^4
-2	-3.150	6.30	4	-12.60	-8	16
-1	-1.390	1.39	1	-1.39	-1	1
0	0.620	0	0	0	0	0
1	2.880	2.88	1	2.88	1	1
2	5.378	10.756	4	21.512	8	16
$\sum x = 0$	$\sum y = 4.338$	$\sum xy = 21.326$	$\sum x^2 = 10$	$\sum x^2y = 10.402$	$\sum x^3 = 0$	$\sum x^4 = 34$

The N.E's become

$$\begin{aligned} 5a + 0.b + 10c &= 4.338 \\ 0.a + 10b + 0.c &= 21.326 \\ 10a + 0.b + 34c &= 10.402 \end{aligned}$$

On Solving, we have

$$a = 0.6210 \qquad b = 2.1326 \qquad \text{and} \qquad c = 0.1233$$

5. Fit a parabola for the following data.

<i>x</i>	1	2	3	4	5	6	7	8	9
<i>y</i>	2	6	7	8	10	11	11	10	9

Sol: Equation of the parabola $y = a + bx + cx^2$

Normal Equations(N.E's):

$$\begin{aligned} na + b\sum x + c\sum x^2 &= \sum y \\ a\sum x + b\sum x^2 + c\sum x^3 &= \sum xy \\ a\sum x^2 + b\sum x^3 + c\sum x^4 &= \sum x^2y \end{aligned}$$

Here n=9



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x	y	xy	x^2	x^2y	x^3	x^4
1	2	2	1	2	1	1
2	6	12	4	24	8	16
3	7	21	9	63	27	81
4	8	32	16	128	64	256
5	10	50	25	250	125	625
6	11	66	36	396	216	1296
7	11	77	49	539	343	2401
8	10	80	64	640	512	4096
9	9	81	81	729	729	6561
$\sum x = 45$	$\sum y = 74$	$\sum xy = 421$	$\sum x^2 = 285$	$\sum x^2y = 2771$	$\sum x^3 = 2025$	$\sum x^4 = 15333$

The N.E's becomes

$$9a + 45b + 285c = 74$$

$$45a + 285b + 2025c = 421$$

$$285a + 2025b + 15333c = 2771$$

On Solving, we have

$$a = -0.9286 \quad b = 3.5232 \quad \text{and} \quad c = -0.2673$$

Thus, (*) $\Rightarrow y = (-0.9286) + (3.5232)x + (-0.2673)x^2$, is the curve of best fit .

6. Fit a second degree parabola to the following data

x	1	1.5	2	2.5	3	3.5	4
y	1.1	1.3	1.6	2	2.7	3.4	4.1

Sol: Equation of the parabola $y = a + bx + cx^2$

Normal Equations(N.E's):

$$\begin{aligned} na + b \sum x + c \sum x^2 &= \sum y \\ a \sum x + b \sum x^2 + c \sum x^3 &= \sum xy \\ a \sum x^2 + b \sum x^3 + c \sum x^4 &= \sum x^2y \end{aligned}$$

Here n=7

x	y	xy	x^2	x^2y	x^3	x^4
1	1.1	1.1	1	1.1	1	1
1.5	1.3	1.95	2.25	2.95	3.375	5.0625
2	1.6	3.2	4	6.4	8	16
2.5	2	5	6.25	12.5	15.625	39.0625
3	2.7	8.1	9	24.3	27	81
3.5	3.4	11.9	12.25	41.65	42.875	150.0625
4	4.1	16.4	16	65.6	64	256
$\sum x = 17.5$	$\sum y = 16.2$	$\sum xy = 47.65$	$\sum x^2 = 50.75$	$\sum x^2y = 154.475$	$\sum x^3 = 161.875$	$\sum x^4 = 548.1875$



The N.E's becomes

$$\begin{aligned}7a + 17.5b + 50.75c &= 16.2 \\17.5a + 50.75b + 161.875c &= 47.65 \\50.75a + 161.875b + 548.1875c &= 154.475\end{aligned}$$

On Solving, we have

$$a = 1.0357 \quad b = -0.1928 \quad \text{and} \quad c = 0.2428$$

Thus, (*) $\Rightarrow y = (1.0357) + (-0.1928)x + (0.2428)x^2$, is the curve of best fit .

Fitting of a curve of the form $y = ax^b$

Consider $y = ax^b$

Taking $\log_e$ on both sides we get,

$$\log_e y = \log_e(ax^b)$$

$$\log_e y = \log_e a + \log_e x^b$$

$$\log_e y = \log_e a + b \log_e x$$

$$Y = A + bX \quad , \text{ where } Y = \log_e y \quad , A = \log_e a \quad X = \log_e x$$

Normal Equations(N.E's)for the st.line $Y = A + bX$:

$$\begin{aligned} nA + b \sum X &= \sum Y \\ A \sum X + b \sum X^2 &= \sum XY \end{aligned}$$

Solving above equations we get A, b . But $\log_e a = A \Rightarrow a = e^A$

Problems

1. Fit a least square geometric curve $y = ax^b$ for the following data

x	1	2	3	4	5
y	0.5	2	4.5	8	12.5

Sol: Consider $y = ax^b$ --- (*)

Taking $\log_e$ on both sides we get,

$$\log_e y = \log_e(ax^b)$$

$$\log_e y = \log_e a + \log_e x^b$$

$$\log_e y = \log_e a + b \log_e x$$

$$Y = A + bX \quad , \text{ where } Y = \log_e y \quad , A = \log_e a \quad X = \log_e x$$

Normal Equations(N.E's)for the st.line $Y = A + bX$

$$\begin{aligned} nA + b \sum X &= \sum Y \\ A \sum X + b \sum X^2 &= \sum XY \end{aligned}$$

Here $n=5$

x	y	$X = \log_e x$	$Y = \log_e y$	XY	x^2
1	0.5	0	-0.6931	0	0
2	2	0.6931	0.6931	0.4804	0.4804
3	4.5	1.0986	1.5041	1.6524	1.2069
4	8	1.3863	2.0794	2.8827	1.9218
5	12.5	1.6094	2.5257	4.0649	2.5902
		$\sum X = 4.7874$	$\sum Y = 6.1092$	$\sum XY = 9.0804$	$\sum x^2 = 6.1933$

$N.E$'s becomes

$$5A + 4.7874b = 6.1092$$

$$4.7874A + 6.1993b = 9.0804$$

On Solving, we have

$$A = -0.6932 \quad b = 2$$

But $\log_e a = -0.6932 \Rightarrow a = e^{-0.6932} = 0.5$

Thus, (*) $\Rightarrow y = ax^b$

$y = (0.5)x^2$, is the curve of best fit.

2. An experiment of lifetime t of cutting tool at different cutting speeds v (units) are given below

Speed (v)	350	400	500	600
Life (t)	61	26	7	2.6

Fit a relation of the form $v = at^b$.



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Sol: Consider $v = at^b$ --- (*)

Taking $\log_e$ on both sides we get,

$$\log_e v = \log_e(at^b)$$

$$\log_e v = \log_e a + \log_e t^b$$

$$\log_e v = \log_e a + b \log_e t$$

$$V = A + bT, \text{ where } V = \log_e v, A = \log_e a, T = \log_e t$$

Normal Equations(N.E's)for the st.line $V = A + bT$

$$\begin{aligned} nA + b \sum T &= \sum V \\ A \sum T + b \sum T^2 &= \sum VT \end{aligned}$$

Here n=4

v	t	$V = \log_e v$	$T = \log_e t$	VT	T^2
350	61	5.8579	4.1109	24.0812	16.8995
400	26	5.9915	3.2581	19.5209	10.6152
500	7	6.2146	1.9459	12.093	3.7865
600	2.6	6.3969	0.9555	6.1122	0.913
		$\sum V$ = 24.4609	$\sum T$ = 10.2704	$\sum VT$ = 61.8073	$\sum T^2$ = 32.2142

N.E's becomes

$$4A + 10.2704b = 24.4609$$
$$10.2704A + 32.2142b = 61.8073$$

Solving above equations we get,

$$A = 6.5539 \quad b = -0.1709$$

But $\log_e a = 6.5539 \Rightarrow a = e^{6.5539} = 701.9766$

Thus, (*) $\Rightarrow v = at^b$

$$v = (701.9766)t^{-0.1709}, \text{ is the curve of best fit.}$$

3. Fit a curve of the form $y = ax^b$ for the following data

x	1	2	3	4	5	6
y	2.98	4.26	5.21	6.1	6.8	7.5



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Statistical Methods



Statistics is a science of facts and figures which may be readily available or obtained through the process of direct enquiry or enumeration. It deals with the methods of collecting, classifying and analysing the data so as to draw some valid conclusion.

‘Correlation’ can be understood as co-relation. Co-variance of two independent magnitudes is known as correlation. ‘Regression’ is an estimation of one independent variable interms of the other.

Correlation and Regression

This topic deals with data concerning independent observations.

Examples

- 1. Marks of individuals in two subjects*
- 2. Height and weight of individuals*
- 3. Amount involved in advertising a product and sale of product etc.,*

We discuss the aspect of inter-relation between the independent variables.

Correlation and Correlation Coefficient

Co-variation of two independent magnitudes is known as **correlation**. If two variables x and y are related in such a way that increase or decrease in one of them corresponds to increase or decrease in the other, we say that the variables are positively correlated. Also if increase or decrease in one of them corresponds to decrease or increase in the other, the variables are negatively correlated.

The numerical measure of correlation between two variables x and y is known as Karl Pearson's coefficient of correlation usually by r and is defined as follows

$$r = \frac{\sum XY}{\sqrt{\sum X^2} \cdot \sqrt{\sum Y^2}} \quad \text{where, } X = x - \bar{x}, Y = y - \bar{y}, \bar{x} = \frac{\sum x}{n}, \bar{y} = \frac{\sum y}{n}$$

Note:

$$1. \sigma_x^2 = \frac{\sum X^2}{n} \quad ; \quad \sigma_y^2 = \frac{\sum Y^2}{n} \quad ; \quad \sigma_x \sigma_y = \frac{\sqrt{\sum X^2} \cdot \sqrt{\sum Y^2}}{n} \quad \text{or} \quad n \sigma_x \sigma_y = \sqrt{\sum X^2} \cdot \sqrt{\sum Y^2}$$

2. The coefficient of correlation numerically does not exceed unity i.e., $-1 \leq r \leq 1$
3. If $r = \pm 1$, we say that x and y are perfectly correlated and if $r = 0$ we say that x and y are non correlated.
4.
$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y}$$

Regression

Regression is an estimation of one independent variable in terms of the other. If x and y are correlated, the best fitting straight line in the least square sense gives reasonably a good relation between x and y .

The best fitting straight line of the form $y = ax + b$ is called the regression line of y on x and $x = ay + b$ is called the regression line of x on y .

The regression line of **y on x** is

$$(y - \bar{y}) = r \frac{\sigma_y}{\sigma_x} (x - \bar{x})$$

The regression line of **x on y** is

$$(x - \bar{x}) = r \frac{\sigma_x}{\sigma_y} (y - \bar{y})$$

$r \frac{\sigma_x}{\sigma_y}$; $r \frac{\sigma_y}{\sigma_x}$ are known as regression coefficients.

Note:

1. $r = \pm \sqrt{(\text{coeff. of } x) \cdot (\text{coeff. of } y)}$, the sign of r will be positive or negative according as the regression coefficients are positive or negative.
2. The lines of regression are also of the form $Y = \frac{\sum XY}{\sum X^2} (X)$ and $X = \frac{\sum XY}{\sum Y^2} (Y)$

Problems

1. Compute the coefficient of correlation and the equation of the lines of regression for the data

x	1	2	3	4	5	6	7
y	9	8	10	12	11	13	14

Sol: Now $\bar{x} = \frac{\sum x}{n}$, $\bar{y} = \frac{\sum y}{n}$

Here $\sum x = 28$ $\sum y = 77$, $n = 7$

$$\bar{x} = \frac{28}{7} \qquad \bar{y} = \frac{77}{7}$$

$$\bar{x} = 4 \qquad \bar{y} = 11$$

x	y	$X = x - \bar{x}$ $= x - 4$	$Y = y - \bar{y}$ $= y - 11$	XY	X^2	Y^2
1	9	-3	-2	6	9	4
2	8	-2	-3	6	4	9
3	10	-1	-1	1	1	1
4	12	0	1	0	0	1
5	11	1	0	0	1	0
6	13	2	2	4	4	4
7	14	3	3	9	9	9
				$\sum XY = 26$	$\sum X^2 = 28$	$\sum Y^2 = 28$



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$$\text{Wkt, } r = \frac{\sum XY}{\sqrt{\sum X^2} \cdot \sqrt{\sum Y^2}}$$

$$r = \frac{26}{\sqrt{28} \cdot \sqrt{28}}$$

$$r = \frac{26}{28}$$
$$r = \mathbf{0.9286}$$

The lines regression are given by

$$(y - \bar{y}) = r \frac{\sigma_y}{\sigma_x} (x - \bar{x})$$

$$(y - 11) = 0.9286(x - 4)$$

$$r \frac{\sigma_y}{\sigma_x} = \frac{\sum XY}{\sum X^2} = \frac{26}{28} = \mathbf{0.9286}$$

$$y - 11 = 0.9286x - 3.7144$$

$$y = 0.9286x + 7.2856$$

$$\text{Also, } (x - \bar{x}) = r \frac{\sigma_x}{\sigma_y} (y - \bar{y})$$

$$(x - 4) = 0.9286(y - 11)$$

$$x - 4 = 0.9286y - 10.2146$$

$$x = 0.9286y - 6.2146$$

$$r \frac{\sigma_x}{\sigma_y} = \frac{\sum XY}{\sum Y^2} = \frac{26}{28} = 0.9286$$

2. Obtain the lines of regression and hence find the coefficient of correlation for the following data

x	1	2	3	4	5	6	7
y	9	8	10	12	11	13	14

Sol: Now $\bar{x} = \frac{\sum x}{n}$, $\bar{y} = \frac{\sum y}{n}$

Here $\sum x = 28$ $\sum y = 77$, $n = 7$, $\bar{x} = \frac{28}{7}$, $\bar{x} = 4$; $\bar{y} = \frac{77}{7}$, $\bar{y} = 11$

x	y	$X = x - \bar{x}$ $= x - 4$	$Y = y - \bar{y}$ $= y - 11$	XY	X^2	Y^2
1	9	-3	-2	6	9	4
2	8	-2	-3	6	4	9
3	10	-1	-1	1	1	1
4	12	0	1	0	0	1
5	11	1	0	0	1	0
6	13	2	2	4	4	4
7	14	3	3	9	9	9
				$\sum XY = 26$	$\sum X^2 = 28$	$\sum Y^2 = 28$

The lines regression are given by

$$(y - \bar{y}) = \frac{\sum XY}{\sum X^2} (x - \bar{x})$$

$$(y - 11) = 0.9286(x - 4)$$

$$y - 11 = 0.9286x - 3.7144$$

$$y = \mathbf{0.9286x + 7.2856}$$

$$\frac{\sum XY}{\sum X^2} = \frac{26}{28} = \mathbf{0.9286}$$

Also , $(x - \bar{x}) = \frac{\sum XY}{\sum Y^2} (y - \bar{y})$

$$(x - 4) = 0.9286(y - 11)$$

$$x - 4 = 0.9286y - 10.2146$$

$$x = \mathbf{0.9286y - 6.2146}$$

$$\frac{\sum XY}{\sum Y^2} = \frac{26}{28} = \mathbf{0.9286}$$

Wkt $r = \sqrt{(\text{coeff. of } x) \cdot (\text{coeff. of } y)}$

r must be positive since regression coefficients are +ve

$$r = \sqrt{(0.9286) \cdot (0.9286)}$$

$$r = 0.9286$$

3. Find the correlation coefficient for the following two groups

A	92	89	87	86	83	77	71	63	53	50
B	86	83	91	77	68	85	52	82	37	57

Sol: Now $\bar{x} = \frac{\sum x}{n}$, $\bar{y} = \frac{\sum y}{n}$

Here $\sum x = 751$ $\sum y = 718$, $n = 10$

$$\bar{x} = \frac{751}{10} , \bar{x} = 75.1 \quad ; \quad \bar{y} = \frac{718}{10} , \bar{y} = 71.8$$

x	y	$X = x - \bar{x}$ $= x - 75.1$	$Y = y - \bar{y}$ $= y - 71.8$	XY	X^2	Y^2
92	86	16.9	14.2	239.98	285.61	201.64
89	83	13.9	11.2	155.68	193.21	125.44
87	91	11.9	19.2	228.48	141.61	368.64
86	77	10.9	5.2	56.68	118.81	27.04
83	68	7.9	-3.8	-30.02	62.41	14.44
77	85	1.9	13.2	25.08	3.61	174.24
71	52	-4.1	-19.8	81.18	16.81	392.04
63	82	-12.1	10.2	-123.42	146.41	104.04
53	37	-22.1	-34.8	769.08	488.41	1211.04
50	57	-25.1	-14.8	371.48	630.01	219.04
				$\sum XY = 1774.2$	$\sum X^2 = 2086.59$	$\sum Y^2 = 2837.6$

Wkt, $r = \frac{\sum XY}{\sqrt{\sum X^2} \cdot \sqrt{\sum Y^2}}$

$$r = \frac{1774.2}{\sqrt{2086.59} \cdot \sqrt{2837.6}}$$

$$r = \frac{1774.2}{2433.29}$$

$$r = 0.7291$$

4. Find the correlation coefficient and the equation of the lines of regression for the following values of x and y

x	1	2	3	4	5
y	2	5	3	8	7

Sol: Now $\bar{x} = \frac{\sum x}{n}$, $\bar{y} = \frac{\sum y}{n}$

Here $\sum x = 15$ $\sum y = 25$, $n = 5$

$\bar{x} = \frac{15}{5}$, $\bar{x} = 3$; $\bar{y} = \frac{25}{5}$, $\bar{y} = 5$

x	y	$X = x - \bar{x}$ $= x - 3$	$Y = y - \bar{y}$ $= y - 5$	XY	X^2	Y^2
1	2	-2	-3	6	4	9
2	5	-1	0	0	1	0
3	3	0	-2	0	0	4
4	8	1	3	3	1	9
5	7	2	2	4	4	4
				$\sum XY = 13$	$\sum X^2 = 10$	$\sum Y^2 = 26$

$$\text{Wkt, } r = \frac{\sum XY}{\sqrt{\sum X^2} \cdot \sqrt{\sum Y^2}}$$

$$r = \frac{13}{\sqrt{10} \cdot \sqrt{26}}$$

$$r = 0.8062$$

$$\text{Now, } \sigma_x^2 = \frac{\sum X^2}{n} = \frac{10}{5} = 2 \quad ; \quad \sigma_x^2 = 2 \quad ; \quad \sigma_x = \sqrt{2}$$

$$\sigma_y^2 = \frac{\sum Y^2}{n} = \frac{26}{5} = 5.2 \quad ; \quad \sigma_y^2 = 5.2 \quad ; \quad \sigma_y = \sqrt{5.2}$$

The lines regression are given by

$$(y - \bar{y}) = r \frac{\sigma_y}{\sigma_x} (x - \bar{x})$$



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$$(y - 5) = (0.8062) \frac{\sqrt{2}}{\sqrt{5.2}} (x - 3)$$

$$(y - 5) = (1.3) (x - 3)$$

$$y - 5 = 1.3x - 3.9$$

$$y = 1.3x - 3.9 + 5$$

$$\mathbf{y = 1.3x + 1.1}$$

$$\text{Also , } (x - \bar{x}) = r \frac{\sigma_x}{\sigma_y} (y - \bar{y})$$

$$(x - 3) = (0.8062) \frac{\sqrt{2}}{\sqrt{5.2}} (y - 5)$$

$$x - 3 = 0.5(y - 5)$$

$$x = 0.5y - 2.5 + 3$$

$$\mathbf{x = 0.5y + 0.5}$$

5. The following data gives the age of husband (x) and the age of wife (y) in years. Form the two regression lines and calculate the age of husband corresponding to 16 years age of wife.

x	36	23	27	28	28	29	30	31	33	35
y	29	18	20	22	27	21	29	27	29	28

Sol: Now $\bar{x} = \frac{\sum x}{n}$, $\bar{y} = \frac{\sum y}{n}$

Here $\sum x = 300$ $\sum y = 250$, $n = 10$

$\bar{x} = \frac{300}{10}$
 $\bar{y} = \frac{250}{10}$

$\bar{x} = 30$
 $\bar{y} = 25$



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x	y	$X = x - \bar{x}$ $= x - 30$	$Y = y - \bar{y}$ $= y - 25$	XY	X^2	Y^2
36	29	6	4	24	36	16
23	18	-7	-7	49	49	49
27	20	-3	-5	15	9	25
28	22	-2	-3	6	4	9
28	27	-2	2	-4	4	4
29	21	-1	-4	4	1	16
30	29	0	4	0	0	16
31	27	1	2	2	1	4
33	29	3	4	12	9	16
35	28	5	3	15	25	9
				$\sum XY = 123$	$\sum X^2 = 138$	$\sum Y^2 = 164$

The lines regression are given by

$$(y - \bar{y}) = \frac{\sum XY}{\sum X^2} (x - \bar{x})$$

$$(y - 25) = 0.8913(x - 30)$$

$$y - 25 = 0.8913x - 26.739$$

$$y = 0.8913x - 1.739$$

Also , $(x - \bar{x}) = \frac{\sum XY}{\sum Y^2} (y - \bar{y})$

$$(x - 30) = 0.75(y - 25)$$

$$x - 30 = 0.75y - 18.75$$

$$x = 0.75y + 11.25$$

$$\frac{\sum XY}{\sum X^2} = \frac{123}{138} = 0.8913$$

$$\frac{\sum XY}{\sum Y^2} = \frac{123}{164} = 0.75$$

When $y = 16$, $x = 0.75(16) + 11.25$ $x = 23.25 \cong 23$

Thus husband's age is 23 years corresponding to wife age of 16 years.

6. In a partially destroyed lab record, only the lines of regression of y on x and x on y are available as $4x - 5y + 33 = 0$ and $20x - 9y = 107$ respectively. Calculate the coefficient of correlation between x and y .

Sol: Lines of regression of y on x and x on y are

$$4x - 5y + 33 = 0$$

$$20x - 9y = 107$$

$$5y = 4x + 33$$

$$20x = 9y + 107$$

$$y = \frac{4}{5}x + \frac{33}{5}$$

$$x = \frac{9}{20}y + \frac{107}{20}$$

$$\text{Coefficient of } x = \frac{4}{5}$$

$$\text{Coefficient of } y = \frac{9}{20}$$

The coefficient of correlation is

$$r = \sqrt{(\text{coeff. of } x) \cdot (\text{coeff. of } y)}$$

$$r = \sqrt{\left(\frac{4}{5}\right) \cdot \left(\frac{9}{20}\right)}$$

r must be positive since regression coefficients are +ve

$$r = \sqrt{\frac{36}{100}}$$

$$r = \frac{6}{10}$$

$$r = 0.6$$

7. Given the equation of the regression lines $x = 19.13 - 0.87y$, $y = 11.64 - 0.5x$.

Compute the mean of x 's, mean of y 's and the coefficient of correlation.

Sol: To find $\bar{x}$, $\bar{y}$ and r .

We know that the regression lines passes through $(\bar{x}, \bar{y})$.

$$\therefore \bar{x} = 19.13 - 0.87\bar{y} \quad \text{and} \quad \bar{y} = 11.64 - 0.5\bar{x}$$

That is , $\bar{x} + 0.87\bar{y} = 19.13$
 $0.5\bar{x} + \bar{y} = 11.64$

Solving above equations , we get

$\bar{x} = 15.94$ and $\bar{y} = 3.67$

Rewriting the equations

$y = -0.5x + 11.64$;

Coefficient of $x = -0.5$

$x = -0.87y + 19.13$

Coefficient of $y = -0.87$

The coefficient of correlation is

$r = -\sqrt{(\text{coeff. of } x) \cdot (\text{coeff. of } y)}$ r must be negative since regression coefficients are –ve

$r = -\sqrt{(-0.5) \cdot (-0.87)}$

$r = -0.6595$



8. Compute $\bar{x}$, $\bar{y}$ and r from the following equation of the regression lines $2x + 3y + 1 = 0$ and $x + 6y - 4 = 0$

Sol: We know that the regression lines passes through $(\bar{x}, \bar{y})$.

$$\therefore 2\bar{x} + 3\bar{y} = -1 \quad \text{and} \quad \bar{x} + 6\bar{y} = 4$$

Solving above equations , we get

$$\bar{x} = -2 \quad \text{and} \quad \bar{y} = 1$$

Consider the regression lines

$$2x + 3y + 1 = 0 \quad \text{and}$$

$$x + 6y - 4 = 0$$

$$3y = -2x - 1 \quad \text{and}$$

$$x = -6y + 4$$

$$y = -\frac{2}{3}x - \frac{1}{3}$$

$$\text{Coefficient of } x = -\frac{2}{3}$$

$$\text{Coefficient of } y = -6$$

The coefficient of correlation is

$$r = -\sqrt{(\text{coeff. of } x) \cdot (\text{coeff. of } y)} \quad r \text{ must be negative since regression coefficients are } -ve$$

$$r = -\sqrt{\left(-\frac{2}{3}\right) \cdot (-6)}$$

$$r = -\sqrt{4}$$

$r = -2$, this is absurd. Since r lies between -1 and +1.

Rewriting the regression lines

$$x + 6y - 4 = 0 \quad \text{and}$$

$$2x + 3y + 1 = 0$$

$$6y = -x + 4 \quad \text{and}$$

$$2x = -3y - 1$$

$$y = -\frac{1}{6}x + \frac{4}{6} \quad \text{and}$$

$$x = -\frac{3}{2}y - \frac{1}{2}$$

Coefficient of $x = -\frac{1}{6}$

Coefficient of $y = -\frac{3}{2}$

The coefficient of correlation is

$$r = -\sqrt{(\text{coeff. of } x) \cdot (\text{coeff. of } y)}$$

r must be negative since regression coefficients are –ve

$$r = -\sqrt{\left(-\frac{1}{6}\right) \cdot \left(-\frac{3}{2}\right)}$$

$$r = -\sqrt{\frac{1}{4}}$$

$$r = -0.5$$

Thus, $\bar{x} = -2$, $\bar{y} = 1$ and $r = -0.5$

9. Given _____ and $r = 0.8$, write down the equation of the lines of _____

	x-series	y-series
Mean	18	100
SD	14	20

regression and hence find the most probable value of y when $x = 70$.

Sol: By data $\bar{x} = 18$ and $\bar{y} = 100$; $\sigma_x = 14$ and $\sigma_y = 20$; $r = 0.8$

The equations of regression lines are

$(y - \bar{y}) = r \frac{\sigma_y}{\sigma_x} (x - \bar{x})$
;
 $(x - \bar{x}) = r \frac{\sigma_x}{\sigma_y} (y - \bar{y})$

$(y - 100) = (0.8) \frac{20}{14} (x - 18)$
;
 $(x - 18) = (0.8) \frac{14}{20} (y - 100)$

$y - 100 = 1.14(x - 18)$
;
 $x - 18 = 0.56(y - 100)$

$y = 1.14x + 79.48$
;
 $x = 0.56y - 38$

When $x = 70$, $y = 1.14(70) + 79.48$; $y = 159.28$

10. Show that if θ is the angle between the lines of regression, then $\tan\theta = \frac{\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2} \left(\frac{1-r^2}{r} \right)$

Sol:

We know that if θ is acute, the angle between the lines $y = m_1x + c_1$ and $y = m_2x + c_2$ is given by,

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1m_2} \quad \text{--- (*)}$$

The equations of regression lines are ,

$$(y - \bar{y}) = r \frac{\sigma_y}{\sigma_x} (x - \bar{x}) \quad \text{--- (1)} \quad \text{and} \quad (x - \bar{x}) = r \frac{\sigma_x}{\sigma_y} (y - \bar{y}) \quad \text{--- (2)}$$

We rewrite the equation (2) as $(y - \bar{y}) = \frac{\sigma_y}{r \cdot \sigma_x} (x - \bar{x})$

Slopes of (1) and (2) are $m_1 = r \frac{\sigma_y}{\sigma_x}$ and $m_2 = \frac{\sigma_y}{r \cdot \sigma_x}$



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Substituting the slopes in (*), we get

$$\tan\theta = \frac{\frac{\sigma_y}{r \cdot \sigma_x} - r \frac{\sigma_y}{\sigma_x}}{1 + r \frac{\sigma_y}{\sigma_x} \cdot \frac{\sigma_y}{r \cdot \sigma_x}}$$

$$\tan\theta = \frac{\frac{\sigma_y}{\sigma_x} \left(\frac{1}{r} - r \right)}{1 + r \cdot \frac{1}{r} \frac{\sigma_y^2}{\sigma_x^2}}$$

$$\tan\theta = \frac{\frac{\sigma_y}{\sigma_x} \left(\frac{1 - r^2}{r} \right)}{1 + \frac{\sigma_y^2}{\sigma_x^2}}$$

$$\tan\theta = \frac{\frac{\sigma_y}{\sigma_x} \left(\frac{1-r^2}{r} \right)}{\frac{\sigma_x^2 + \sigma_y^2}{\sigma_x^2}}$$

$$\tan\theta = \frac{\sigma_y}{\sigma_x} \cdot \frac{\sigma_x^2}{(\sigma_x^2 + \sigma_y^2)} \left(\frac{1-r^2}{r} \right)$$

$$\tan\theta = \frac{\sigma_x \sigma_y}{(\sigma_x^2 + \sigma_y^2)} \left(\frac{1-r^2}{r} \right)$$

Note: If $r = \pm 1$, $\tan\theta = 0 \Rightarrow \theta = 0$, which implies that the two regression lines coincide and hence the variables are perfectly correlated. Also if $r = 0$, $\tan\theta = \infty \Rightarrow \theta = \frac{\pi}{2}$, which implies that the lines are perpendicular and hence the variables are uncorrelated.

11. In a bivariate distribution $\sigma_x = \sigma_y$ and the angle between the regression lines is $\tan^{-1}(3)$. Find the correlation coefficient.

Sol: Let θ be the angle between the lines of regression, we have

$$\tan\theta = \frac{\sigma_x\sigma_y}{(\sigma_x^2 + \sigma_y^2)} \left(\frac{1-r^2}{r} \right) \quad \text{---- (1)}$$

By data, $\theta = \tan^{-1}(3)$ or $\tan\theta = 3$ and $\sigma_x = \sigma_y$

$$\text{Hence (1)} \Rightarrow 3 = \frac{\sigma_x \cdot \sigma_x}{(\sigma_x^2 + \sigma_x^2)} \left(\frac{1-r^2}{r} \right)$$

$$3 = \frac{\sigma_x^2}{2\sigma_x^2} \left(\frac{1-r^2}{r} \right)$$

$$6 = \left(\frac{1-r^2}{r} \right)$$

$$6r = 1 - r^2$$



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$$r^2 + 6r - 1 = 0$$

$$r = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1}$$

$$r = \frac{-6 \pm \sqrt{36 + 4}}{2}$$

$$r = \frac{-6 \pm \sqrt{40}}{2}$$

$$r = \frac{-6 \pm 2\sqrt{10}}{2}$$

$$r = -3 \pm \sqrt{10}$$

$$r = +0.1623 \quad \text{or} \quad r = -6.1623(\text{neglect})$$

Since $-1 \leq r \leq 1$, **$r = 0.1623$**

12. If the coefficient of correlation between two variables x and y is 0.5 and the acute angle between their lines of regressions is $\tan^{-1} \left(\frac{3}{5} \right)$, show that $\sigma_x = 2\sigma_y$ or $\sigma_y = 2\sigma_x$.

Sol: By data $r = 0.5$ and $\theta = \tan^{-1} \left(\frac{3}{5} \right)$ or $\tan \theta = \frac{3}{5}$

The angle between the lines of regression is given by

$$\tan \theta = \frac{\sigma_x \sigma_y}{(\sigma_x^2 + \sigma_y^2)} \left(\frac{1 - r^2}{r} \right)$$

$$\frac{3}{5} = \frac{\sigma_x \sigma_y}{(\sigma_x^2 + \sigma_y^2)} \left(\frac{1 - \left(\frac{1}{2} \right)^2}{\frac{1}{2}} \right)$$



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$$\frac{3}{5} = \frac{\sigma_x \sigma_y}{(\sigma_x^2 + \sigma_y^2)} \left(\frac{1 - \frac{1}{4}}{\frac{1}{2}} \right)$$

$$\frac{3}{5} = \frac{\sigma_x \sigma_y}{(\sigma_x^2 + \sigma_y^2)} \left(\frac{3}{4} \cdot \frac{2}{1} \right)$$

$$\frac{3}{5} = \frac{3\sigma_x \sigma_y}{2(\sigma_x^2 + \sigma_y^2)}$$

$$\frac{1}{5} = \frac{\sigma_x \sigma_y}{2(\sigma_x^2 + \sigma_y^2)}$$

$$2(\sigma_x^2 + \sigma_y^2) = 5\sigma_x \sigma_y$$



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$$2\sigma_x^2 + 2\sigma_y^2 = 5\sigma_x\sigma_y$$

$$2\sigma_x^2 - 5\sigma_x\sigma_y + 2\sigma_y^2 = 0$$

$$2\sigma_x^2 - \sigma_x\sigma_y - 4\sigma_x\sigma_y + 2\sigma_y^2 = 0$$

$$\sigma_x(2\sigma_x - \sigma_y) - 2\sigma_y(2\sigma_x - \sigma_y) = 0$$

$$(2\sigma_x - \sigma_y)(\sigma_x - 2\sigma_y) = 0$$

Thus, we get

$$\sigma_y = 2\sigma_x \quad \text{or} \quad \sigma_x = 2\sigma_y$$



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Thank You